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# TRANSFER THEOREMS AND RIGHT-CONTINUOUS PROCESSES

A counterexample to a transfer result in [5] (Theorem 2.4, Chap. 4) is given. A new result, which provides a reasonable substitute for the disproved one, is proved as well. This result yields, in particular, a transfer theorem for processes whose paths are right-continuous but not necessarily cadlag.

## 1. Introduction

A *transfer theorem* is a result of the following type; see e.g. [3, page 112] and [5, pages 135 and 152]; see also [1].

Let $(\mathcal{X},\mathcal{E})$ and $(\mathcal{Y},\mathcal{F})$ be measurable spaces, $X$ and $X_0$ random variables with values in $(\mathcal{X},\mathcal{E})$, and $Y_0$ a random variable with values in $(\mathcal{Y},\mathcal{F})$. Suppose $X$ is defined on the probability space $(\Omega,\mathcal{A},P)$ while $X_0$ and $Y_0$ are defined on the probability space $(\Omega_0,\mathcal{A}_0,P_0)$. A transfer theorem gives conditions for the existence of a random variable $Y$, defined on an extension of $(\Omega,\mathcal{A},P)$, taking values in $(\mathcal{Y},\mathcal{F})$, and such that $(X,Y)$ is a *copy* of $(X_0,Y_0)$, namely

$$(X,Y)\sim(X_0,Y_0). \tag{1}$$

By an *extension* of $(\Omega,\mathcal{A},P)$, we mean a probability space $(\Omega_1,\mathcal{A}_1,P_1)$ such that

$$\Omega_1=\Omega\times T,\quad A\times T\in\mathcal{A}_1\quad\text{and}\quad P_1(A\times T)=P(A)$$

for all $A\in\mathcal{A}$ and some set $T$. Then, with a slight abuse of notation, $X$ can be regarded as a random variable on $(\Omega_1,\mathcal{A}_1,P_1)$. Note that $X\sim X_0$ is a necessary condition for (1).

Apart from foundational interest, transfer theorems are particularly useful in *coupling* constructions. A typical application can be outlined as follows. Some key aspect of the coupling construction is isolated and treated on a conveniently chosen probability space, say $(\Omega_0,\mathcal{A}_0,P_0)$, supporting both a copy $X_0$ of a random variable $X$ from the original probability space $(\Omega,\mathcal{A},P)$ and also a "new" random variable $Y_0$. Subsequently, $Y_0$ is "transferred" to (an extension of) the original probability space $(\Omega,\mathcal{A},P)$. One example is the construction of distributional coupling times for two versions of a classical regenerative process with inter-regeneration times that have an absolutely continuous component with respect to Lebesgue measure. Distributional coupling times can then be constructed for copies of the regeneration times and transferred to the original regenerative processes; see [5, Chap. 10, Sect. 3.4 – 3.5]. Another example is the turning of distributional couplings into nondistributional couplings; see [5, Chap. 4 – 7].

A well known transfer theorem states that, for $Y$ satisfying condition (1) to exist, it suffices that $X\sim X_0$ and $Y_0$ admits a regular conditional distribution (r.c.d.) given $X_0$. A r.c.d. is a function $K$ on $\mathcal{X}\times\mathcal{F}$ such that

- $K(x,\cdot)$ is a probability measure on $\mathcal{F}$ for fixed $x\in\mathcal{X}$;
- The map $x\mapsto K(x,F)$ is $\mathcal{E}$-measurable for fixed $F\in\mathcal{F}$;

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– $P_0\big(X_0 \in E, Y_0 \in F\big) = \int_E K(x,F)\,\mu(dx)$ for all $E \in \mathcal{E}$ and $F \in \mathcal{F}$, where $\mu$ denotes the probability distribution of $X_0$.

We recall that a r.c.d. for $Y_0$ given $X_0$ exists whenever $\mathcal{F}$ is countably generated and the probability distribution of $Y_0$ is perfect (see Section 2).

The proof of this transfer theorem is straightforward. With $K$ as above, it suffices to let

$$\Omega_1 = \Omega \times \mathcal{Y}, \quad \mathcal{A}_1 = \mathcal{A} \otimes \mathcal{F}, \quad Y(\omega, y) = y \quad \text{for } (\omega, y) \in \Omega \times \mathcal{Y}, \text{ and}$$
$$P_1(H) = \int \int I_H(\omega, y)\, K\big(X(\omega),\, dy\big)\, P(d\omega) \quad \text{for } H \in \mathcal{A} \otimes \mathcal{F}.$$

Then, $P_1(A \times F) = \int_A K(X,F)\,dP$ for all $A \in \mathcal{A}$ and $F \in \mathcal{F}$. In particular, $P_1(A \times \mathcal{Y}) = P(A)$. Further, since $X \sim \mu$,

$$P_1\big(X \in E,\, Y \in F\big) = P_1\big(\{X \in E\} \times F\big) = \int_{\{X \in E\}} K(X,F)\,dP$$
$$= \int_E K(x,F)\,\mu(dx) = P_0\big(X_0 \in E, Y_0 \in F\big) \quad \text{for all } E \in \mathcal{E}.$$

While $X \sim X_0$ is a necessary condition for (1), the existence of a r.c.d. is not. Thus, a natural question is whether the existence of a r.c.d. may be replaced by some weaker condition. Say that $Y_0$ admits a *weak-sense*-r.c.d. given $X_0$ if:

> There are a measurable space $(\mathcal{Z}, \mathcal{G})$, a subset $G \subset \mathcal{Z}$, and a bijective bi-measurable function
> $$f : (\mathcal{Y}, \mathcal{F}) \to (G, \mathcal{G} \cap G)$$
> such that $f(Y_0)$ (regarded as a $(\mathcal{Z}, \mathcal{G})$-valued random variable) admits a r.c.d. given $X_0$.

According to Theorem 2.4 of [5, Chap. 4], a random variable $Y$ satisfying (1) exists provided $X \sim X_0$ and $Y_0$ admits a weak-sense-r.c.d. given $X_0$. As it stands, however, this assertion fails to be true.

The aim of this note is to present a counterexample to Theorem 2.4 and to establish a reasonable substitute for that incorrect result; see Example 1 and Theorem 2. In [5], the flawed transfer theorem was used to turn distributional couplings of right-continuous processes on a Polish state space into nondistributional couplings. A transfer result for a right-continuous (but not necessarily cadlag) process $Y_0$ follows easily from Theorem 2; see Corollary 3.

## 2. A counterexample and two transfer results

For any probability space $(V, \mathcal{V}, Q)$, the *outer* measure $Q^*$ and the *inner* measure $Q_*$ are defined as

$$Q^*(A) = \inf\big\{Q(B) : A \subset B \in \mathcal{V}\big\} \quad \text{and} \quad Q_*(A) = 1 - Q^*(A^c) \quad \text{for } A \subset V.$$

Also, $Q$ is *perfect* if, for each measurable function $f : V \to \mathbb{R}$, there is a Borel subset $B$ of $\mathbb{R}$ such that $B \subset f(V)$ and $Q(f \in B) = 1$. If $V$ is separable metric and $\mathcal{V}$ the Borel $\sigma$-field, then $Q$ is perfect if and only if it is tight. We refer to [4] for more information on perfect probability measures.

We begin with a counterexample to Theorem 2.4.

*Example* 1. Let $\mathcal{B}$ and $\mathcal{B}^2$ be the Borel $\sigma$-fields on $[0,1]$ and $[0,1]^2$, respectively, and let $m$ be the Lebesgue measure on $\mathcal{B}$. Fix a subset $I \subset [0,1]$ with $m^*(I) = 1$ and $m_*(I) = 0$, and define $J = [0,1] \setminus I$ and

$$\Omega = J, \quad \Omega_0 = [0,1] \times I, \quad \mathcal{X} = [0,1], \quad \mathcal{Y} = I,$$
$$X(\omega) = \omega, \quad X_0(x,y) = x, \quad Y_0(x,y) = y$$

for all $\omega \in \Omega$ and $(x,y) \in \Omega_0$. All spaces are equipped with the corresponding Borel $\sigma$-fields, namely,

$$\mathcal{A} = \mathcal{B} \cap \Omega, \quad \mathcal{A}_0 = \mathcal{B}^2 \cap \Omega_0, \quad \mathcal{E} = \mathcal{B}, \quad \mathcal{F} = \mathcal{B} \cap \mathcal{Y}.$$

Define also $P = m^*$ on $\mathcal{A}$ and

$$P_0\big(H \cap \Omega_0\big) = m^*\big\{x \in I : (x,x) \in H\big\} \quad \text{for all } H \in \mathcal{B}^2.$$

Since $m^*(J) = 1 - m_*(I) = 1$, then $P$ is a probability measure on $\mathcal{A}$. Similarly, $P_0$ is a probability measure on $\mathcal{A}_0$.

It remains to see that: (i) $X \sim X_0$; (ii) $Y_0$ admits a weak-sense-r.c.d. given $X_0$; (iii) No random variable $Y$, defined on an extension of $(\Omega, \mathcal{A}, P)$ and measurable with respect to $(\mathcal{Y}, \mathcal{F})$, satisfies condition (1).

(i) Just note that

$$P(X \in B) = m^*(B \cap J) = m(B) = m^*(B \cap I) = P_0(X_0 \in B) \quad \text{for all } B \in \mathcal{B}.$$

(ii) Take $(\mathcal{Z}, \mathcal{G}) = ([0,1], \mathcal{B})$, $G = \mathcal{Y} = I$, and $f(y) = y$ for all $y \in \mathcal{Y}$. Define $K_0(x,B) = \delta_x(B)$ for all $x \in \mathcal{X}$ and $B \in \mathcal{G}$. Since $\mu = m$, where $\mu$ is the probability distribution of $X_0$, then

$$\int_A K_0(x,B)\,\mu(dx) = m(A \cap B) = m^*(A \cap B \cap I) = P_0\big(X_0 \in A,\ f(Y_0) \in B\big)$$

for all $A$, $B \in \mathcal{B}$. Hence, $K_0$ is a r.c.d. for $f(Y_0)$, regarded as a $(\mathcal{Z}, \mathcal{G})$-valued random variable, given $X_0$.

(iii) Let $Y$ be a random variable, with values in $(\mathcal{Y}, \mathcal{F})$, defined on an extension $(\Omega_1, \mathcal{A}_1, P_1)$ of $(\Omega, \mathcal{A}, P)$. Since $X \in J$ and $Y \in I$, then $X \neq Y$ everywhere on $\Omega_1$. Thus, if $(X,Y) \sim (X_0, Y_0)$, one gets the contradiction

$$P_1(\emptyset) = P_1\big(X = Y\big) = P_0\big(X_0 = Y_0\big) = 1.$$

Example 1 disproves Theorem 2.4. The parts of [5] where that flawed theorem was used concern assertions about existence of nondistributional couplings. Accordingly, these parts need to be adjusted. The conservative way of doing this is to replace the condition of existence of weak-sense-r.c.d. by r.c.d. and to add the condition of existence of left-hand limits wherever assertions are made about nondistributional couplings of right-continuous processes. We also mention that Section 3.4, page 84, does not work as it stands.

Let us turn now to a possible substitute of Theorem 2.4, partially saving the results affected by that incorrect assertion. The idea is to place a mild condition on $\mathcal{F}$ plus a condition on the probability space $(\Omega, \mathcal{A}, P)$ where $X$ lives. Our main result is the following.

**Theorem 2.** *Let $\mathcal{R} \subset \mathcal{F}$ be a countably generated sub $\sigma$-field and $\nu$ the restriction on $\mathcal{R}$ of the probability distribution of $Y_0$. Given $C \subset \mathcal{Y}$, suppose*

$$\mathcal{R} \cap C = \mathcal{F} \cap C, \quad X \sim X_0, \quad P \text{ and } \nu \text{ are perfect.}$$

*If the range of $Y_0$ is contained in $C$, then $(X,Y) \sim (X_0, Y_0)$ for some random variable $Y$, defined on an extension of $(\Omega, \mathcal{A}, P)$, measurable with respect to $(\mathcal{Y}, \mathcal{F})$, and with range included in $C$.*

*Proof.* To avoid misunderstandings, we write $Z_0$ when $Y_0$ is regarded as a $(\mathcal{Y}, \mathcal{R})$-valued random variable. The probability distribution of $Z_0$ is $\nu$. Hence, since $\mathcal{R}$ is countably generated and $\nu$ is perfect, $Z_0$ admits a r.c.d. given $X_0$. It follows that

$$(X, Z) \sim (X_0, Z_0)$$

for some random variable $Z$, defined on an extension $(\Omega_1, \mathcal{A}_1, P_1)$ of $(\Omega, \mathcal{A}, P)$ and measurable with respect to $(\mathcal{Y}, \mathcal{R})$. Also, as shown in Section 1, we can take

$$\Omega_1 = \Omega \times \mathcal{Y}, \quad \mathcal{A}_1 = \mathcal{A} \otimes \mathcal{R}, \quad Z(\omega, y) = y \text{ for all } (\omega, y) \in \Omega \times \mathcal{Y}.$$

Since $Z$ is a canonical projection, the marginal of $P_1$ on $\mathcal{R}$ is $\nu$. Since the range of $Y_0$ is contained in $C$, we have $\nu^*(C) = 1$. Therefore,

$$P_1^*(Z \in C) = P_1^*(\Omega \times C) = \nu^*(C) = 1$$

where the second equality depends on $P$ is perfect and [1, Lemma 6].

Next, define $\Omega_2 = \Omega_1$ and

$$\mathcal{A}_2 = \sigma\Big(\mathcal{A}_1 \cup \{Z \in C\}\Big).$$

Let $P_2$ be the probability on $\mathcal{A}_2$ such that $P_2 = P_1$ on $\mathcal{A}_1$ and $P_2(Z \in C) = 1$. Also, fix $c \in C$ and define $Y = c$ on $\{Z \notin C\}$ and $Y = Z$ on $\{Z \in C\}$. Then, $(\Omega_2, \mathcal{A}_2, P_2)$ is an extension of $(\Omega, \mathcal{A}, P)$ and the range of $Y$ is contained in $C$. Fix $E \in \mathcal{E}$ and $F \in \mathcal{F}$. Since $\mathcal{R} \cap C = \mathcal{F} \cap C$, we have $F \cap C = B \cap C$ for some $B \in \mathcal{R}$. Since the ranges of $Y$ and $Y_0$ are both included in $C$, we have $\{Y \in F\} = \{Y \in B\}$ and $\{Y_0 \in F\} = \{Y_0 \in B\}$. Thus, if $c \in B$, one obtains

$$\{Y \in F\} = \{Y \in B\} = \{Z \notin C\} \cup \{Z \in B \cap C\} \in \mathcal{A}_2.$$

Similarly, $\{Y \in F\} \in \mathcal{A}_2$ if $c \notin B$. Hence, $Y$ is measurable with respect to $(\mathcal{Y}, \mathcal{F})$. Finally, since $(X, Z) \sim (X_0, Z_0)$, we obtain

$$\begin{aligned} &P_2\big(X \in E,\, Y \in F\big) = P_2\big(X \in E,\, Y \in B,\, Z \in C\big) = P_1\big(X \in E,\, Z \in B\big) \\ &= P_0\big(X_0 \in E,\, Z_0 \in B\big) = P_0\big(X_0 \in E,\, Y_0 \in B\big) = P_0\big(X_0 \in E,\, Y_0 \in F\big). \end{aligned}$$

□

We conclude by applying Theorem 2 to the case when $Y_0$ is a process with right-continuous (but not necessarily cadlag) paths.

**Corollary 3.** *Suppose $X \sim X_0$, $P$ is perfect, and $Y_0 = \{Y_0(t) : t \geq 0\}$ is an $S$-valued process with right-continuous paths, where $S$ is a Polish space. Then, $(X, Y) \sim (X_0, Y_0)$ for some $S$-valued process $Y = \{Y(t) : t \geq 0\}$, with right-continuous paths, defined on an extension of $(\Omega, \mathcal{A}, P)$.*

*Proof.* Let $\mathcal{Y}$ be the set of all functions $y : [0, \infty) \to S$ and

$$f_t(y) = y(t) \quad \text{for all } t \geq 0 \text{ and } y \in \mathcal{Y}.$$

Fix an enumeration $q_1, q_2, \ldots$ of the non-negative rationals, and define

$$\phi(y) = \big(y(q_1), y(q_2), \ldots\big) \quad \text{for all } y \in \mathcal{Y}.$$

Take $\mathcal{R}$ to be the $\sigma$-field generated by $\phi : \mathcal{Y} \to S^\infty$ and $\mathcal{F}$ the $\sigma$-field generated by the evaluation maps $f_t$ for all $t \geq 0$. Clearly, $\mathcal{R} \subset \mathcal{F}$ and $\mathcal{R}$ is countably generated. Since $S$ is Polish, every Borel probability on $S$ is perfect. Hence, every Borel probability on $S^\infty$ is perfect as well. Since $\phi$ is surjective, it follows that each probability on $\mathcal{R}$ is perfect.

Now, let

$$C = \big\{y \in \mathcal{Y} : \, y \text{ right-continuous}\big\}.$$

To prove $\mathcal{R} \cap C = \mathcal{F} \cap C$, it suffices to show that $f_t|C$ (i.e., the restriction of $f_t$ to $C$) is measurable with respect to $\mathcal{R} \cap C$ for each $t \geq 0$. Fix $t \geq 0$ and take a sequence $(r_n)$ of non-negative rationals such that $r_n \downarrow t$ as $n \to \infty$. Then,

$$f_t(y) = y(t) = \lim_n y(r_n) = \lim_n f_{r_n}(y) \quad \text{whenever } y \in C,$$

and $f_{r_n}|C$ is $\mathcal{R} \cap C$-measurable since $f_{r_n}$ is $\mathcal{R}$-measurable. Thus, $f_t|C$ is measurable with respect to $\mathcal{R} \cap C$. An application of Theorem 2 completes the proof.

□

Actually, the above proof yields slightly more than asserted. Even if the state space $S$ of $Y_0$ is not Polish, Corollary 3 applies provided $S$ is a separable metric space and each Borel probability on $S$ is perfect. This happens if (and only if) $S$ is a universally measurable subset of a Polish space; see [2, Lemma 4].

## References

1. P. Berti, L. Pratelli and P. Rigo, *Gluing lemmas and Skorohod representations*, Electr. Comm. Probab. **20** (2015), 1–11.
2. P. Berti, L. Pratelli and P. Rigo, *A survey on Skorokhod representation theorem without separability*, Theory Stoch. Proc. **20(36)** (2015), no. 2, 1–12.
3. O. Kallenberg, *Foundations of modern probability*, Second Edition, Springer, New York, 2002.
4. D. Ramachandran, *Perfect measures I and II*, ISI-*Lect. Notes Series* 5 and 7, New Delhi: Macmillan, 1979.
5. H. Thorisson, *Coupling, stationarity, and regeneration*, Springer, New York, 2000.

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