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MULTIDIMENSIONAL DIFFUSION PROCESS WITH PARTIAL REFLECTION ON A FIXED HYPERPLANE AND WITH GENERALIZED DIFFUSION CHARACTERISTICS

We investigate the problem on pasting two parts of a diffusion process with variable coefficients on a hyperplane with additional conjugation condition of the Wentzel type given on it. A semigroup of operators that describe the unknown generalized diffusion process is obtained by using the method of classical potential theory.

1. Introduction and problem posing

By $\mathbb{R}^d$, $d \geq 2$, we denote the real d-dimensional Euclidean space. Let $\mathcal{D}_m = \{x = (x_1, \dots, x_d) \in \mathbb{R}^d : (-1)^m x_d > 0\}, m = 1, 2$, be domains in $\mathbb{R}^d$, and let $S = \mathbb{R}^{d-1} = \{x \in \mathbb{R}^d : x_d = 0\}$ and $\overline{\mathcal{D}}_m = \mathcal{D}_m \cup S$ be the boundary and the closure of $\mathcal{D}_m$, respectively. We suppose that, on $\mathcal{D}_m$, $m = 1, 2$, a diffusion homogeneous process with bounded continuous symmetric and nonnegative definite diffusion matrix $b(x) = (b_{ij}(x))_{i,j=1}^d$ and a bounded continuous vector of transposition $a(x) = (a_i(x))_{i=1}^d$, $x \in \mathbb{R}^d$ are defined.

We denote the generating differential operator of this process by L. We suppose also that, on S, the bounded continuous functions $q_1(x)$, $q_2(x)$, $\beta_{kl}(x)$, $\alpha_k(x)$, $(k, l = 1, \dots, d-1)$ are defined and are such that $\beta(x) = \big(\beta_{kl}(x)\big)$ is a symmetric nonnegative definite matrix, $q_1(x) \geq 0$, $q_2(x) \geq 0$, and $q_1(x) + q_2(x) \neq 0$, $x \in S$. These functions will describe the corresponding continuations of the diffusion process after going out on S and, under the common Wentzel condition (see [1,2,3]), will be responsible for the partial reflection in one of the domains $\mathcal{D}_1$ or $\mathcal{D}_2$ and for the diffusion and the transposition along the border.

Our aim is to construct the semigroup of operators $(T_t)_{t\geq 0}$ that describes a continuous unprecipice process of Feller on $\mathbb{R}^d$ such that its parts in $\mathcal{D}_1$ and $\mathcal{D}_2$ coincide with the diffusion process operated by L, and the behavior of the process at the points of S is defined under the given Wentzel conjugation condition. The solution of this problem with some additional estimations of operator's coefficients L and the boundary Wentzel operator are first obtained by the author by applying some analytic methods with the use of parabolic potentials that were constructed using the common fundamental solution for a uniformly parabolic operator.

We prove that the obtained process can be interpreted as a generalized diffusion process in the sense of N.I. Portenko [4].

Let us recall [2-4] that, within the analytic approach, the problem of existence of the required semigroup actually reduces to the solution of the corresponding problem of conjugation for a linear parabolic equation of the second order, in which one of two conditions of conjugation, like the equation in a domain, is described by a linear second-order differential operator. In the considered case, the Wentzel condition is set by a uniform elliptic second-order operator by variables tangential to the border that also include the derivatives in the direction of a normal to S. Let us mention that the

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earlier common problem was investigated by analytic methods in [3,5], where an integral representation of the required semigroup was used in the construction of the special parabolic potential of a simple layer.

Moreover, the parabolic case of the initial-boundary-value Wentzel problem with second derivatives with respect to tangential variables in the boundary condition was considered also in [6] and investigated therein in the Hölder class with the use of a local method. We mention also works [7,8,9], where the problem of construction of a generalized diffusion was investigated by methods of stochastic analysis.

In this paper, we use the following notations: T is a fixed positive number; $x' = (x_1, \dots, x_{d-1})$ is a point in $\mathbb{R}^{d-1}$; by x', we sometimes denote a point like $(x_1, \dots, x_{d-1}, 0)$; $(x, y) = \sum_{i=1}^{d} x_i y_i$, $(x', y') = \sum_{i=1}^{d-1} x_i y_i$; $\nu(x') = (\nu_i(x'))_{i=1}^{d}$, $\nu_i \equiv 0$, $i = 1, \dots, d-1$, $\nu_d \equiv 1$, is a single vector of the normal to the surface S at the point x', $N(x') = b(x')\,\nu(x')$ is the normal vector; $D_t = D_t^1 = \partial/\partial t$, $D_i = \partial/\partial x_i$, $D_{ij} = \partial^2/\partial x_i \partial x_j$, $i, j = 1, \dots, d$, are the operations of differentiation; D_t^r and D_x^p are, respectively, the symbols of partial derivatives with respect to t of order r and with respect to x of order p, where r and p are integer nonnegative numbers; $\nabla = (D_1, \dots, D_d)$, $\nabla' = (D_1, \dots, D_{d-1})$ are the "spatial" gradients; $\triangle_x^{\widetilde{x}} f(\cdot, x) = f(\cdot, x) - f(\cdot, \widetilde{x})$, $\triangle_t^{\widetilde{t}} f(t, \cdot) = f(t, \cdot) - f(\widetilde{t}, \cdot)$; $\mathcal{B}(\mathbb{R}^d)$ is the Banach space of bounded and measurable functions φ with the norm $\|\varphi\| = \sup_{x \in \mathbb{R}^d} |\varphi(x)|$; $C^l(\mathcal{D})$ $\left(C^l(\overline{\mathcal{D}})\right)$, $l = 0, 1, 2$, $\left(C^0(\mathcal{D}) = C(\mathcal{D})\right)$ is the set of functions continuous in $\mathcal{D}$ (in $\overline{\mathcal{D}}$) that have derivatives D_x^p, $p \le l$, continuous in $\mathcal{D}$ (in $\overline{\mathcal{D}}$,) where $\mathcal{D}$ is the subset $\mathbb{R}^d$; $C(\Omega)$ $\left(C(\overline{\Omega})\right)$ is the set of functions continuous in Ω $(\overline{\Omega})$, where Ω is a subset from the domain $(0, \infty) \times \mathbb{R}^d$; $C^{1,2}(\Omega)$ $\left(C^{1,2}(\overline{\Omega})\right)$ is the set of functions continuous in Ω $(\overline{\Omega})$ that have derivatives D_t^r, D_x^p, $r = 1, p \le 2$, continuous in Ω (in $\overline{\Omega}$); $H^{l+\lambda}(\overline{\mathcal{D}})$ and $H^{(l+\lambda)/2,\, l+\lambda}(\overline{\Omega})$, $l = 0, 1, 2$, $\lambda \in (0, 1)$, are the spaces of Hölder's functions with norms $||\varphi||_{H^{l+\lambda}(\overline{\mathcal{D}})}$ and $||\varphi||_{H^{(l+\lambda)/2,\, l+\lambda}(\overline{\Omega})}$, respectively (see [10, Ch. II]); $\underset{0}{H}^{(l+\lambda)/2,\, l+\lambda}([0, T] \times \mathbb{R}^{d-1})$ is the subset of functions from $H^{(l+\lambda)/2,\, l+\lambda}([0, T] \times \mathbb{R}^{d-1})$, that together with its permissible derivatives with re4spect to t vanish for $t = 0$. Everywhere below, C and c are positive constants, whose specific values are usually irrelevant. The other notations will be explained in the places, where they appear for the first time.

2. Fundamental solution (f.s.) for a uniformly parabolic operator and potentials generated by it

Let us consider a uniformly parabolic operator in the domain $(t, x) \in (0, \infty) \times \mathbb{R}^d$ of the form

$$D_t - L \equiv D_t - \frac{1}{2} \sum_{i,j=1}^{d} b_{ij}(x) D_{ij} - \sum_{i=1}^{d} a_i(x) D_i, \tag{1}$$

whose real-valued coefficients satisfy the conditions

$A1)$ $(b(x)\Theta, \Theta) \ge C_0 |\Theta|^2$, $C_0 > 0$, $\forall x \in \mathbb{R}^d$, $\forall \Theta \in \mathbb{R}^d$;

$A2)$ $b_{ij} \in H^\lambda(\mathbb{R}^d)$, $a_i \in H^\lambda(\mathbb{R}^d)$, $b_{ij} = b_{ji}$, $i, j = 1, \dots, d$.

Let $g(t, x, y)$ $(t > 0,\ x, y \in \mathbb{R}^d)$ be the f.s. for the operator $D_t - L$ constructed by the Levy method (see [10, Ch. IV; 4, Ch. II]):

$$g(t, x, y) = g_0(t, x, y) + g_1(t, x, y),$$

where

$$g_0(t, x, y) = g_0^{(y)}(t, x - y) = (2\pi t)^{-d/2} (\det b(y))^{-\frac{1}{2}} \exp\Big\{ -\frac{1}{2t} \big(b^{-1}(y)(y - x), y - x\big) \Big\}$$

(by $b^{-1}(y)$, we denote here the inverse matrix to the matrix $b(y)$), and the function $g_1(t,x,y)$ is written in view of the integral operator with kernel g_0 and density Φ_0 that can be found from some integral equation.

The function $g(t,x,y)$ is nonnegative, continuous by the set of variables, continuously differentiable with respect to t, and twice continuously differentiable with respect to x. Moreover, at $t \in (0,T]$, $x, y \in \mathbb{R}^d$, the inequality

$$|D_t^r D_x^p g(t,x,y)| \le Ct^{-\frac{d+2r+p}{2}} \exp\left\{-c\frac{|y-x|^2}{t}\right\}, \quad 2r+p \le 2, \tag{2}$$

holds.

The main part of the f.s. g_0, as a function of the arguments t and x, is infinitely differentiable for $t > 0$ and satisfies inequality (2) for any integer nonnegative values of r and p. The function $g_0^{(y)}(t,z)$ and its derivatives with respect to t and z satisfy the Hölder condition by the variable y (see inequality (11.4) in [10]). We will use also estimates (13.2) and (13.3) from [10] that will be applied to differences $\triangle_x^{\widetilde{x}}(D_t^r D_x^p\, g(t,x,y))$ and $\triangle_t^{\widetilde{t}}(D_t^r D_x^p\, g(t,x,y))$, respectively, and relations (2.38) and (2.39) from [4].

We now define the following integrals using the f.s. g:

$$u_0(t,x) = \int_{\mathbb{R}^d} g(t,x,y)\varphi(y)dy, \qquad u_1(t,x) = \int_0^t d\tau \int_{\mathbb{R}^{d-1}} g(t-\tau,x,y')V(\tau,y')dy'. \tag{3}$$

Here, $\varphi(x)\,(x \in \mathbb{R}^d)$, $V(t,x')\,(t > 0,\ x' \in \mathbb{R}^{d-1})$ are given functions. The functions $u_0(t,x)$ and $u_1(t,x)$ are, respectively, the Poisson potential and the potential of a simple layer.

It is worth to mention some properties of parabolic potentials (see [10-12]). First, we consider the function $u_1(t,x)$. If $V(t,x')$ is a continuous bounded function, then the double integral in (3) exists, and the function $u_1(t,x)$ is continuous on $[0,\infty) \times \mathbb{R}^d$, satisfies the equation $(D_t - L)u_1 = 0$ in the domain $(0,\infty) \times \mathcal{D}_m$, $m = 1,2$, and the initial condition $u_1(0,x) = 0$. In the case where $V \in \underset{0}{H}^{\lambda/2,\,\lambda}([0,T] \times \mathbb{R}^{d-1})$, we have $u_1 \in \underset{0}{H}^{\frac{1+\lambda}{2},\,1+\lambda}([0,T] \times \overline{\mathcal{D}}_m)$. Below, we will use also the formula of "jump" for the potential of a simple layer (see [2,4,12]). Under our conditions, it has the form

$$\frac{\partial u_1(t,x',0\pm)}{\partial N(x')} = \int_0^t d\tau \int_{\mathbb{R}^{d-1}} \frac{\partial g(t-\tau,x',y')}{\partial N(x')} V(\tau,y')dy' \mp V(t,x'), \quad t > 0. \tag{4}$$

Here, the expressions $\frac{\partial u_1(t,x',0-)}{\partial N(x')}$ and $\frac{\partial u_1(t,x',0+)}{\partial N(x')}$ mean the limit of the conormal derivative $\frac{\partial u_1(t,x)}{\partial N(x')}$ at the point $x' \in \mathbb{R}^{d-1}$, when x tends to x' from the side of the domain $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively. The integral on the right-hand side of (4) is called a direct value of the conormal derivative of the potential of a simple layer. Its existence follows from the inequality

$$\left|\frac{\partial g(t,x',y')}{\partial N(x')}\right| \le C(t-\tau)^{-\frac{d+1-\lambda}{2}} \exp\left\{-c\frac{|x'-y'|}{t-\tau}\right\}, \quad 0 \le \tau < t \le T, x', y' \in \mathbb{R}^{d-1}.$$

Obviously, the existence of the potential u_1 can be considered under more general conditions concerning the density V.

We now consider the potential u_0 from (3). If we suppose that φ is a bounded continuous function on $\mathbb{R}^d$, then the function $u_0(t,x)$ is continuous at $t \ge 0$, $x \in \mathbb{R}^d$, satisfies the equation $(D_t - L)u_0 = 0$ in the domain $(t,x) \in (0,\infty) \times \mathbb{R}^d$, and the initial condition $u_0(0,x) = \varphi(x)$ for $x \in \mathbb{R}^d$. In addition, if $\varphi \in H^{2+\lambda}(\mathbb{R}^d)$, then $u_0 \in H^{(2+\lambda)/2,\,2+\lambda}([0,T] \times \mathbb{R}^d)$.

Remark 2.1. During establishing the existence of the classical solution of the conjugation problem that will be formulated in Section 3, we should impose additional restrictions on higher coefficients b_{ij} of the operator L from (1); this is related to the necessity to differentiate the function g with respect to the variable y. Further, we suppose that the coefficients of the operator L satisfy the condition

$$A2')\ \ b_{ij} \in H^{1+\lambda}(\mathbb{R}^d),\ a_i \in H^{\lambda}(\mathbb{R}^d),\ b_{ij} = b_{ji},\ i,j = 1,\dots,d,$$

instead of condition $A2$). In monograph [10, Ch. IV, §16], it is noted that, under conditions $A1$) and $A2'$), the f.s. g has the derivatives $\frac{\partial g}{\partial y_j}$, $\frac{\partial^2 g}{\partial x_k \partial y_j}$, $\frac{\partial^3 g}{\partial x_i \partial x_k \partial y_j}$, $\frac{\partial^2 g}{\partial t \partial y_j}$, $i,j,k = 1,\dots,d$, that are continuous functions at $x \neq y$ which satisfy inequalities of the form (16.3) and (16.4). In addition, we suppose that the density V in the integral from (3) belongs to the Hölder space $H^{(1+\lambda)/2,\,1+\lambda}([0,T]\times S)$. Then it can be shown that the function u_1 has the derivatives $D^p_{x'}u$, $p \le 2$, at $t > 0, x \in \overline{\mathcal{D}}_m$, $m = 1,2$, that change continuously during the transition through the boundary S.

3. The conjugation problem for the second-order parabolic equation and its solution

Let us consider the operator L defined in (1) and the functions $q_m\,(m = 1,2)$, $\alpha_k, \beta_{kl}\,(k,l = 1,\dots,d-1)$, defined on S. We suppose that conditions $A1$), $A2'$) are true for the coefficients of the operator L and the functions q_m, α_k, and β_{kl} satisfy the conditions

$B1$) $(\beta(x')\Theta',\Theta') \ge \delta_0|\Theta'|^2,\ \delta_0 > 0,\ \forall x' \in \mathbb{R}^{d-1}, \forall \Theta' \in \mathbb{R}^{d-1}$;

$B2$) $\beta_{kl},\ \alpha_k,\ q_m \in H^{\lambda}(\mathbb{R}^{d-1}),\ \beta_{kl} = \beta_{lk},\quad k,l = 1,\dots,d-1,\ m = 1,2$;

$B3$) $q_1(x') \ge 0,\ q_1(x') \ge 0,\ \inf(q_1(x') + q_2(x')) > 0,\ x' \in \mathbb{R}^{d-1}$.

Our aim consists in finding the solution (in the classical sense) of the conjugation problem:

$$(D_t - L)u(t,x) = 0, \quad (t,x) \in (0,\infty)\times\mathcal{D}_m, \quad m = 1,2\,, \tag{5}$$

$$u(0,x) = \varphi(x), \quad x \in \mathbb{R}^d\,, \tag{6}$$

$$u(t,x',0-) = u(t,x',0+), \quad (t,x') \in (0,+\infty)\times\mathbb{R}^{d-1}, \tag{7}$$

$$L_0 u(t,x',0) \equiv \frac{1}{2}\sum_{k,l=1}^{d-1}\beta_{kl}(x')D_{kl}u(t,x',0) + \sum_{k=1}^{d-1}\alpha_k(x')D_k u(t,x',0) -$$

$$- q_1(x')D_d u(t,x',0-) + q_2(x')D_d u(t,x',0+) = 0, \quad (t,x') \in (0,\infty)\times\mathbb{R}^{d-1}, \tag{8}$$

where $\varphi \in C(\mathbb{R}^d)\cap\mathcal{B}(\mathbb{R}^d)$ is the given function.

We will suppose that the solution $u(t,x)$ bounded by the spatial variable is a continuous function in the domain $[0,+\infty)\times\mathbb{R}^d$, belongs to the set $C^{1,2}((0,\infty)\times\mathcal{D}_m)$, $m = 1,2$, and satisfies conditions (6)-(8). In addition, the derivatives with respect to the tangential variables $D_k u, D_{kl}u,\ k,l = 1,\dots,d-1$, as well as the function u, must also change continuously during the transition through S.

Let us determine the classical solubility of problem (5)-(8) in assumption that the initial function φ from (6) satisfies the conditions

$$\varphi \in H^{2+\lambda}(\mathbb{R}^d), \quad L_0\varphi(x',0) = 0,\ x' \in \mathbb{R}^{d-1}. \tag{9}$$

Theorem 3.1. *Let the coefficients of the operators L and L_0 satisfy conditions $A1$), $A2'$) and $B1$)-$B3$), respectively, and let the function φ satisfy condition (9). Then there* *exists a unique classical solution of problem (5)-(8), for which the estimation*

$$|u(t,x)| + \sum_{i=1}^{d-1} |D_i u(t,x)| + \sum_{i,j=1}^{d-1} |D_{ij} u(t,x)| \le C||\varphi||_{H^{2+\lambda}} \tag{10}$$

holds at $(t,x) \in [0,T] \times \overline{\mathcal{D}}_m,\ m = 1,2.$

Proof. First of all, we will prove the existence of the solution $u(t,x)$. We will search it in the form

$$u(t,x) = u_0(t,x) + u_1(t,x), \quad (t,x) \in (0,\infty) \times \mathbb{R}^d, \tag{11}$$

where the function V included into u_1 is unknown and should be determined.

Let us consider the condition of conjugation (8). Extracting the conormal derivative in the expression for L_0 and using formula (4), we transform it to the form

$$L_0' u(t,x',0) \equiv \frac{1}{2} \sum_{k,l=1}^{d-1} \beta_{kl}^{(0)}(x') D_{kl} u(t,x',0) + \sum_{k=1}^{d-1} \alpha_k^{(0)}(x') D_k u(t,x',0) -$$

$$- u(t,x',0) = \Theta^{(0)}(t,x'), \quad (t,x') \in (0,\infty) \times \mathbb{R}^{d-1}, \tag{12}$$

where

$$\beta_{kl}^{(0)}(x') = (q_1(x') + q_2(x'))^{-1} (b_{dd}(x'))^{\frac{1}{2}} \beta_{kl}(x'),$$

$$\alpha_k^{(0)}(x') = (b_{dd}(x'))^{\frac{1}{2}} \Big((q_1(x') + q_2(x'))^{-1} \alpha_k(x') - q(x')\ b_{kd}(x') \Big),$$

$$q(x') = \frac{q_2(x') - q_1(x')}{q_1(x') + q_2(x')}, \quad |q(x')| \le 1,\ x' \in \mathbb{R}^{d-1},$$

$$\Theta^{(0)}(t,x') = (b_{dd}(x'))^{-\frac{1}{2}} V(t,x') - \int_0^t d\tau \int_{\mathbb{R}^{d-1}} \Big[\frac{q(x')}{(b_{dd}(x'))^{\frac{1}{2}}} \frac{\partial g(t-\tau,x',y')}{\partial N(x')} +$$

$$+ g(t-\tau,x',y') \Big] V(\tau,y') dy' - \frac{q(x')}{(b_{dd}(x'))^{\frac{1}{2}}} \frac{\partial u_0(t,x',0)}{\partial N(x')} - u_0(t,x',0).$$

We will consider equality (12) as an autonomous elliptic equation on $S = \mathbb{R}^{d-1}$ for the function $u(t,x',0) = v(t,x')$, and the variable t will be interpreted here as a parameter. From the conditions of Theorem 1, it follows that the coefficients of this equation belong to the space $H^{\lambda}(\mathbb{R}^{d-1})$. If we will suppose in advance that the unknown function V is Hölder by both variables with the coefficient λ, then it is obvious that the function $\Theta^{(0)}$ will belong to the same Hölder class (by variable x'). In addition, the conditions of Theorem 1 and the presence of the term $(-1) \cdot u(t,x',0)$ on the left-hand side of Eq. (12) guarantee also the existence, for the uniformly elliptic operator L_0', the main f.s. $\Gamma(x',y')$ $(x',y' \in \mathbb{R}^{d-1},\ x' \ne y')$ (see [13, Ch. III, §20], [14]) that can be presented in our case by the formula

$$\Gamma(x',y') = \int_0^\infty e^{-s}\, G(s,x',y') ds,$$

where $G(s,x',y')$ $\quad (s > 0,\ x',y' \in \mathbb{R}^{d-1})$ is the f.s. of the uniformly parabolic operator

$$L' = \frac{1}{2} \sum_{k,l=1}^{d-1} \beta_{kl}^{(0)}(x') D_{kl} + \sum_{k=1}^{d-1} \alpha_k^{(0)}(x') D_k - D_s.$$

Note that, for the f.s. G, one can formulate properties analogous to those formulated in Section 2 for the f.s. g.

Using the f.s. Γ, the unique solution of Eq. (12) can be written in the form

$$v(t,x') = -\int_{\mathbb{R}^{d-1}} \Gamma(x',z')\ \Theta^{(0)}(t,z')dz' =$$

$$= -\int_0^\infty e^{-s}ds \int_{\mathbb{R}^{d-1}} G(s,x',z')\ \Theta^{(0)}(t,z')dz', \quad (t,x') \in (0,\infty)\times S. \tag{13}$$

Thus, in addition to formula (11), where we should take $(t,x) = (t,x')$, we have obtained also relation (13) for the function $u(t,x',0) = v(t,x')$. Comparing the right-hand sides of these relations, we obtain the integral equation for V in the form

$$\int_0^t d\tau \int_{\mathbb{R}^{d-1}} K(t-\tau,x',y')\ V(\tau,y')dy' + \int_0^\infty e^{-s}ds\times$$

$$\times \int_{\mathbb{R}^{d-1}} G(s,x',z')(b_{dd}(z'))^{-\frac{1}{2}}V(t,z')dz' = \psi(t,x'), \quad (t,x') \in (0,\infty)\times\mathbb{R}^{d-1}, \tag{14}$$

where

$$K(t-\tau,x',y') = g(t-\tau,x',y') - \int_0^\infty e^{-s}ds\times$$

$$\times \int_{\mathbb{R}^{d-1}} G(s,x',z') \left(\frac{q(z')}{(b_{dd}(z'))^{\frac{1}{2}}} \frac{\partial g(t-\tau,z',y')}{\partial N(z')} + g(t-\tau,z',y') \right) dz',$$

$$\psi(t,x') = \int_0^\infty e^{-s}ds \int_{\mathbb{R}^{d-1}} G(s,x',z')\times$$

$$\times \left(\frac{q(z')}{(b_{dd}(z'))^{\frac{1}{2}}} \frac{\partial u_0(t,z',0)}{\partial N(z')} + u_0(t,z',0) \right) dz' - u_0(t,x',0).$$

Equation (14) is an integral first-kind equation. For the purpose of its transformation, we introduce an operator $\mathcal{E}$ acting by the formula

$$\mathcal{E}(t,x')\psi = \sqrt{\frac{2}{\pi}} \Big\{ D_t \int_0^t (t-\tau)^{-\frac{1}{2}}d\tau \int_{\mathbb{R}^{d-1}} \Big[h(\hat{t}-\tau,x',y') +$$

$$+ \int_0^\infty (1-\frac{u}{t-\tau}) e^{-\frac{u^2}{2(t-\tau)}}du \int_{\mathbb{R}^{d-1}} h(\hat{t}-\tau,x',v')\ G(u,v',y')dv' \Big] \psi(\tau,y')dy' \Big\}\Big|_{\hat{t}=t},$$

where $h(t,x',y')\quad (t>0,\ x',y' \in \mathbb{R}^{d-1})$ is the f.s. of the operator

$$D_t - \frac{1}{2}\sum_{i,j=1}^{d-1} \tilde{b}_{ij}(x')D_{ij}\,, \quad \tilde{b}_{ij} = b_{ij} - \frac{b_{id}\ b_{jd}}{b_{dd}}, \quad i,j = 1,\dots,d-1.$$

Let us prove that the function $\widehat{\psi}(t,x') = \mathcal{E}(t,x')\psi$ satisfies the condition

$$\widehat{\psi} \in \underset{0}{H}^{(1+\lambda')/2,\,1+\lambda'}([0,T]\times\mathbb{R}^{d-1}),\ \lambda' = \lambda/2. \tag{15}$$

This can be easily done using the relations

$$\sqrt{\frac{\pi}{2}}\widehat{\psi}(t,x') = -\Big\{ D_t \int_0^t (t-\tau)^{-\frac{1}{2}}d\tau \int_{\mathbb{R}^{d-1}} \Big[h(\hat{t}-\tau,x',y') + \int_0^\infty D_u\, e^{-\frac{u^2}{2(t-\tau)}}du\times \tag{16}$$

$$\times \int_{\mathbb{R}^{d-1}} h(\hat{t}-\tau,x',v')\ G(u,v',y')dv' \Big] \triangle_{y'}^{x'} u_0(\tau,y',0)dy' \Big\}\Big|_{\hat{t}=t} + \Big\{ D_t \int_0^t (t-\tau)^{-\frac{1}{2}}d\tau\times$$

$$\times \int_{\mathbb{R}^{d-1}} \frac{q(y')}{(b_{dd}(y'))^{\frac{1}{2}}} \frac{\partial u_0(\tau,y',0)}{\partial N(y')} dy' \int_0^\infty e^{-\frac{u^2}{2(t-\tau)}}du\times$$

$$\times \int_{\mathbb{R}^{d-1}} h(\hat{t}-\tau,x',v')\ G(u,v',y')dv' \Big\}\Big|_{\hat{t}=t}$$

$$(17)\quad \sqrt{\frac{\pi}{2}}D_i\widehat{\psi}(t,x') = -\Big\{D_t\int_0^t (t-\tau)^{-\frac12}d\tau\int_{\mathbb{R}^{d-1}}\Big[D_ih(\hat t-\tau,x',y')+ + \int_0^\infty D_u\, e^{-\frac{u^2}{2(t-\tau)}}du\int_{\mathbb{R}^{d-1}}D_ih(\hat t-\tau,x',v')\,\triangle_{v'}^{x'}G(u,v',y')dv'\Big]\triangle_{y'}^{x'}u_0(\tau,y',0)dy'\Big\}\Big|_{\hat t=t}+ + \Big\{D_t\int_0^t (t-\tau)^{-\frac12}d\tau\int_{\mathbb{R}^{d-1}}\frac{q(y')}{(b_{dd}(y'))^{\frac12}}\frac{\partial u_0(\tau,y',0)}{\partial N(y')}dy'\int_0^\infty e^{-\frac{u^2}{2(t-\tau)}}du\times \times\int_{\mathbb{R}^{d-1}}D_ih(\hat t-\tau,x',v')\,\triangle_{v'}^{x'}G(u,v',y')dv'\Big\}\Big|_{\hat t=t},\ i=1,\dots,d-1.$$

Before estimating the integral on the right-hand sides of (16) and (17), we note that conditions (9) yield

$$\psi(0,x)=\int_0^\infty e^{-s}ds\int_{\mathbb{R}^{d-1}}G(s,x',z')\Big(\frac{q(z')}{(b_{dd}(z'))^{\frac12}}\frac{\partial\varphi(z',0)}{\partial N(z')}+\varphi(z',0)\Big)dz'-\varphi(x',0)=0.$$

That is why in the expression for $\psi(t,x')$ and, as a result, in expressions for $\widehat{\psi}(t,x')$ and $D_i\widehat{\psi}(t,x')$, the function $u_0(t,x)$ can be replaced by the function $\Phi(t,x)=u_0(t,x)-\varphi(x)$, $t>0,\ x\in\mathbb{R}^d$. Whence and from the properties of u_0, we obtain

$$(18)\qquad \Phi\in H^{(2+\lambda)/2,\,2+\lambda}([0,T]\times\mathbb{R}^d),\qquad D_x^p\Phi(0,x)=0,\ p\le 2.$$

Now we can estimate $\widehat{\psi}$ and $D_i\widehat{\psi}$, $i=1,\dots,d-1$. First, we consider $\widehat{\psi}(t,x')$. Taking the derivative with respect to t on the right-hand side of (16), we obtain the formula

$$(19)\quad \sqrt{\frac{\pi}{2}}\widehat{\psi}(t,x') = = \frac12\int_0^t(t-\tau)^{-\frac32}d\tau\int_{\mathbb{R}^{d-1}}h(t-\tau,x',y')[\triangle_{y'}^{x'}\Phi(\tau,y',0)-(y'-x',\nabla'\Phi(\tau,x',0)]dy'- -\int_0^t d\tau\int_{\mathbb{R}^{d-1}}\triangle_{y'}^{x'}\Phi(\tau,y',0)dy'\int_0^\infty D_u\,D_t\Big((t-\tau)^{-\frac12}e^{-\frac{u^2}{2(t-\tau)}}\Big)du\times \times\int_{\mathbb{R}^{d-1}}h(t-\tau,x',v')\ [\triangle_{v'}^{x'}G(u,v',y')-(v'-x',\nabla'G(u,x',y'))]dv'- -\int_0^t d\tau\int_{\mathbb{R}^{d-1}}\triangle_{y'}^{x'}D_t\Phi(t-\tau,y',0)dy'\int_0^\infty D_u\Big(\tau^{-\frac12}e^{-\frac{u^2}{2\tau}}\Big)G(u,x',y')du+ +\int_0^t d\tau\int_{\mathbb{R}^{d-1}}\frac{q(y')}{(b_{dd}(y'))^{\frac12}}\frac{\partial\Phi(\tau,y',0)}{\partial N(y')}dy'\int_0^\infty D_t\Big((t-\tau)^{-\frac12}e^{-\frac{u^2}{2(t-\tau)}}\Big)du\times \times\int_{\mathbb{R}^{d-1}}h(t-\tau,x',v')\,\triangle_{v'}^{x'}G(u,v',y')dv'+\int_0^t d\tau\int_{\mathbb{R}^{d-1}}\frac{q(y')}{(b_{dd}(y'))^{\frac12}}\ \triangle_\tau^t\frac{\partial\Phi(\tau,y',0)}{\partial N(y')}dy'\times \times\int_0^\infty D_t\Big((t-\tau)^{-\frac12}e^{-\frac{u^2}{2(t-\tau)}}\Big)G(u,x',y')du+\int_0^\infty t^{-\frac12}e^{-\frac{u^2}{2t}}du\times \times\int_{\mathbb{R}^{d-1}}G(u,x',y')\frac{q(y')}{(b_{dd}(y'))^{\frac12}}\frac{\partial\Phi(t,y',0)}{\partial N(y')}dy'=\sum\nolimits_{i=1}^6 M_i.$$

Estimating the terms M_i on the right-hand side of (19), we consider condition (18), the theorem on average, the inequality $\sigma^\mu e^{-\varepsilon\sigma}\le const$ for $0\le\sigma<\infty$, $\varepsilon>0$, $\mu>0$, and estimation (2) for the f.s. h and G. We have

$$|\widehat{\psi}(t,x')|\le C||\varphi||_{H^{2+\lambda}(\mathbb{R}^d)}\ t^{\frac{1+\lambda'}{2}},\quad (t,x')\in[0,T]\times\mathbb{R}^{d-1},\ \lambda'=\lambda/2.$$

Similarly, using (16)-(18) and properties of the fundamental solutions, we derive the relations

$$\left|\triangle_t^{\widetilde{t}}\psi(t,x')\right| \le C||\varphi||_{H^{2+\lambda}(\mathbb{R}^d)}(t-\widetilde{t})^{\frac{1+\lambda'}{2}},\ \ 0\le\widetilde{t}<t\le T,\ x'\in\mathbb{R}^{d-1},$$
$$\left|\triangle_t^{\widetilde{t}}D_i\psi(t,x')\right| \le C||\varphi||_{H^{2+\lambda}(\mathbb{R}^d)}(t-\widetilde{t})^{\frac{\lambda'}{2}},\ \ 0\le\widetilde{t}<t\le T,\ x'\in\mathbb{R}^{d-1},$$
$$\left|\triangle_{x'}^{\widetilde{x}'}D_i\psi(t,x')\right| \le C||\varphi||_{H^{2+\lambda}(\mathbb{R}^d)}|x'-\widetilde{x}'|^{\lambda'},\ \ t\in[0,T],\ x',\widetilde{x}'\in\mathbb{R}^{d-1},$$

whence (15) follows.

We now prove that the application of the operator $\mathcal{E}$ to both sides of Eq. (14) transforms this equation to an equivalent Volterra equation of the second type

$$V(t,x')+\int_0^t d\tau\int_{\mathbb{R}}^{d-1}K_0(t-\tau,x',y')V(\tau,y')dy'=\psi_0(t,x'),\ \ t>0,\ x'\in\mathbb{R}^{d-1}, \tag{20}$$

where $\psi_0(t,x')=(b_{dd}(x'))^{\frac{1}{2}}\mathcal{E}(t,x')\psi$. For the kernel $K_0(t-\tau,x',y')$ at $0\le\tau<t\le T,x',y'\in\mathbb{R}^{d-1}$, the estimation

$$\begin{aligned}\left|K_0(t-\tau,x',y')\right| &\le C\Big[(t-\tau)^{-\frac{d+1-\lambda}{2}}e^{-c\,\frac{|x'-y'|^2}{t-\tau}}+\\ &\quad+(t-\tau)^{-1+\frac{\lambda-\gamma}{4}}\Phi_{c,\gamma}(t-\tau,x',y')\Big],\ \ 0<\gamma<\lambda,\\ \Phi_{c,\gamma}(t-\tau,x',y') &= \int_0^\infty u^{-1+\gamma/2}e^{-c\frac{u^2}{t-\tau}}(t-\tau+u)^{-\frac{d-1}{2}}e^{-c\,\frac{|x'-y'|^2}{t-\tau+u}}du\end{aligned} \tag{21}$$

holds.

Inequality (21) for the kernel K_0 allows us to apply the method of successive approximations to Eq. (20) and, as a result, to obtain V. Additionally, we check that the solution V has the same smoothness as the function ψ_0, i.e., V satisfies condition (15). Actually, this condition in combination with conditions $A2'$), (9), and estimation (2) ensure the existence of all the derivatives from Eq. (5) for u_1 and, therefore, for u, as well as the condition of conjugation (8) and, hence, the validity of inequality (10).

After we have proved that $u(t,x)$ satisfies (8), let us pass to the proof of (5)-(7). These equations follow directly from properties of the potentials u_0 and u_1 mentioned in Section 2.

Finally, proving the statement of Theorem 1 on the uniqueness of the solution, we note that the function $u(t,x)$ constructed by formulas (11) and (20) can be considered in each of the domains $(t,x)\in(0,\infty)\times\mathcal{D}_m,\ m=1,2,$ as a solution of the parabolic first boundary-value problem

$$\begin{aligned}&(D_t-L)u=0,\quad (t,x)\in(0,\infty)\times\mathcal{D}_m,\quad m=1,2,\\ &u(0,x)=\varphi(x),\quad x\in\mathcal{D}_m,\quad m=1,2,\\ &u(t,x')=v(t,x'),\quad (t,x')\in(0,\infty)\times\mathbb{R}^{d-1},\end{aligned}$$

under the concordance condition $v(0,x')=\varphi(x',0),\ x'\in\mathbb{R}^{d-1}$, where the function $v(t,x')$ is defined by formula (13).

Theorem 1 is proved. □

Remark 3.1. The existence of a unique classical solution of problem (5)-(8) can be established also without the assumption about the concordance condition $L_0\varphi(x',0)=0$ for the function φ (see (9)), leaving other conditions of Theorem 1 without changes. In this case, the solution $u(t,x)$ will be also defined by formulas (11) and (20). For it, the estimation

$$|u(t,x)|\le C||\varphi||_{H^{2+\lambda}(\mathbb{R}^d)},\ \ (t,x)\in[0,T]\times\mathbb{R}^d \tag{22}$$

will be true.

4. Construction of a generalized diffusion process

From Theorem 1, it follows that a family of linear operators $(T_t)_{t\geq 0}$ can be defined on the set of smooth functions φ. The action of the operators can be determined by the formula

$$T_t\varphi(x) = T_t^{(0)}\varphi(x) + T_t^{(1)}\varphi(x), \tag{23}$$

where $T_t^{(0)}\varphi(x) = u_0(t,x)$, $\quad T_t^{(1)}\varphi(x) = u_1(t,x)$, the functions u_0 and u_1 are presented in (3), and the density V in the potential of a simple layer u_1 is a solution of the integral equation (20). Let us prove now that the operator T_t can be applied to functions φ from the class $\mathcal{B}(\mathbb{R}^d)$. For that, it is enough to establish the existence of the double integral in (3), since the existence of the function $T_t^{(0)}\varphi(x)$ is a simple consequence of the validity of the inequality ($(t,x)\in(0,T]\times\mathbb{R}^d$)

$$\left|D_t^r\, D_x^p\; u_0(t,x)\right| \leq C\,||\varphi||\, t^{-\frac{2r+p}{2}}, \quad 2p+r\leq 2, \tag{24}$$

where we should set $r=p=0$.

Thus, we suppose that $\varphi\in\mathcal{B}(\mathbb{R}^d)$ and consider the integral equation (20). To estimate its right-hand side ψ_0, we can use (19) once again, by replacing the function Φ by u_0 everywhere and the term M_3 by the term

(25)

$$M_3' = -\int_0^{t/2} d\tau \int_{\mathbb{R}^{d-1}} \triangle_{y'}^{x'} D_t u_0(t-\tau,y',0)dy' \int_0^\infty G(u,x',y')D_u\,\left(\tau^{-\frac{1}{2}}e^{-\frac{u^2}{2\tau}}\right)du-$$
$$-\int_{t/2}^t d\tau \int_{\mathbb{R}^{d-1}} \triangle_{y'}^{x'} D_t u_0(t-\tau,y',0)dy' \int_0^\infty G(u,x',y')\triangle_\tau^t D_u\,\left(\tau^{-\frac{1}{2}}e^{-\frac{u^2}{2\tau}}\right)du-$$
$$-\int_0^\infty D_u\,\left(t^{-\frac{1}{2}}e^{-\frac{u^2}{2t}}\right)du\int_{\mathbb{R}^{d-1}} G(u,x',y')\triangle_{y'}^{x'}u_0(t/2,y',0)dy'.$$

In a similar way, we split the integrals in the expressions for M_1 and M_5 from (19) into two terms. Then, by using (2) and (24), we find

$$|\psi_0(t,x')| \leq C\,||\varphi||\,t^{-1/2}, \quad (t,x')\in(0,T]\times\mathbb{R}^{d-1}. \tag{26}$$

It follows from inequalities (21) and (26) that the method of successive approximations can be applied to Eq. (20) in this case as well.

Thus, if $\varphi\in\mathcal{B}(\mathbb{R}^d)$, then there exists a unique solution $V(t,x')$ of the integral equation (20) which is continuous at $t>0$, $x'\in\mathbb{R}^{d-1}$ and, in each domain of the form $(t,x')\in(0,T]\times\mathbb{R}^{d-1}$, allows estimation (26). Inequalities (2) and (26) will ensure the existence of the function $T_t^{(1)}\varphi(x)$ and the validity of the estimation

$$|T_t^{(1)}\varphi(x)| \leq C\,||\varphi||, \quad (t,x)\in(0,T]\times\mathbb{R}^d.$$

Uniting (24) (at $r=p=0$) and (26), the same estimation can be obtained for the function $T_t\varphi(x)$.

Next, basing on (19) and taking (25) and inequalities (2), (24), and (26) into account, we can formulate one more important property of the solution of the integral equation (20) and, as a result, of the family of operators (T_t): if for a sequence of functions $\varphi_n(x)$ on $\mathbb{R}^d$ such that $\sup\limits_n ||\varphi_n|| < \infty$, $\lim\limits_{n\to\infty}\varphi_n(x)=\varphi(x)$ for all $x\in\mathbb{R}^d$, then $\lim\limits_{n\to\infty}V(t,x',\varphi_n) = V(t,x',\varphi)$, $\lim\limits_{n\to\infty}T_t\varphi_n(x) = T_t\varphi(x)$ for all $t>0$, $x'\in\mathbb{R}^{d-1}$, $x\in\mathbb{R}^d$. This allows us to check various properties of the operator T_t only on the smooth functions φ, specifically on those that belong to space $H^{2+\lambda}(\mathbb{R}^d)$. Taking this remark into account and proceeding similarly as in [3,4,5], we make sure easily that, for the family of operators $(T_t)_{t\geq 0}$, the following properties are true: $1^0)||T_t|| \le 1$ for all $t \ge 0$; $2^0) T_t\varphi(x) \ge 0$ for all $t \ge 0, x \in \mathbb{R}^d$, whenever the function $\varphi \in \mathcal{B}(\mathbb{R}^d)$ is nonnegative. In other words, $\varphi(x) \ge 0$ for all $x \in \mathbb{R}^d$; 3^0) and, for all $t \ge 0,\ s \ge 0$, the relation $T_{t+s} = T_t\, T_s$ holds, i.e. the family $(T_t)_{t\ge 0}$ is a semigroup of operators.

From the previous properties of the operator T_t, it follow that there exists the transition probability $P(t, x, dy)$ in $\mathbb{R}^d$ such that

$$T_t\varphi(x) = \int_{\mathbb{R}^d} P(t, x, dy)\, \varphi(y)$$

for all $t > 0,\ x \in \mathbb{R}^d,\ \varphi \in \mathcal{B}(\mathbb{R}^d)$. An additional analysis of the constructed semigroup shows that the respective Markov process is a continuous Feller process and a generalized diffusion process. Its local characteristics of motion are determined by using the relation

$$\lim_{t\downarrow 0} \frac{1}{t} \int_{\mathbb{R}^d} \varphi(x)\Big[\int_{\mathbb{R}^d} (y-x, \Theta) P(t, x, dy)\Big] dx = \int_{\mathbb{R}^d} \varphi(y)\big(\alpha(y), \Theta\big) dy + $$
$$+ \int_S \varphi(y', 0)\big(\hat{\alpha}(y'), \Theta\big) dy',$$
$$\lim_{t\downarrow 0} \frac{1}{t} \int_{\mathbb{R}^d} \varphi(x)\Big[\int_{\mathbb{R}^d} (y-x, \Theta)^2 P(t, x, dy)\Big] dx = \int_{\mathbb{R}^d} \varphi(y)(b(y)\Theta, \Theta) dy + $$
$$+ \int_S \varphi(y', 0)\big(\hat{\beta}(y')\Theta', \Theta'\big)\, dy', \tag{27}$$

where $\Theta \in \mathbb{R}^d$, $\Theta' \in \mathbb{R}^{d-1}$, and φ is any continuous finite function defined on $\mathbb{R}^d$

$$\hat{\alpha}(y') = (\hat{\alpha}_i(y'))_{i=1}^d\,,\ \hat{\alpha}_i(y') = \frac{\alpha_i(y')}{q_1(y') + q_2(y')},\quad i = 1, \dots, d-1, \quad \hat{\alpha}_d(y') = q(y'),$$

$$\hat{\beta}(y') = \Big(\hat{\beta}_{kl}(y')\Big)_{k,l=1}^{d-1}, \qquad \hat{\beta}_{kl}(y') = \frac{\beta_{kl}(y')}{q_1(y') + q_2(y')} = \frac{\beta_{kl}^{(0)}(y')}{\sqrt{b_{dd}(y')}}.$$

Thus, we have proved such a theorem.

Theorem 4.1. *Let, for coefficients of the operator L from (1) and operator L_0 from (8), conditions $A1$), $A2'$), and $B1$)-$B3$) hold, respectively. Then the semigroup of operators constructed by formulas (23) and (20) determines uniquely a continuous Feller process in $\mathbb{R}^d$, i.e., a generalized diffusion one with characteristics that are expressed by relations (27).*

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