Theory of Stochastic Processes
Vol. 16 (32), no. 1, 2010, pp. 1–11

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THE MAXIMUM PRINCIPLE FOR SOME NONLINEAR STOCHASTIC CONTROL SYSTEM WITH VARIABLE STRUCTURE

Necessary conditions of optimality are derived for the stochastic control problem for a dynamical system with variable structure. The system is described by stochastic differential equations, when a control enters the drift and diffusion coefficients. The maximum principle for some non-linear stochastic control system with endpoint constraint is proved.

1. Introduction

Systems with stochastic uncertainties have raised a lot of interest in problems of nuclear fission, communication systems, self-oscillating systems, etc., where the influences of random disturbances cannot be ignored [1-3]. Variations of the structure of a system mean that it may go over at some moment from one law of movement to another. After a variation of the structure, the parameters of the initial state of the system depend on its previous ones. This joins them into a single system with variable structure [4-5]. The modern optimal stochastic control theory has been developed along the lines of Pontryagin's maximum principle and Bellman's dynamic programming [6]. The earliest papers on the extension of Pontryagin's maximum principle to stochastic control problems are [7-10]. The necessary conditions of optimality for stochastic control systems with the controlled diffusion coefficient are considered in [11, 12]. At the early stage of researches, the necessary condition of optimality in terms of the maximum principle for variable-structure stochastic control systems with the uncontrolled diffusion coefficient was obtained in [13, 14]. The present paper is dedicated to a stochastic optimal control problem for a system with variable structure, when the diffusion coefficient also contains a control.

2. Statement of the problem

Let $(\Omega, F^l, P), l = 1, ..., r$ be the probability spaces with filtration $\{F_t^l, t \in [t_{l-1}, t_l], l = 1, ..., r\}$, $0 = t_0 < t_2 < \ldots < t_r = T$. Let $w_t^1, w_t^2, \ldots, w_t^r$ be independent Wiener processes, $F_t^l = \overline{\sigma}(w_q^l, t_{l-1} \le q \le t \le t_l), l = 1, ..., r$, let $L^2_{F^l}(a, b; R^n)$ be the space of all predictable processes such that $E\int_a^b |x_t(\omega)|^2 dt < +\infty$, and let $R^{m\times n}$ be the space of linear transformations from R^m to R^n. Let also $O_l \subset R^{n_l}, Q_l \subset R^{m_l}, l = 1, \ldots, r$, be open sets, R^n be an n-dimensional Euclidean space, and $\boldsymbol{T} = [0, T]$ be a finite interval.

Consider the following stochastic control system with variable structure:

$$dx_t^l = g^l(x_t^l, u_t^l, t)dt + f^l(x_t^l, u_t^l, t)dw_t^l,\ t \in (t_{l-1}, t_l],\ l = \overline{1, r}; \tag{1}$$

$$x_{t_{l-1}}^l = x_{t_{l-1}}^{l-1},\ l = \overline{2, r};\ x_{t_0}^1 = x_0; \tag{2}$$

$$u_t^l \in U_\partial^l \equiv \{u^l(\cdot,\cdot) \in L_F^2(t_{l-1}, t_l; R^m) | u^l(t,\cdot) \in U^l \subset R^m, l = \overline{1, r}\ a.c.\}, \tag{3}$$

where $U^l, l = 1, r$ are non-empty bounded sets.

2000 *Mathematics Subject Classification.* Primary 93E20.

Key words and phrases. Variable structure system, nonlinear stochastic differential equations, stochastic optimal control problem, maximum principle, admissible controls, adjoint stochastic differential equations, optimal control problem with constraint, Ekeland variation principle.

The problem is concluded in the minimization of a cost functional:

$$J(u)=\sum_{l=1}^{r}J^l(u^l)=\sum_{l=1}^{r}E\left[\varphi^l(x_{t_l}^l)+\int_{t_{l-1}}^{t_l}p^l(x_t^l,u_t^l,t)dt\right] \tag{4}$$

which is defined on the solutions of system (1)-(3) generated by all admissible controls $U=U^1\times U^2\times\ldots\times U^r$ under the condition

$$Eq(x_T^r)\in G\subset R^k, \tag{5}$$

where G is a closed convex set in R^k.

Definition 1. The set of functions $\{x_t^l=x^l(t,\pi^l),t\in[t_{l-1},t_l],l=1,\ldots,r\}$, is said to be a solution of the equation with variable structure which corresponds to an element $\pi^r\in A^r$, if the function $x_t^l\in O_l$ at the point t_l satisfies condition (2), while it is absolutely continuous on the interval $[t_l,t_{l+1}]$ with probability 1 and satisfies Eq. (1) almost everywhere.

Consider the sets

$$A_i=\prod_{j=1}^{i}O_j\times\prod_{j=1}^{i}Q_j,\ i=1,\ldots,r, \tag{6}$$

with the elements $\pi^l=(x_{t_1}^1,x_{t_2}^2,\ldots,x_{t_l}^l,u^1,u^2,\ldots,u^l),\ l=1,\ldots,r.$

Definition 2. The element $\pi^r\in A_r$ is said to be admissible if the corresponding solution $\{x_t^l,t\in[t_{l-1},t_l],l=1,\ldots,r\}$, of system (1)-(3) satisfies condition (5).

By A_r^0, we denote the set of admissible controls.

Definition 3. The element $\widetilde{\pi}^r\in A_r^0$, is said to be an optimal solution of problem (1)-(5) if there exists a solution $\{\tilde{x}_t^l,t\in[t_{l-1},t_l],l=1,\ldots,r\}$, of system (1)-(2) and admissible controls $\tilde{u}_t^l,t\in[t_{l-1},t_l],l=1,\ldots,r$, such that the pairs $(\tilde{x}_t^l,\tilde{u}_t^l),l=1,\ldots,r$, minimize functional (4).

Let us assume that the following requirements are satisfied:

I. Functions $g^l,f^l,p^l,l=1,...,r$ and their derivatives are continuous in $(x,u,t):$

$$g^l(x,u,t):\ O_l\times Q_l\times T\to R^{n_l};$$
$$f^l(x,u,t):\ O_l\times Q_l\times T\to R^{n_l\times n_l};$$
$$p^l(x,u,t):\ O_l\times Q_l\times T\to R^1.$$

II. When (t,u) are fixed, then the functions $g^l,f^l,p^l,l=1,...,r$, are twice continuously differentiable with respect to $x,g_{xx}^l,f_{xx}^l,p_{xx}^l$ are bounded, and the following condition of linear growth is satisfied:

$$(1+|x|)^{-1}(|g^l(x,u,t)|+|g_x^l l(x,u,t)|+|f^l(x,u,t)|+$$
$$+|f_x^l(x,u,t)|+|p^l(x,u,t)|+|p_x^l(x,u,t)|)\le N.$$

III. Functions $\varphi^l(x):\ R^{n_l}\to R^1,l=1,...,r$, are twice continuously differentiable, and

$$|\varphi^l(x)|+|\varphi_x^l(x)|\le N(1+|x|),\ |\varphi_{xx}^l|\le N.$$

IV. Function $q(x):\ R^{n_r}\to R^k$ is twice continuously differentiable, and

$$|q(x)|+|q_x(x)|\le N(1+|x|),\ |q_{xx}(x)|\le N.$$

3. Main result

The following result that is a necessary condition of optimality for problem (1)–(5) has been obtained. At first, the stochastic optimal control problem (1)–(4) is being considered.

Theorem 1. *Let conditions I-III hold, and let* $(x_t^l, u_t^l), l = 1, ..., r$ *be a solution of problem (1)–(4). Then there exist the random processes* $(\psi_t^l, \beta_t^l) \in L^2_{F^l}(t_{l-1}, t_l; R^{n_l}) \times L^2_{F^l}(t_{l-1}, t_l; R^{n_l \times n_l})$ *and* $(\Phi_t^l, K_t^l) \in L^2_{F^l}(t_{l-1}, t_l; R^{n_l}) \times L^2_{F^l}(t_{l-1}, t_l; R^{n_l \times n_l})$*, which are the solutions of the adjoint equations*

$$\begin{cases} d\psi_t^l = -H_x^l(\psi_t^l, x_t^l, u_t^l, t)dt + \beta_t^l dw_t^l, \ t_{l-1} \le t < t_l, \ l = 1, ..., r; \\ \psi_{t_l}^l = -\varphi_x^l(x_{t_l}^l), \end{cases} \tag{7}$$

$$\begin{cases} d\Phi_t^l = -[g_x^{l*}(x_t^l, u_t^l, t)\Phi_t^l + \Phi_t^l g_x^l(x_t^l, u_t^l, t) + \\ + f_x^{l*}(x_t^l, u_t^l, t)\Phi_t^l f_x^l(x_t^l, u_t^l, t)dt + f_x^{l*}(x_t^l, u_t^l, t)K_t^l + K_t^l f_x^l(x_t^l, u_t^l, t) + \\ + H_{xx}^l(\psi_t^l, x_t^l, u_t^l, t)]dt + K_t^l dw_t^l, \ t_{l-1} \le t < t_l, \\ \Phi_{t_l}^l = -p_{xx}^l(x_{t_l}^l), \end{cases} \tag{8}$$

and $\forall\ u^l \in U^l, \ l = 1, ..., r,$ *a.c. fulfills the following:*

$$H^l(\psi_\theta^l, x_\theta^l, u^l, \theta) - H^l(\psi_\theta^l, x_\theta^l, u_\theta^l, \theta) +$$

$$+0.5\Delta_{u^l} f^{l*}(x_\theta^l, u_\theta^l, \theta)\Phi_\theta^l \Delta_{u^l} f^l(x_\theta^l, u_\theta^l, \theta) \le 0, \ \text{a.e. } \theta \in [t_{l-1}, t_l]. \tag{9}$$

Here,

$$H^l(\psi_t, x_t, u_t, t) = \psi_t g^l(x_t, u_t, t) + \beta_t f^l(x_t, u_t, t) - p^l(x_t, u_t, t), \ t \in [t_{l-1}, t_l], \ l = \overline{1, r},$$

and M^* *denotes the transpose of the element* M.

Proof. The existence and uniqueness of solutions of the stochastic adjoint systems (7),(8) stem from the following results [7]. □

Let A_t and B_t be the predictable bounded matrices.

Lemma 1. *The equation*

$$d\Phi_t = A_t\Phi_t dt + B_t\Phi_t dw_t, 0 < t \le 1$$
$$\Phi_0 = I,$$

has a unique solution Φ_t *with* $E\sup\|\Phi_t\|^{2s} < \infty, \ s \ge 1$*. The matrix* Φ_t *has an inverse one, and* $\Psi_t = \Phi_t^{-1}$ *is a solution of the equation*

$$d\Psi_t = -(\Psi_t A_t - \Psi_t B_t B_t) - \Psi_t B_t dw_t,$$
$$\Psi_0 = I.$$

Proof. See [7]. □

Theorem 2. *Let* $\xi : \Omega \to R^n$ *be an* $\mathfrak{T}_1$*-measurable square integrable variable, and let* $a_t \in L^2$*. Then the stochastic differential equation*

$$dp_t = -(A_t^* p_t + B_t^* q_t - a_t) + q_t dw_t, 0 \le t < 1,$$
$$p_1 = \xi$$

has a unique solution $(p_t, q_t) \in L^2 \times L^2$*. Moreover,* p_t *and* q_t *can be represented as*

$$p_t = -\Psi_1^* E\left\{\Phi_1^*\xi + \int_t^1 \Phi_s^* a_s ds|\mathfrak{T}\right\},$$
$$q_t = -B_t^* p_t - \Psi_t^* g_t,$$

where g_t is obtained from the relation

$$E\left\{\Phi_1^*\xi+\int_t^1\Phi_s^*a_sds|\mathfrak{T}_t\right\}=E\left\{\Phi_1^*\xi+\int_t^1\Phi_s^*a_sds\right\}+\int_0^t g_sdw_s.$$

Proof. See [7]. □

Here, L_2 is the space of all predictable processes in $[0,1]$.

Now we will show the fulfillment of (9). Let $\overline{u}_t^l=u_t^l+\Delta\overline{u}_t^l,\ l=\overline{1,r}$ be some admissible controls, and let $\overline{x}_t^l=x_t^l+\Delta\overline{x}_t^l,\ l=\overline{1,r}$, be the corresponding trajectories of system (1)–(3). Then

$$\begin{cases} d\Delta\overline{x}_t^l=[g^l(\overline{x}_t^l,\overline{u}_t^l,t)-g^l(x_t^l,u_t^l,t)]dt+[f^l(\overline{x}_t^l,\overline{u}_t^l,t)-f^l(x_t^l,u_t^l,t)]dw_t^l=\\ =\{\Delta_{\overline{u}^l}g^l(x_t^l,u_t^l,t)+g_x^l(x_t^l,u_t^l,t)\Delta\overline{x}_t^l+0.5\Delta x_t^*g_{xx}^l(x_t^l,u_t^l,t)\Delta x_t\}dt+\\ +\{\Delta_{\overline{u}^l}f^l(x_t^l,u_t^l,t)+f_x^l(x_t^l,u_t^l,t)\Delta\overline{x}_t^l+0.5\Delta x_t^*f_{xx}^l(x_t^l,u_t^l,t)\Delta x_t\}dtw_t^l\\ +\eta_t^1,\ t\in(t_{l_1},t_l],\\ \Delta\overline{x}_{t_{l-1}}^l=0,\ l=1,...,r, \end{cases} \tag{10}$$

where

$$\begin{aligned}\eta_t^1=&\left\{\int_0^1[g_x^{l*}(x_t^l+\mu\Delta\overline{x}_t^l,\overline{u}_t^l,t)-g_x^{l*}(x_t^l,u_t^l,t)]\Delta\overline{x}_t^ld\mu+\right.\\ &\left.+0.5\int_0^1\Delta\overline{x}_t^{l*}[g_{xx}^{l*}(x_t^0+\mu\Delta\overline{x}_t^l,u_t^l,t)-g_{xx}^{l*}(x_t^l,u_t^l,t)]\Delta\overline{x}_t^ld\mu\right\}dt+\\ &+\left\{\int_0^1[f_x^{l*}(x_t^l+\mu\Delta\overline{x}_t^l,\overline{u}_t^l,t)-f_x^{l*}(x_t^l,u_t^l,t)]\Delta\overline{x}_t^ld\mu+\right.\\ &\left.+0.5\int_0^1\Delta\overline{x}_t^{l*}[f_{xx}^{l*}(x_t^0+\mu\Delta\overline{x}_t^l,u_t^l,t)-f_{xx}^{l*}(x_t^l,u_t^l,t)]\Delta\overline{x}_t^ld\mu\right\}dw_t^l.\end{aligned}$$

According to Itô's formula [2], we have

$$\begin{aligned}&d(\psi_t^{l*}\Delta\overline{x}_t^l)=d\psi_t^{l*}\Delta\overline{x}_t^l+\psi_t^{l*}d\Delta\overline{x}_t^l+\{\beta_t^{l*}[\Delta_{\overline{u}^l}f^l(x_t^l,u_t^l,t)+\\ &f_x^l(x_t^l,u_t^l,t)\Delta\overline{x}_t^l+0.5\Delta\overline{x}_t^{l*}f_{xx}^l(x_t^l,u_t^l,t)\Delta\overline{x}_t^l]+\\ &+\beta_t^{l*}\int_0^1[f_x^l(x_t^0+\mu\Delta\overline{x}_t^l,\overline{u}_t^l,t)-f_x^l(x_t^l,\overline{u}_t^l,t)]\Delta\overline{x}_t^ld\mu+\\ &+0.5\beta_t^{l*}\int_0^1\Delta\overline{x}_t^{l*}[f_{xx}^l(x_t^0+\mu\Delta\overline{x}_t^l,\overline{u}_t^l,t)-f_{xx}^l(x_t^0,\overline{u}_t^l,t)]\Delta\overline{x}_t^l\}dt\end{aligned} \tag{11}$$

and

$$\begin{aligned}&d(\Delta\overline{x}_t^{l*}\Phi_t^l\Delta\overline{x}_t^l)=\Delta\overline{x}_t^{l*}d\Phi_t^l\Delta\overline{x}_t^l+\Delta\overline{x}_t^{l*}\Phi_t^l\Delta\overline{x}_t^l+d\Delta\overline{x}_t^{l*}\Phi_t^l\Delta\overline{x}_t^l+\\ &+\{K_t^{l*}[\Delta_{\overline{u}}f^l(x_t^l,u_t^l,t)+f_x^l(x_t^l,u_t^l,t)\Delta\overline{x}_t^l+\\ &+0.5\Delta\overline{x}_t^{l*}f_{xx}^l(x_t^l,u_t^l,t)\Delta\overline{x}_t^l]+[\Delta_{\overline{u}}f_x^l(x_t^l,u_t^l,t)+\\ &+f_x^l(x_t^l,u_t^l,t)\Delta\overline{x}_t^l+0.5\Delta\overline{x}_t^{l*}f_{xx}^l(x_t^l,u_t^l,t)\Delta\overline{x}_t^l]\Phi_t^l[\Delta_{\overline{u}}f^l(x_t^l,u_t^l,t)+\\ &+f_x^l(x_t^l,u_t^l,t)\Delta\overline{x}_t^l+0.5\Delta\overline{x}_t^{l*}f_{xx}^l(x_t^l,u_t^l,t)\Delta\overline{x}_t^l]\}dt.\end{aligned} \tag{12}$$

Almost certainly, the uniqueness of solutions of the adjoint stochastic equations (7) and (8) follows from Lemma 1 and Theorem 2 [7].

We define the stochastic processes ψ_t^l and $\Phi_t^l,\ l=1,\dots,r$, at the point t_l as follows:

$$\begin{aligned}\psi_{t_l}^l&=-\varphi_x(x_{t_l}^l)\\ \Phi_{t_l}^l&=-p_{xx}^l(x_{t_l}^l).\end{aligned}$$

With regard for (10)–(12), the expression for an increment of the cost functional (4) along the admissible control looks like

$$\Delta J(u)=\sum_{l=1}^{r}E\left\{\varphi^l(\overline{x}_{t_l}^l)-\varphi^l(x_{t_l}^l)+\int_{t_{l-1}}^{t_l}[p^l(\overline{x}_t^l,\overline{u}_t^l,t)-p^l(x_t^l,u_t^l,t)]dt\right\}=$$

$$=-\sum_{l=1}^{r}E\int_{t_{l-1}}^{t_l}[\psi_t^{l*}\Delta_{\overline{u}^l}g^l(x_t^l,u_t^l,t)+\beta_t^{l*}\Delta_{\overline{u}^l}f^l(x_t^l,u_t^l,t)-\Delta_{\overline{u}^l}p^l(x_t^l,u_t^l,t)+$$

$$+0.5\Delta_{\overline{u}^l}f^{l*}(x_t^l,u_t^l,t)\Phi_t^l\Delta_{\overline{u}^l}f^l(x_t^l,u_t^l,t)-$$

$$-0.5\Delta\overline{x}_{t_l}^{l*}f^{l*}(x_t^l,u_t^l,t)\Phi_t^lf_x^l(x_t^l,u_t^l,t)\Delta\overline{x}_{t_l}^l+\Delta\overline{x}_{t_l}^{l*}\Delta_{\overline{u}^l}g^l(x_t^l,u_t^l,t)\Phi_t^l\Delta\overline{x}_{t_l}^l-$$

$$-\Delta\overline{x}_{t_l}^{l*}g_x^l(x_t^l,u_t^l,t)\Phi_t^l\Delta\overline{x}_{t_l}^l+\Delta\overline{x}_{t_l}^{l*}\Delta_{\overline{u}^l}f^l(x_t^l,u_t^l,t)K_t^l\Delta\overline{x}_{t_l}^l-\Delta\overline{x}_{t_l}^{l*}f_x^l(x_t^l,u_t^l,t)K_t^l\Delta\overline{x}_{t_l}^l+$$

$$+\psi_t^{l*}\Delta_{\overline{u}^l}g_x^l(x_t^l,u_t^l,t)\Delta\overline{x}_{t_l}^l+\beta_t^{l*}\Delta_{\overline{u}^l}f_x^l(x_t^l,u_t^l,t)\Delta\overline{x}_{t_l}^l-\Delta_{\overline{u}^l}p_x^l(x_t^l,u_t^l,t)\Delta\overline{x}_{t_l}^l]dt+\sum_{l=1}^{r}\eta_{t_{l-1}}^{t_l},\tag{13}$$

where

$$\begin{aligned}\eta_{t_{l-1}}^{t_l}&=-E\int_0^1\Delta\overline{x}_{t_l}^{l*}(1-\mu)[\varphi_{xx}^{l*}(x_{t_l}^l+\mu\Delta\overline{x}_{t_l}^l)-\varphi_{xx}^*(x_{t_l}^l)]\Delta\overline{x}_{t_l}^ld\mu-\\&-E\int_{t_{l-1}}^{t_l}\left\{\int_0^1\Delta\overline{x}_t^{l*}(1-\mu)[p_{xx}^{l*}(x_t^l+\mu\Delta\overline{x}_t^l,u_t^l,t)-p_{xx}^{l*}(x_t^l,u_t^l,t)]\Delta\overline{x}_t^ld\mu\right\}dt+\\&+E\int_{t_{l-1}}^{t_l}\left\{\int_0^1\Delta\overline{x}_t^{l*}(1-\mu)\psi_t^{l*}[g_{xx}^l(x_t^l+\mu\Delta\overline{x}_t^l,u_t^l,t)-g_{xx}^l(x_t^l,u_t^l,t)]\Delta\overline{x}_t^ld\mu\right\}dt+\\&+E\int_{t_{l-1}}^{t_l}\int_0^1\Delta\overline{x}_t^{l*}(1-\mu)\beta_t^{l*}[f_{xx}^l(x_t^l+\mu\Delta\overline{x}_t^l,u_t^l,t)-f_{xx}^l(x_t^l,u_t^l,t)]\Delta x_td\mu dt.\end{aligned}\tag{14}$$

Performing a simple transformation of expression (14), we have

$$\Delta J(u)=\sum_{l=1}^{r}\Delta J^l(u^l)=-\sum_{l=1}^{r}E\int_{t_{l-1}}^{t_l}[\Delta_{\overline{u}^l}H^l(\psi_t^l,x_t^l,u_t^l,t)+\Delta_{\overline{u}^l}H_{x^l}^l(\psi_t^l,x_t^l,u_t^l,t)\Delta\overline{x}_t^l+$$

$$+0.5\Delta_{\overline{u}^l}f^{l*}(x_t^l,u_t^l,t)\Phi_t^l\Delta_{\overline{u}^l}f^l(x_t^l,u_t^l,t)-0.5\Delta\overline{x}_{t_l}^{l*}f^{l*}(x_t^l,u_t^l,t)\Phi_t^lf_x^l(x_t^l,y_t^l,t)\Delta\overline{x}_{t_l}^l+$$

$$+\Delta\overline{x}_{t_l}^{l*}\Delta_{\overline{u}^l}g^l(x_t^l,u_t^l,t)\Phi_t^l\Delta\overline{x}_{t_l}^l-\Delta\overline{x}_{t_l}^{l*}g_x^l(x_t^l,u_t^l,t)\Phi_t^l\Delta\overline{x}_{t_l}^l+\Delta\overline{x}_{t_l}^{l*}\Delta_{\overline{u}^l}f^l(x_t^l,u_t^l,t)K_t^l\Delta\overline{x}_{t_l}^l-$$

$$-\Delta\overline{x}_{t_l}^{l*}f_x^l(x_t^l,u_t^l,t)K_t^l\Delta\overline{x}_{t_l}^l]dt+\sum_{l=1}^{r}\eta_{t_{l-1}}^{t_l}.\tag{15}$$

Consider the spike variation

$$\Delta u_t^l=\Delta u_{t,\varepsilon^l}^{\theta^l}=\begin{cases}0, & t\notin[\theta_l,\theta_l+\varepsilon_l),\varepsilon_l\in[t_{l-1},t_l)\\ \widetilde{u}^l-u_t^l, & t\in[\theta_l,\theta_l+\varepsilon_l),\ \widetilde{u}^l\in L^2(\Omega,F^{\theta_l},P;R^m),\end{cases}$$

where $\theta_l,l=\overline{1,r}$, are Lebesgue points, and ε_l are enough small numbers.

Then expression (15) takes the form

$$\Delta_\theta J(u)=\sum_{l=1}^{r}E\int_{\theta_l}^{\theta_l+\varepsilon_l}[\Delta_{\overline{u}^l}H^l(\psi_t^l,x_t^l,u_t^l,t)+$$

$$+0.5\Delta_{\overline{u}^l}f^{l*}(x_t^l,u_t^l,t)\Phi_t^l\Delta_{\overline{u}^l}f^l(x_t^l,u_t^l,t)]dt+\sum_{l=1}^{r}\eta_{\theta_l}^{\theta_l+\varepsilon_l}.\tag{16}$$

The following lemma will be used in the estimation of increment (16) .

Lemma 2. *Let us assume that conditions I-III are satisfied. If* $\varepsilon_l \to 0,\ l = 1, ..., r,$ *then* $E|x^{\theta_l}_{t,\varepsilon_l} - x^l_t|^2 \leq N\varepsilon_l$, *where* $x^{\theta_l}_{t,\varepsilon_l}$ *are trajectories of system (1)–(2) corresponding to the controls* $u^{\theta_l}_{t,\varepsilon_l} = u^l_t + \Delta u^{\theta_l}_{t,\varepsilon_l}$, *respectively.*

Proof. We denote $\widetilde{x}^l_{t,\varepsilon_l} = x^{\theta_l}_{t,\varepsilon_l} - x^l_t$. It is clear that $\forall\ t \in [t_{l-1}, \theta_l)\ \widetilde{x}^l_{t,\varepsilon_l} = 0, l = 1, r$. Then, for $\forall\ t \in [\theta_l, \theta_l + \varepsilon_l)$,

$$d\widetilde{x}^l_{t,\varepsilon_l} = [g^l(x^l_t + \varepsilon^l \widetilde{x}^l_{t,\varepsilon_l}, \widetilde{u}^l, t) - g^l(x^l_t, u^l_t, t)]dt+$$

$$+[f^l(x^l_t + \varepsilon^l \widetilde{x}^l_{t,\varepsilon_l}, \widetilde{u}^l, t) - f^l(x^l_t, u^l_t, t)]dw^l_t,\ t \in (\theta_l, \theta_l + \varepsilon_l)$$

$$\widetilde{x}^l_{\theta_l,\varepsilon_l} = -(g^l(x^l_{\theta_l}, \widetilde{u}^l, \theta_l) - g(x^l_{\theta_l}, u^l_{\theta_l}, \theta_l))$$

or

$$\widetilde{x}^l_{\theta_l+\varepsilon_l,\varepsilon_l} = \int_{\theta_l}^{\theta_l+\varepsilon_l} [g^l(x^l_s + \varepsilon_l \widetilde{x}^l_{s,\varepsilon_l}, u^l, s) - g^l(x^l_s, u^l_s, s)]ds+$$

$$+\int_{\theta_l}^{\theta_l+\varepsilon_l} [g^l(x^l_\theta, u^l_{\theta_1}, \theta_l) - g^l(x^l_s, u^l_s, s)]ds + \int_{\theta_l}^{\theta_l+\varepsilon_l} [f^l(x^l_s + \varepsilon^l \widetilde{x}^l_{s,\varepsilon_l}, u^l_s, s)-$$

$$-f^l(x^l_s, u^l_s, s)]dw^l_s + \int_{\theta_l}^{\theta\nu+\varepsilon_l} [g^l(x^l_s, \widetilde{u}^l, s) - g^l(x^l_{\theta_l}, \widetilde{u}^l, \theta_l)]ds.$$

Therefore, conditions I-II and the Gronwall's inequality yield

$$E|\widetilde{x}^l_{\theta_l+\varepsilon_l,\varepsilon_l}|^2 \leq N\Bigg[\varepsilon^2_l E \sup_{\theta_l \leq t \leq \theta_l+\varepsilon_l} |x^{\theta_l}_{t,\varepsilon_l} - x^l_t|^2 + \varepsilon^2_l E \sup_{\theta_l \leq t \leq \theta_l+\varepsilon_l} |x^l_t - x^l_{\theta_l}|^2+$$

$$+\sup_{\theta_l \leq t \leq \theta_l+\varepsilon_l} \varepsilon^2_l E|g^l(x^l_t, \widetilde{u}^l, t) - g^l(x^l_{\theta_l}, \widetilde{u}^l, \theta_l)|^2+$$

$$+\varepsilon_l E \int_{\theta_l}^{\theta_l+\varepsilon_l} |f^l(x^l_t, u^l_t, t) - f^l(x^l_{\theta_l}, u^l_{\theta_l}, \theta_l)|^2 dt+$$

$$+\varepsilon^2_l E \int_{\theta_l}^{\theta_l+\varepsilon_l} |g^l(x^l_t, u^l_t, t) - g^l(x^l_{\theta_l}, u^l_{\theta_l}, \theta_l)|^2 dt\Bigg].$$

Hence,

$$E|\widetilde{x}^l_{t+\varepsilon^l,\varepsilon^l}|^2 \leq \varepsilon^l N,\ \varepsilon^l \to 0,\ \forall\ t \in [\theta^l, \theta^l + \varepsilon^l).$$

For $\forall\ t \in [\theta^l + \varepsilon^l, t_l]$, we have

$$d\widetilde{x}^l_{t,\varepsilon^l} = [g^l(x^l_t + \varepsilon^l \widetilde{x}^l_{t,\varepsilon}, u^l_t, t) - g^l(x^l_t, u^l_t, t)]dt + [f^l(x^l_t + \varepsilon^l \widetilde{x}^l_{t,\varepsilon}, u^l_t, t) - f(x^l_t, u^l_t, t)]dw^l_t.$$

Consequently, we obtain

$$d\widetilde{x}^l_{t,\varepsilon^l} = \int_0^1 g^l_x(x^l_t + \mu\varepsilon^l \widetilde{x}^l_{t,\varepsilon^l}, u^l_t, t)\widetilde{x}^l_{t,\varepsilon^l} d\mu dt + \int_0^1 f^l_x(x^l_t + \mu\varepsilon^l \widetilde{x}^l_{t,\varepsilon^l}, u^l_t, t)\widetilde{x}^l_{t,\varepsilon^l} d\mu dt,$$

$$\widetilde{x}^l_{\theta^l+\varepsilon^l,\varepsilon^l} = -(g^l(x^l_{\theta^l+\varepsilon^l}, u^l_{\theta^l+\varepsilon^l}, \theta^l) - g(x^l_{\theta^l+\varepsilon^l}, \widetilde{u}^l, \theta^l)).$$

Hence,

$$E|\widetilde{x}^l_{t,\varepsilon^l}|^2 \leq \varepsilon^l N,\ \text{for}\ \forall\ t \in [\theta^l + \varepsilon^l, t_l],\ \text{if}\ \varepsilon_l \to 0.$$

Thus,

$$\sup_{t_{l-1} \leq t \leq t_l} E|\widetilde{x}^l_{t,\varepsilon^l}|^2 \leq N\varepsilon_l,\ l = 1, r.$$

Lemma 2 is proved. □

Due to Lemma 2, expression (14) yields the estimation

$$\eta_{\theta_l}^{\theta_l+\varepsilon_l} = o(\varepsilon_l).$$

Then, according to the optimality of $u_t^l,\ l=\overline{1,r}$, it follows from (16) that

$$\Delta_{\theta^l} J(u) = -E[\psi_{\theta^l}^{l*}\Delta_{\overline{u}^l} g^l(x_{\theta^l}^l, u_{\theta^l}^l, \theta^l) - \Delta_{\overline{u}^l} p^l(x_{\theta^l}^l, u_{\theta^l}^l, \theta^l)+$$

$$+0.5\Delta_{\overline{u}^l} f^{l*}(x_{\theta^l}^l, u_{\theta^l}^l, \theta^l)\Phi_{\theta^l}^l \Delta_{\overline{u}^l} f^l(x_{\theta^l}^l, u_{\theta^l}^l, \theta^l) + o(\varepsilon_l) \geq 0.$$

Hence, due to the sufficient smallness of ε_l, relation (9) is satisfied. Theorem 1 is proved.

Then using Theorem 1 and the Ekeland variation principle [15], we obtain the following necessary condition of optimality for the stochastic control problem with the endpoint constraint (5).

Theorem 3. *Let conditions I-IV hold, and let* $(x_t^l, u_t^l), l=\overline{1,r}$ *be solutions of problem (1)–(5). Then exist the nonzero* $(\lambda_0,\lambda_1)\in R^{k+1}$ *and the random processes* $(\psi_t^l,\beta_t^l)\in L_{F^l}^2(t_{l-1},t_l;R^{n_l})\times L_{F^l}^2(t_{l-1},t_l;R^{n_l\times n_l})$ *and* $(\Phi_t^l,K_t^l)\in L_{F^l}^2(t_{l-1},t_l;R^{n_l})\times L_{F^l}^2(t_{l-1},t_l;R^{n_l\times n_l})$ *which are solutions of the adjoint equations*

$$\begin{cases} d\psi_t^l &= -H_x^l(\psi_t^l, x_t^l, u_t^l, t)dt + \beta_t^l dw_t^l,\ t_{l_1}\leq t< t_l,\ l=\overline{1,r};\\ \psi_{t_l}^l &= -\varphi_x^l(x_{t_l}^l),\ l=1,...,r-1;\\ \psi_T^r &= \lambda_0\varphi_x^r(x_T^r) - \lambda_1 q_x^r(x_T^r).\end{cases} \tag{17}$$

$$\begin{cases} d\Phi_t^l = -[g_x^{l*}(x_t^l,u_t^l,t)\Phi_t^l + \Phi_t^l g_x^l(x_t^l,u_t^l,t)+\\ +f_x^{l*}(x_t^l,u_t^l,t)\Phi_t^l f_x^l(x_t^l,u_t^l,t)dt + f_x^{l*}(x_t^l,u_t^l,t)K_t^l + K_t^l f_x^l(x_t^l,u_t^l,t)+\\ +H_{xx}^l(\psi_t^l,x_t^l,u_t^l,t)]dt + K_t^l dw_t^l,\ t_{l-1}\leq t<t_l, l=\overline{1,r};\\ \Phi_{t_l}^l = -\varphi_{xx}^l(x_{t_l}^l),\ l=\overline{1,r-1};\\ \Phi_T^r = -\lambda_0\varphi_{xx}^r(x_T^r) - \lambda_1 q_{xx}^r(x_T^r),\end{cases} \tag{18}$$

and $\forall\ u^l\in U^l, l=\overline{1,r}$*, a.c. fulfills the following:*

$$H^l(\psi_\theta^l, x_\theta^l, u^l, \theta) - H^l(\psi_\theta^l, x_\theta^l, u_\theta^l, \theta)+$$

$$+0.5\Delta_{u^l} f^{l*}(x_\theta^l, u_\theta^l, \theta)\Phi_\theta^l \Delta_{u^l} f^l(x_\theta^l, u_\theta^l, \theta) \leq 0\ \text{a.e. } \theta\in[t_{l-1},t_l]. \tag{19}$$

Proof. For any natural j, we introduce the approximating functional

$$J_j(u) = S_j\left(\sum_{l=1}^r E\left[\varphi^l(x_{t_l}^l) + \int_{t_{l-1}}^{t_l} p(x_t^l,u_t^l,t)dt\right], Eq(x_T^r)\right) =$$

$$= \min_{(c,y)\in\mathcal{E}} \sqrt{\left|c - 1/j - \sum_{l=1}^r E\left[\varphi^l(x_{t_l}^l) + \int_{t_{l-1}}^{t_l} p(x_t^l,u_t^l,t)dt\right]\right|^2 + \|y - Eq(x_T^r)\|^2},$$

$\mathcal{E} = \{(c,y) : c\leq I^0, y\in G\}$, and I^0 is the minimal value of the functional in (1)–(5). Let $V^l \equiv (U_t^l, d^l)$ be the space of controls obtained by introducing the metric $d^l(u,v) = (L\otimes P)\{(t,\omega)\in[t_{l-1},t_l]\times\Omega : v_t^l\neq u_t^l\}$, and let $V = V^1\times V^2\times\ldots\times V^r$ be the complete metric space.

It is easy to show that

Lemma 3. *Let us assume that conditions I-IV hold,* $u_t^{l,n}, l=1,...,r$ *is a sequence of admissible controls from* V^l*, and* $x_t^{l,n}$ *is a sequence of the corresponding trajectories of system (1)–(3). If* $d(u_t^{l,n},u_t^l)\to 0,\ n\to\infty$*, then* $\lim_{n\to\infty}\{\sup_{t_{l-1}\leq t\leq t_l} E|x_t^{l,n} - x_t^l|^2\} = 0$*, where* x_t^l *is a trajectory corresponding to an admissible control* $u_t^l, l=\overline{1,r}$.

Due to Lemma 3, we obtain the continuity of the functional $J_j : V \to R^n$,. Then, according to the Ekeland variation principle, we have that there exists a control $u_t^{l,j}$: $d(u_t^{l,j}, u_t^l) \le \sqrt{\varepsilon_j}$ and, $\forall\, u^l \in V^l$, $J_j(u^{l,j}) \le J_j(u^l) + \sqrt{\varepsilon_j} d(u^{l,j}, u^l)$, $\varepsilon_j = \frac{1}{j}$. This inequality means that $(x_t^{l,j}, u_t^{l,j})$ is a solution of the following problem:

$$\begin{cases} I_j(u) = J_j(u) + \sqrt{\varepsilon_j} \sum_{l=1}^r E \int_{t_{l-1}}^{t_l} \delta(u_t^l, u_t^{l,j}) dt \to \min \\ dx_t^l = g(x_t^l, u_t^l, t)dt + f(x_t^l, u_t^l, t)dw_t^l,\ t \in (t_{l-1}, t_l] \\ x_{t_l-1}^{l-1} = x_{t_l}^{l-1},\ l = \overline{2,r};\ x_{t_0}^0 = x_0 \\ u_t^l \in U_\partial^l. \end{cases} \tag{20}$$

Let $(x_t^{l,j}, u_t^{l,j})$ be a solution of problem (20). Then, according to Theorem 1, there exist the random processes $\psi_t^{l,j} \in L_{F^t}^2(t_{l-1}, t_l; R^{n_l})$, $\beta_t^{l,j} \in L_{F^t}^2(t_{l-1}, t_l; R^{n_l \times n_l})$ which are solutions of the system

$$\begin{cases} d\psi_t^{l,j} = -H_x^l[\psi_t^{l,j}, x_t^{l,j}, u_t^{l,j}, t]dt + \beta_t^{l,j} dw_t,\ t \in [t_{l-1}, t_l),\ l = \overline{1,r}; \\ \psi_{t_l}^{l,j} = -\varphi_x^l(x_{t_l}^{l,j}),\ l = \overline{1, r-1}; \\ \psi_T^r = -\lambda_0^j \varphi_x^r(x_T^{r,j}) - \lambda_1^j q_x(x_T^{r,j}) \end{cases} \tag{21}$$

and the random processes $(\Phi_t^{l,j}, K_t^{l,j}) \in L_{F^l}^2(t_{l-1}, t_l; R^{n_l}) \times L_{F^l}^2(t_{l-1}, t_l; R^{n_l \times n_l})$ which are solutions of the adjoint equations

$$\begin{cases} d\Phi_t^{l,j} = -[g_x^{l*}(x_t^{l,j}, u_t^{l,j}, t)\Phi_t^{l,j} + \Phi_t^{l,j} g_x^l(x_t^{l,j}, u_t^{l,j}, t) + \\ + f_x^{l*}(x_t^{l,j}, u_t^{l,j}, t)\Phi_t^{l,j} f_x^l(x_t^{l,j}, u_t^{l,j}, t)dt + + f_x^{l*}(x_t^{l,j}, u_t^{l,j}, t)K_t^{l,j} + K_t^{l,j} f_x^l(x_t^{l,j}, u_t^{l,j}, t) + \\ + H_{xx}^l(\psi_t^{l,j}, x_t^{l,j}, u_t^{l,j}, t)]dt + K_t^{l,j} dw_t^l,\ t_{l-1} \le t < t_l,\ l = 1, ..., r; \\ \Phi_{t_l}^{l,j} = -\varphi_{xx}^l(x_{t_l}^{l,j}),\ l = 1, ..., r-1; \\ \Phi_T^{r,j} = -\lambda_0^j \varphi_{xx}^r(x_T^{r,j}) - \lambda_1^j q_{xx}^r(x_T^{r,j}), \end{cases} \tag{22}$$

where non-zero (λ_0^j and $\lambda_1^j) \in R^{k+1}$ meet the requirement

$$(\lambda_0^j, \lambda_1^j) = \left(-c_j + 1/j + \sum_{l=1}^r E\left[\varphi^l(x_{t_l}^{l,j}) + \int_{t_{l-1}}^{t_l} p^l(x_t^{l,j}, u_t^{l,j}, t)dt\right], -y_j + Eq(x_T^{r,j}) \right) / I_j^0,$$

and, almost certainly for any $\overline{u}^l \in U^l, l = \overline{1,r}$, we have

$$\begin{aligned} &H^l(\psi_t^{l,j}, x_t^{l,j}, \overline{u}_t^l, t) - H^l(\psi_t^{l,j}, x_t^{l,j}, u_t^{l,j}, t) + \\ &+ 0.5\Delta_{\overline{u}^l} f^{l*}(x_t^{l,j}, u_t^{l,j}, t)\Phi_t^{l,j} \Delta_{\overline{u}^l} f^l(x_t^{l,j}, u_t^{l,j}, t) \le 0,\ \text{a.e. } t \in [t_{l-1}, t_l],\ l = \overline{1,r}. \end{aligned} \tag{23}$$

We now show some results which are needed in what follows. Since $\|(\lambda_0^j, \lambda_1^j)\| = 1$, we may think in view of conditions I-IV that $(\lambda_0^j, \lambda_1^j) \to (\lambda_0$, and $\lambda_1)$, if $j \longrightarrow \infty$.

Lemma 4. *Let $\psi_{t_l}^{l,j}$ be a solution of system (17), and let $\psi_{t_l}^l$ be a solution of system (21). Then*

$$E\int_{t_{l-1}}^{t_l} |\psi_t^{l,j} - \psi_t^l|^2 dt + E\int_{t_{l-1}}^{t_l} |\beta_t^{l,j} - \beta_t^l|^2 dt \to 0,\ l = \overline{1,r},\ \text{if } d(u_t^{l,j}, u_t^l) \to 0,\ j \to \infty.$$

Proof. It is clear that, $\forall\ t \in [t_{l-1}, t_l]$,

$$\begin{aligned} d(\psi_t^{l,j} - \psi_t^l) &= -(H_x^l[\psi_t^{l,j}, x_t^{l,j}, u_t^{l,j}, t] - H_x^l[\psi_t^l, x_t^l, u_t^l, t])dt + (\beta_t^{l,j} - \beta_t^l)dw_t = \\ &= -(\psi_t^{l,j} g_x^l(x_t^{l,j}, u_t^{l,j}, t) + \beta_t^{l,j} f_x^l(x_t^{l,j}, u_t^{l,j}, t) - p_x^l(x_t^{l,j}, u_t^{l,j}, t) - \psi_t^l g_x^l(x_t^l, u_t^l, t) - \\ &\quad - \beta_t^l f_x^l(x_t^l, u_t^l, t) + p_x^l(x_t^l, u_t^l, t))dt + (\beta_t^{l,j} - \beta_t^l)dw_t. \end{aligned}$$

Let us square both sides of the equation. According to the Itô formula, $\forall\ s \in [t_{r-1}, T]$,

$$E|\psi_T^{r,j} - \psi_T^r|^2 - E|\psi_s^{r,j} - \psi_s^r| = 2E\int_s^T [\psi_t^{r,j} - \psi_t^r][(g_x^{r*}(x_t^{r,j}, u_t^{r,j}, t) - (g_x^{r*}(x_t^r, u_t^r, t))\psi_t^{r,j} +$$

$$+ g_x^{r*}(x_t^r, u_t^r, t)(\psi_t^{r,j} - \psi_t^r) + (f_x^{r*}(x_t^{r,j}, u_t^{r,j}, t) - f_x^{r*}(x_t^r, u_t^r, t))\beta_t^{r,j} + f_x^{r*}(x_t^r, u_t^r, t)\times$$

$$\times(\beta_t^{r,j} - \beta_t^r) - p_x^r(x_t^{r,j}, u_t^{r,j}, t) + p_x^r(x_t^r, u_t^r, t)]dt + E\int_s^T |\beta_t^{r,j} - \beta_t^r|^2 dt.$$

Taking assumptions I-IV into account and using simple transformations, we obtain

$$E\int_s^T |\beta_t^{r,j} - \beta_t^r|^2 dt + E|\psi_s^{r,j} - \psi_s^r|^2 \leq EN\int_s^T |\psi_t^{r,j} - \psi_t^r|^2 dt +$$

$$+ EN\varepsilon\int_s^T |\beta_t^{r,j} - \beta_t^r|^2 dt + E|\psi_T^{r,j} - \psi_T^r|^2.$$

Hence, according to the Gronwall inequality, we have

$$E|\psi_s^{l,j} - \psi_s^l|^2 \leq De^{N(T-s)} \text{ a.e. in } [t_{r-1}, T], \tag{21}$$

where the constant D is determined in the following way: $D = E|\psi_T^{l,j} - \psi_T^l|^2$. According to (17) and (21), we obtain $\psi_T^{r,j} \to \psi_T^r$ and $D \to 0$. Consequently, it follows from (21) that $\psi_s^{r,j} \to \psi_s^r$ in $L^2_{F^r}(t_{r-1}, T; R^{n_r})$ and $\beta_s^{r,j} \to \beta_s^r$ in $L^2_{F^r}(t_{r-1}, T; R^{n_r \times n_r})$. Then, from the expression

$$E|\psi_{t_l}^{l,j} - \psi_{t_l}^l|^2 - E|\psi_s^{l,j} - \psi_s^l|^2 = 2E\int_s^{t_l} (\psi_t^{l,j} - \psi_t^l)[(g_x^{l*}(x_t^{l,j}, u_t^{l,j}, t) -$$

$$- g_x^{l*}(x_t^l, u_t^l, t))\psi_t^{l,j} + g_x^{l*}(x_t^l, u_t^l, t)(\psi_t^{l,j} - \psi_t^l) + (f_x^{l*}(x_t^{l,j}, u_t^{l,j}, t) -$$

$$- f_x^{l*}(x_t^l, u_t^l, t))\beta_t^j + f_x^{l*}(x_t^l, u_t^l, t))(\beta_t^{l,j} - \beta_t^l) +$$

$$+ p_x^l(x_t^l, u_t^l, t) - p_x^l(x_t^{l,j}, u^{,j}l_t, t)]dt + E\int_s^{t_l} |\beta_t^{l,j} - \beta_t^l|^2 dt,$$

making simple transformations, and taking assumptions I-IV into account, we obtain

$$E\int_s^{t_l} |\beta_t^{l,j} - \beta_t^l|^2 dt + E|\psi_s^{l,j} - \psi_s^l|^2 \leq EN\int_s^{t_l} |\psi_t^{l,j} - \psi_t^l|^2 dt +$$

$$+ EN\varepsilon\int_s^{t_l} |\beta_t^{l,j} - \beta_t^l|^2 dt + E|\psi_{t_l}^{l,j} - \psi_{t_l}^l|^2.$$

Hence, according to the Gronwall inequality, we have

$$E|\psi_s^{l,j} - \psi_s^l|^2 \leq De^{N(t_l - s)} \text{ a.e. in } [t_{l-1}, t_l],\ l = 1, r-1,$$

where the constant D is determined as $D = E|\psi_{t_l}^{l,j} - \psi_{t_l}^l|^2$ with $D \to 0$. Then, it follows from (21) that $\psi_s^{l,j} \to \psi_s^l$ in $L^2_{F^l}(t_{l-1}, t_l; R^{n_l})$ and $\beta_s^{l,j} \to \beta_s^l$ in $L^2_{F^l}(t_{l-1}, t_l; R^{n_l \times n_l})$.

Lemma 4 is proved. □

Lemma 5. *Let $\Phi_{t_l}^{l,j}$ be a solution of system (18), and let $\Phi_{t_l}^l$ be a solution of system (22). Then*

$$E\int_{t_{l-1}}^{t_l} |\Phi_t^{l,j} - \Phi_t^l|^2 dt + E\int_{t_{l-1}}^{t_l} |K_t^{l,j} - K_t^l|^2 dt \to 0,\ l = 1, ..., r,\ \text{if } j \to \infty.$$

Proof.

$$d(\Phi_t^{l,j}-\Phi_t^l)=-\{(g_x^{l*}(x_t^{l,j},u_t^{l,j},t))\Phi_t^{l,j}-g_x^{l*}(x_t^j,u_t^j,t))\Phi_t^l)+(\Phi_t^{l,j}g_x^l(x_t^{l,j},u_t^{l,j},t)-$$

$$-\Phi_t^l g_x^l(x_t^j,u_t^j,t))+(f_x^{l*}(x_t^{l,j},u_t^{l,j},t)\Phi_t^{l,j}f_x^l(x_t^{l,j},u_t^{l,j},t)-f_x^{l*}(x_t^l,u_t^l,t)\Phi_t^l f_x^l(x_t^l,u_t^l,t))+$$

$$+(f_x^{l*}(x_t^{l,j},u_t^{l,j},t)K_t^{l,j}-f_x^{l*}(x_t^l,u_t^l,t)K_t^l)+(K_t^{l,j}f_x^l(x_t^{l,j},u_t^{l,j},t)-K_t^l(x_t^l,u_t^l,t))+$$

$$+H_{xx}^l(\Phi_t^{l,j},x_t^{l,j},u_t^{l,j},t)-H_{xx}^l(\Phi_t^l,x_t^l,u_t^l,t)\}dt+(K_t^{l,j}-K_t^l)dw_t^l.$$

Due to the Itô formula for $\forall\ s\in[t_{l-1},t_l)$, we have

$$E|\Phi_{t_l}^{l,j}-\Phi_{t_l}^l|^2-E|\Phi_s^{l,j}-\Phi_s^l|^2\le 2E\int_s^{t_l}[\Phi_t^{l,j}-\Phi_t^l][(g_x^{l*}(x_t^{l,j},u_t^{l,j},t)-$$

$$-g_x^{l*}(x_t^j,u_t^j,t))\Phi_t^{l,j}+g_x^{l*}(x_t^j,u_t^j,t)(\Phi_t^{l,j}-\Phi_t^l)+$$

$$+(f_x^{l*}(x_t^{l,j},u_t^{l,j},t)-f_x^{l*}(x_t^l,u_t^l,t))K_t^{l,j}+f_x^{l*}(x_t^l,u_t^l,t)(K_t^{l,j}-K_t^l)+$$

$$+H_{xx}^l(\Phi_t^{l,j},x_t^{l,j},u_t^{l,j},t)-H_{xx}^l(\Phi_t^j,x_t^j,u_t^j,t)+H_{xx}^l(\Phi_t^{l,j},x_t^l,u_t^l,t)-$$

$$-H_{xx}^l(\Phi_t^l,x_t^l,u_t^l,t)]dt+E\int_s^{t_l}|K_t^{l,j}-K_t^l|^2dt.$$

Then, after simple transformations, we obtain

$$E\int_s^{t_l}|K_t^{l,j}-K_t^l|^2dt+E|\Phi_t^{l,j}-\Phi_t^l|^2\le$$

$$\le EN\int_s^{t_l}|\Phi_t^{l,j}-\Phi_t^l|^2dt+EN\varepsilon\int_s^{t_l}|K_t^{l,j}-K_t^l|^2dt+E|\Phi_{t_l}^{l,j}-\Phi_{t_l}^l|^2.$$

According to the Gronwall inequality, we have

$$E|\Phi_s^{l,j}-\Phi_s^l|^2\le De^{-N(t_l-s)}\ \text{a.e. in}\ \ [t_{l-1},t_l),$$

where the constant D is defined as

$$D=E|\Phi_{t_l}^{l,j}-\Phi_{t_l}^l|^2+EN\varepsilon\int_s^{t_l}|K_t^{l,j}-K_t^l|^2dt.$$

So that $\Phi_T^{r,j}\to\Phi_T^r$. Hence, according to assumptions I-IV and expressions (18) and (22), we have

$$\Phi_s^{l,j}\to\Phi_s^l\ \text{in}\ L^2_{F^r}(t_{r-1},T;R^{n_r})\ \text{if}\ j\to\infty.$$

Then, according to the sufficient smallness of ε, we have $D\to 0$. Consequently,

$$\Phi_t^{l,j}\to\Phi_t^l\ \text{in}\ L^2_{F^l}(t_{l-1},t_l;R^{n_l})\ \text{and}$$
$$K_t^{l,j}\to K_t^l\ \text{in}\ L^2_{F^l}(t_{l-1},t_l;R^{n_l\times n_l}),l=\overline{1,r-1},\ \text{for}\ j\to\infty.$$

Lemma 5 is proved. □

It is follow from Lemmas 4 and 5 that we can pass to the limit in systems (21) and (22) and prove the validity of (17) and (18). In a similar way, by passing to the limit in (23), we obtain that (19) is true. Theorem 3 is proved. □

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