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CONVOLUTED BROWNIAN MOTION: A SEMIMARTINGALE APPROACH

In this paper we analyse semimartingale properties of a class of Gaussian periodic processes, called convoluted Brownian motions, obtained by convolution between a deterministic function and a Brownian motion. A classical example in this class is the periodic Ornstein-Uhlenbeck process. We compute their characteristics and show that in general, they are never Markovian nor satisfy a time-Markov field property. Nevertheless, by enlargement of filtration and/or addition of a one-dimensional component, one can in some case recover the Markovianity. We treat exhaustively the case of the bidimensional trigonometric convoluted Brownian motion and the multidimensional monomial convoluted Brownian motion.

1. Introduction

In this note we focus our attention on a class of processes constructed as the convolution between a deterministic - possibly vector-valued - L^2-function and a Brownian motion. More precisely, for a (scalar) function φ in $L^2\big(0,1\big)$, we first define

$$X_t^{\varphi} := \int_0^t \varphi(t-s)\, dB_s + \int_t^1 \varphi(1+t-s)\, dB_s, \quad t \in [0,1], \tag{1}$$

where $(B_t, t \in [0,1])$ is a real-valued Brownian motion. We will call the process X^{φ} a (scalar) **convoluted Brownian motion**. As example, when $\varphi(s) := \frac{1}{1-e^{-\lambda}} e^{-\lambda s}$, one gets the exponential convoluted Brownian motion.

For a given φ, some properties of $\big(X_t^{\varphi}, t \in [0,1]\big)$ are immediate: the process X^{φ} is stationary, centered, Gaussian and belongs to the first chaos of B. It is periodic on the time interval $[0,1]$ and its law is also time reversal invariant.

One key point of the paper is to study the linear map $\varphi \mapsto X^{\varphi}$. We propose in Proposition 3.3, for φ smooth enough, a decomposition of X^{φ} as a $d\varphi$-mixture of simple Gaussian processes $(Z(r,\cdot))_{r\in[0,1]}$ which satisfy interesting properties. In particular we prove that, for any r in $[0,1]$, $Z(r,\cdot)$ is itself a convoluted Brownian motion associated with the indicator function of a suitable interval. It also corresponds to the random concatenation of Brownian bridges, see Proposition 3.2.

Then, when φ is differentiable, the processes X^{φ} and $X^{\varphi'}$ are linked via Equation (38). This key identity will play an important role in our study. It first permits to interpret the exponential convoluted Brownian motion as the solution of the stochastic integral equation (3) and to identify it as the celebrated periodic Ornstein-Uhlenbeck process. When φ is a trigonometric function, due to the proportionality between φ'' and φ, one derives that the pair of processes $(X^{\cos}, X^{\sin})$ is solution of an (autonomous) bidimensional system of stochastic integral equations, see (43). We also consider the scalar process $X^{\sharp k} := X^{\varphi}$, when φ is the monomial function $s \mapsto s^k$. The process $X^{\sharp k}$ is not solution of an autonomous stochastic equation but, since the derivative of a monomial of order k is a monomial of order $k-1$, it makes sense to consider $X^{\sharp k}$

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as the first component of the $(k+1)$-dimensional process $\mathbf{X}^{\sharp k}$ whose coordinates are $X^{\sharp k}, \cdots, X^{\sharp 1}, X^{\sharp 0}$.

Another central and difficult question we address concerns the Markovianity of X^φ. One way to test this property is to make explicit a semimartingale decomposition of X^φ with respect to a suitable filtration and to show that it solves a *nice* SDE. Note that in our framework, if $(\mathcal{F}_t)_{t\in[0,1]}$ stands for the natural filtration of $(B_t)_{t\in[0,1]}$, X_0^φ is in fact $\mathcal{F}_1$-measurable and therefore $(X_t^\varphi)_{t\in[0,1]}$ cannot be a $(\mathcal{F}_t)_{t\in[0,1]}$-semimartingale. Anyway, since X_0^φ belongs to the first Brownian chaos, thanks to the method of initial enlargement of filtration, $(B_t)_t$ is a semimartingale in the Brownian filtration augmented by $\sigma(X_0^\varphi)$, see Section 2.1 and Lemma 4.1 for details. Consequently $(X_t^\varphi)_{t\in[0,1]}$ is a semimartingale with respect to the enlarged filtration.

This approach can be achieved for the exponential convoluted Brownian motion, see Section 2. This process is a semimartingale and solves a SDE in an enlarged filtration. We thus deduce that it is not Markovian but a time-Markov random field, whose bridges coincide with the ones of the Ornstein-Uhlenbeck process.

We have not been able to carry on with the analysis of Markovianity for a *general* convoluted process X^φ, but we present here some relevant partial results. In Section 4, we introduce a class of multidimensional convoluted processes $\mathbf{X}^{A,\phi}$ indexed by a matrix A and a vector ϕ. As example we treat the bidimensional trigonometric convoluted Brownian motion $(X^{\cos}, X^{\sin})$ and the $(k+1)$-dimensional monomial convoluted process $\mathbf{X}^{\sharp k}$. We compute in Proposition 4.2 the covariance matrix of these stationary Gaussian processes. Under two additional assumptions $(\mathfrak{H}_1)$ and $(\mathfrak{H}_2)$, $\mathbf{X}^{A,\phi}$ solves a linear SDE and is a mixture of its bridges, see Theorem 4.1. Since the trigonometric convoluted Brownian motion satisfies $(\mathfrak{H}_1)$ and $(\mathfrak{H}_2)$, Theorem 4.1 applies to it. However, the condition $(\mathfrak{H}_1)$ fails to hold for the monomial convoluted Brownian motion. Completing the vector $\mathbf{X}^{\sharp k}$ with a one-dimensional component, we recover the Markov property, see Section 4.5.1 for details.

The originality of our contribution is based on various representations of convoluted Brownian motions and the use of initial enlargement of filtrations. This powerful tool of stochastic calculus permits to analyse them pathwise, to show their semimartingale decomposition and their (lack of) Markovianity.

2. A new lightening of the periodic Ornstein-Ulhenbeck process

In this section we analyse the special case of the exponential convoluted Brownian motion, that is the convoluted process X^φ obtained for $\varphi(s) := \frac{1}{1-e^{-\lambda}} e^{-\lambda s}, \lambda \in \mathbb{R}$. We denote it by Y. It satisfies by definition

$$(2) \qquad Y_t = \frac{1}{1-e^{-\lambda}} \int_0^t e^{-\lambda(t-s)} dB_s + \frac{1}{1-e^{-\lambda}} \int_t^1 e^{-\lambda(1+t-s)} dB_s, \quad t \in [0,1].$$

We will observe in Proposition 2.1 that $(Y_t, t \in [0,1])$ is the periodic Ornstein-Uhlenbeck process, here shortened as PerOU, known in the literature as the solution of the following stochastic integral equation with periodic boundary conditions

$$(3) \qquad \begin{cases} X_t &= X_0 + B_t - \lambda \int_0^t X_s ds, \quad t \in [0,1], \\ X_1 &= X_0, \end{cases}$$

where $(B_t, t \in [0,1])$ is a one-dimensional standard Brownian motion.

Due to the fact that its initial condition involves its final one, the PerOU process is not adapted to the natural filtration induced by the Brownian motion. Nevertheless, it has various interesting explicit representations.

In the next Section 2.1 we exhibit its Gaussian properties. In Section 2.2 we propose a new semimartingale decomposition of Y with respect to an enlarged (*grossissement de*) filtration, to overcome the adaptibility problem. This property permits to identify the disintegration of the PerOU process along its initial (and final) time marginal in Section 2.3. Finally we discuss in Section 2.4 a non trivial extension of the periodic boundary condition $Y_1 = Y_0$ of the PerOU into the more general one: $Y_1 = f(Y_0)$.

2.1. **PerOU as convoluted Brownian motion.**

Proposition 2.1. *The unique solution of* (3) *is given by*

$$Y_t = \frac{e^{-\lambda}}{1-e^{-\lambda}} \int_0^1 e^{-\lambda(t-s)} dB_s + \int_0^t e^{-\lambda(t-s)} dB_s, \quad t \in [0,1]. \tag{4}$$

with periodic boundary conditions satisfying

$$Y_1 = Y_0 = \frac{1}{1-e^{-\lambda}} \int_0^1 e^{-\lambda(1-s)} dB_s. \tag{5}$$

Y *is a stationary Gaussian process whose covariance function* $R(h) := \mathrm{Cov}(Y_s, Y_{s+h})$ *satisfies*

$$R(h) = \frac{1}{2\lambda(1-e^{-\lambda})}\Big(e^{-\lambda(1-h)} + e^{-\lambda h}\Big) = \frac{1}{2\lambda} \frac{\cosh\big(\lambda(h-1/2)\big)}{\sinh(\lambda/2)}. \tag{6}$$

Since Proposition 2.1 is a particular case of Proposition 2.4 proved below, we do not prove it separately.

One can find in [8] an application of the Gaussian aspect of the PerOU to filtering problems.

2.2. **PerOU as a semimartingale.** One already noticed that the random variable Y_0 is not $\mathcal{F}_0$-measurable. Consequently the process Y is not a $(\mathcal{F}_t)$-semimartingale. Anyway, by (3) and (5), the process $(Y_t)_t$ solves:

$$Y_t = \frac{e^{-\lambda}}{1-e^{-\lambda}} \int_0^1 e^{\lambda s} dB_s + B_t - \lambda \int_0^t Y_s ds, \quad t \in [0,1]. \tag{7}$$

An initial enlargement of filtration will permit to consider Y as solution of a usual SDE.

Proposition 2.2. *Let* $(\mathcal{G}_t)_t$ *be the filtration obtained by an initial enlargement of* $(\mathcal{F}_t)_t$ *with the random variable* $\int_0^1 e^{\lambda s} dB_s$. *Then, the* PerOU *process* Y *solves the SDE*

$$Y_t = Y_0 + \widetilde{B}_t - \lambda \int_0^t \Big(Y_s + \frac{e^{-\lambda(1-s)} Y_s - Y_0}{\sinh\big(\lambda(1-s)\big)}\Big)\, ds, \tag{8}$$

where $\widetilde{B}$ *is a* $(\mathcal{G}_t)_t$*-Brownian motion independent of* Y_0.

Therefore equation (8) corresponds to the $(\mathcal{G}_t)$-semimartingale decomposition of the PerOU process.

Proof of Proposition 2.2. The initial enlargement $(\mathcal{G}_t)_t$ of $(\mathcal{F}_t)_t$ by a random variable ξ is the smallest filtration satisfying the usual conditions such that $\mathcal{F}_t$ is included in $\mathcal{G}_t$ for any $t \in [0,1]$ and $\mathcal{G}_0 = \sigma(\xi)$. In our framework, ξ belongs to the first chaos of $(B_t)_t$ since $\xi := \int_0^1 e^{\lambda s}\, dB_s$. Therefore, applying the result of Théorème I.1.1 in [2], $(\widetilde{B}_t := B_t - V_t,\, t \in [0,1])$ is a standard $(\mathcal{G}_t)_t$-Brownian motion independent of ξ, where the process with bounded variation V is given by

$$V_t = \int_0^t \frac{e^{\lambda s}}{\int_s^1 e^{2\lambda u} du} \Big(\int_s^1 e^{\lambda u} dB_u\Big) ds = \int_0^t \frac{\lambda}{\sinh\big(\lambda(1-s)\big)} \Big(\int_s^1 e^{-\lambda(1-u)} dB_u\Big) ds. \tag{9}$$

Remark that the stochastic integral between time s and time 1 appearing in the definition of V_t is indeed $\mathcal{G}_s$-measurable:

$$\int_s^1 e^{\lambda u} dB_u = \xi - \int_0^s e^{\lambda u} dB_u.$$

Now (8) is a direct consequence of (7), (5) and (9). □

2.3. **PerOU as a Markov field.** The PerOU process is not Markov but satisfies a time-Markov field property, which we will discuss at the end of this subsection. First, in Proposition 2.3, we present the disintegration of the PerOU process along its initial (and final) time marginal, and we identify its bridges.

Proposition 2.3. *Denote by ν the Gaussian distribution of $Y_0 = \frac{1}{1-e^{-\lambda}} \int_0^1 e^{-\lambda(1-s)} dB_s$. Then, the PerOU process is a ν-mixture of its bridges, that is $\mathcal{L}(Y) = \displaystyle\int_{\mathbb{R}} \mathcal{L}(Y^{xx})\nu(dx)$, where the $x \hookrightarrow x$ bridge, denoted by Y_t^{xx}, solves the SDE*

$$\left\{ \begin{array}{lcl} dX_t &=& d\widetilde{B}_t - \lambda X_t\, dt + \dfrac{\lambda}{\sinh\big(\lambda(1-t)\big)} \Big(x - e^{-\lambda(1-t)} X_t\Big)\, dt, \quad t \in [0,1[, \\ X_0 &=& x. \end{array} \right. \tag{10}$$

Therefore the family of bridges $(Y^{xx})_x$ of the PerOU process coincides with those of an Ornstein-Ulhenbeck process.

Proof. (10) is a direct consequence of (8).

Now, consider the linear SDE with fixed initial condition x (but with free final condition),

$$\left\{ \begin{array}{lcl} dX_t &=& d\widetilde{B}_t - \lambda X_t\, dt, \quad t \in [0,1], \\ X_0 &=& x. \end{array} \right. \tag{11}$$

Its unique solution $X^{OU,x}$ is the Ornstein-Ulhenbeck process with initial deterministic condition x, given by

$$X_t^{OU,x} = xe^{-\lambda t} + \int_0^t e^{-\lambda(t-s)} dB_s.$$

One has

$$\begin{aligned} X_1^{OU,x} - e^{-\lambda(1-t)} X_t^{OU,x} &= xe^{-\lambda} + \int_0^1 e^{-\lambda(1-s)} dB_s - e^{-\lambda}\Big(x + \int_0^t e^{-\lambda s} dB_s\Big) \\ &= \int_t^1 e^{-\lambda(1-s)} dB_s. \end{aligned}$$

Using (11) and (9), we get

$$X_t^{OU,x} = x + \widetilde{B}_t - \lambda \int_0^t X_s^{OU,x} ds + \int_0^t \frac{\lambda}{\sinh\big(\lambda(1-s)\big)} \Big(X_1^{OU,x} - e^{-\lambda(1-s)} X_s^{OU,x}\Big) ds.$$

This leads to the identification of the process $Y_\cdot^{xx}$ as the $x \hookrightarrow x$ bridge of the Ornstein-Ulhenbeck process $X^{OU,x}$. □

Thus the PerOU process is a particular mixture of the bridges of the (Markov) Ornstein-Ulhenbeck process, in other words, it belongs to the *reciprocal class* of the Ornstein-Ulhenbeck process. It implies a time-Markov field property, also called *reciprocal property* (formalized by Jamison in [6]) which states that, given the knowledge of the process at any pair of times s and u (with $s \le u$), the dynamics of the process *inside* $[s,u]$ and *outside* (s,u) are conditionally independent. See [10] for a recent review on the relationship between the Markov property and the reciprocal one.

This fact, already mentioned in [1], was proved in [15], via a completely different way. In the latter paper, the reciprocal class is characterized as the set of solutions of an integration by part formula on the path space.

2.4. **Ornstein-Ulhenbeck process with prescripted time-boundaries.** Let us now relax the periodic boundary conditions of the PerOU imposed in (3) and replace it by $X_1 = f(X_0)$ where f is a measurable real-valued map. Consider the process $\breve{Y}_t$ (if it exists) solution of the stochastic integral equation

$$\left\{ \begin{array}{ll} X_t & = X_0 + B_t - \lambda \int_0^t X_s ds, \quad t \in [0,1], \\ X_1 & = f(X_0). \end{array} \right. \tag{12}$$

This class of pinned Ornstein-Ulhenbeck process was treated in [12] under the assumption called $(\mathfrak{H}_1)$ by the authors, which corresponds to the fact that

$$x \mapsto f(x) - e^{-\lambda} x \quad \text{ is a bijective map.} \tag{13}$$

We now solve (12) under a weaker assumption than (13). First, since $t \mapsto \int_0^t \breve{Y}_s ds$ is differentiable and B admits a finite quadratic variation, one can use the generalised stochastic calculus (see [17]) to get $d(\breve{Y}_t e^{\lambda t}) = e^{\lambda t} dB_t$. Therefore $\breve{Y}_t = \breve{Y}_0 e^{-\lambda t} + \int_0^t e^{-\lambda(t-s)} dB_s$. Considering the boundary conditions, one obtains

$$\breve{Y}_1 - e^{-\lambda} \breve{Y}_0 = f(\breve{Y}_0) - e^{-\lambda} \breve{Y}_0 = \int_0^1 e^{-\lambda(1-s)} dB_s. \tag{14}$$

Suppose now that the map defined in (13) is surjective, then there exists a measurable function g such that $\big(f - e^{-\lambda} Id\big) \circ g = Id$. Therefore one solution to (12) is given by

$$\breve{Y}_t = e^{-\lambda t} g\Big(\int_0^1 e^{-\lambda(1-s)} dB_s \Big) + \int_0^t e^{-\lambda(t-s)} dB_s. \tag{15}$$

Notice that, in general, it is no more a Gaussian process. Furthermore, the above representation of solutions of (12) implies their non-uniqueness as soon as the map (13) fails to be injective.

Take:

$$f(x) := e^{-\lambda} x + x \mathbb{1}_{]-\infty,1[}(x) + (2-x) \mathbb{1}_{[1,2]}(x) + (x-2) \mathbb{1}_{[2,+\infty[}(x).$$

Then, both functions g_1 and g_2 defined by

$$\begin{array}{rcl} g_1(y) & = & y \mathbb{1}_{]-\infty,1]}(y) + (2+y) \mathbb{1}_{]1,+\infty]}(y) \\ \text{and} \quad g_2(y) & = & y \mathbb{1}_{]-\infty,0[}(y) + (2-y) \mathbb{1}_{[0,1]}(y) + (2+y) \mathbb{1}_{]1,+\infty]}(y) \end{array}$$

solve the identity $\big(f - e^{-\lambda} Id\big) \circ g = Id$, which induces two non identical solutions for the equation (12). Moreover any random map g_ϵ where $\epsilon \in \{1,2\}$ is a random variable which is measurable with respect to $\mathcal{F}_1$, leads to the following solution to (12):

$$e^{-\lambda t} g_\epsilon \Big(\int_0^1 e^{-\lambda(1-s)} dB_s \Big) + \int_0^t e^{-\lambda(t-s)} dB_s.$$

Let us summarize these results in the following proposition.

Proposition 2.4. *Take any measurable function f. If the map defined by $x \mapsto f(x) - e^{-\lambda} x$ is surjective, there exists at least one pinned Ornstein-Ulhenbeck process solution* *to* (12). *It belongs to the reciprocal class of the Ornstein-Ulhenbeck process since, for all x and $y = f(x)$, its $x \hookrightarrow y$ bridge satisfies the SDE*

$$\left\{ \begin{array}{lcl} dX_t & = & d\widetilde{B}_t - \lambda X_t \, dt + \dfrac{\lambda}{\sinh\big(\lambda(1-t)\big)} \Big(y - e^{-\lambda(1-t)} X_t \Big) \, dt, \quad t \in [0,1[, \\ X_0 & = & x. \end{array} \right. \tag{16}$$

as the $x \hookrightarrow y$ bridge of the Ornstein-Ulhenbeck process does.

3. Convoluted Brownian motion

We now go back to the study of more general processes denoted by X^φ, admitting the representation (1) which is a kind of convolution between a square integrable determinist function φ – not necessarily of exponential type – and the Brownian motion.

Doing that, we consider processes which are no more in the reciprocal class of the Ornstein-Ulhenbeck process but still belong to the first Wiener chaos. Moreover we will see that the constructed processes – among other interesting properties – are stationary and periodic.

3.1. Definition of the process X^φ and first properties.

For any fixed $\varphi \in L^2\big(0,1\big)$, let us recall the definition of the process X^φ.

$$X^\varphi_t := \int_0^t \varphi(t-s)\, dB_s + \int_t^1 \varphi(1+t-s)\, dB_s, \quad t \in [0,1]. \tag{17}$$

The following proposition extends the properties enounced in Proposition 2.1.

Proposition 3.1. *(i) The process $(X^\varphi_t)_{0\le t\le 1}$ is stationary, centered and Gaussian with covariance function $R^\varphi(h) := \operatorname{Cov}\big(X^\varphi_s, X^\varphi_{s+h}\big)$ given by*

$$R^\varphi(h) = \int_0^h \varphi(1-u)\varphi(h-u)du + \int_0^{1-h} \varphi(u)\varphi(h+u)du. \tag{18}$$

More generally, the covariance between X^ψ_s and X^φ_t for $s \le t$, $\psi, \varphi \in L^2\big(0,1\big)$, is:

$$\int_0^s \varphi(t-u)\psi(s-u)du + \int_s^t \varphi(t-u)\psi(1+s-u)du + \int_t^1 \varphi(1+t-u)\psi(1+s-u)du. \tag{19}$$

(ii) $(X^\varphi_t)_{0\le t\le 1}$ is pathwise periodic and satisfies

$$X^\varphi_0 = X^\varphi_1 = \int_0^1 \varphi(1-s)\, dB_s. \tag{20}$$

(iii) $(X^\varphi_t)_{0\le t\le 1}$ is invariant under time reversal:

$$\big(X^\varphi_{1-t}, 0 \le t \le 1\big) \overset{(d)}{=} \big(X^\varphi_t, 0 \le t \le 1\big). \tag{21}$$

It also satisfies

$$\big(X^{\widehat{\varphi}}_t, 0 \le t \le 1\big) \overset{(d)}{=} \big(X^\varphi_t, 0 \le t \le 1\big), \tag{22}$$

where $\widehat{\varphi}(t) := \varphi(1-t)$ denotes the time reversal of the function φ.

(iv) The linear map $\varphi \mapsto X^\varphi_t$ is an isometry from $L^2\big(0,1\big)$ in $L^2(\Omega)$ for any fixed $t \in [0,1]$. Moreover, the linear map $\varphi \mapsto \int_0^t X^\varphi_u \, du$ has a norm bounded by 1.

Remark 3.1. The reversibility (21) of the process X^φ holds not only in law but also pathwise, in the following sense. X^φ admits the symmetric path representation

$$X_t^\varphi \quad = \quad I^\varphi(B)(t) + I^{\widehat{\varphi}}(\widehat{B})(1-t), \quad t \in [0,1], \tag{23}$$

where $I^\varphi(B)(t) := \int_0^t \varphi(t-s)\, dB_s$ $I^\varphi(B)$ is a stochastic convolution and $\big(\widehat{B}_t := B_{1-t} - B_1, t \in [0,1]\big)$ is the time reversal of the Brownian motion B. Indeed, for any $f \in L^2\big(0,1\big)$,

$$\int_0^1 f(s)dB_s = \int_0^1 \widehat{f}(s)d\widehat{B}_s = \int_0^1 f(1-s)d\widehat{B}_s. \tag{24}$$

Consequently,

$$\begin{aligned} \int_t^1 \varphi(1+t-s)dB_s &= \int_0^1 1_{\{s\geq t\}}\varphi(1+t-s)dB_s = \int_0^{1-t} \varphi(t+s)d\widehat{B}_s \\ &= \int_0^{1-t} \widehat{\varphi}(1-t-s)d\widehat{B}_s = I^{\widehat{\varphi}}(\widehat{B})(1-t) \end{aligned}$$

which leads to (23).

Proof of Proposition 3.1. Since

$$\int_0^t \varphi(t-s)^2 ds + \int_t^1 \varphi(1+t-s)^2 ds = \int_0^1 \varphi(s)^2 ds < +\infty$$

then (17) defines a centered Gaussian process.

Identity (20) is a direct consequence of (17).

Let us calculate $\mathrm{Cov}\big(X_t^\varphi, X_s^\psi\big)$ for $0 \leq s \leq t \leq 1$ and $\varphi, \psi \in L^2\big(0,1\big)$. Using (17), we easily get:

$$\begin{aligned} \mathrm{Cov}\big(X_t^\varphi, X_s^\psi\big) &= \int_0^s \psi(s-u)\varphi(t-u)du + \int_s^t \psi(1+s-u)\varphi(t-u)du \\ &\quad + \int_t^1 \psi(1+s-u)\varphi(1+t-u)du. \end{aligned}$$

We now take $\psi = \varphi$. Using the change of variables $r := s-u$ in the first integral, $r := u-s$ in the second and $r := 1-u+s$ in the third one, we get

$$\mathrm{Var}(X_t^\varphi) = \int_0^{1-t+s} \varphi(r)\varphi(t-s+r)dr + \int_0^{t-s} \varphi(1-r)\varphi(t-s-r)dr.$$

Setting $h := t-s$, we obtain (18). The time reversibility of X^φ is a consequence of its Gaussianity and its stationarity:

$$\mathrm{Cov}(X_{1-t}^\varphi, X_{1-s}^\varphi) = R^\varphi\big(|t-s|\big) = \mathrm{Cov}(X_t^\varphi, X_s^\varphi), \quad 0 \leq s,t \leq 1.$$

The identity (22) is a consequence of

$$R^\varphi(1-h) = R^{\widehat{\varphi}}(h), \quad h \in [0,1]. \tag{25}$$

Last, as for assertion (iv), the continuity of $\varphi \mapsto \int_0^t X_u^\varphi\, du$ follows from:

$$E\Big[\Big(\int_0^t X_u^\varphi du\Big)^2\Big] \leq t\int_0^t E\Big[\big(X_u^\varphi\big)^2\Big]du = t^2 \int_0^1 \varphi^2(s)ds.$$

□

Example 3.1.

- For $\varphi(u) = u^k$, monomial of degree $k \in \mathbb{N}$, the corresponding convoluted process, denoted by $X^{\sharp k}$, satisfies

$$X_t^{\sharp k} = \int_0^t (t-s)^k \, dB_s + \int_t^1 (1+t-s)^k \, dB_s. \tag{26}$$

  In particular, for $\varphi \equiv 1$ (monomial of degree 0) one has $X_t^{\sharp 0} \equiv B_1$ which is a constant process.
- When $\varphi = \mathbb{1}_{[a,1]}, a \in [0,1]$, the corresponding convoluted process denoted by $Z(a,\cdot)$ is given by

$$Z(a,t) = \begin{cases} B_{t+1-a} - B_t & \text{if } \ t \in [0,a] \\ B_{t-a} - B_t + B_1 & \text{if } \ t \in [a,1]. \end{cases} \tag{27}$$

Qualitative and quantitative analysis of the process $Z(a,\cdot)$.

For a fixed, $t \mapsto Z(a,t)$ is a stationary Gaussian process whose covariance function, which we denote by R^a, depends on the value of a.

$$\text{If } 0 \le a \le 1/2, \qquad R^a(h) := \begin{cases} 1-a-h & \text{if } \ h \in [0,a] \\ 1-2a & \text{if } \ h \in [a, 1-a] \\ h-a & \text{if } \ h \in [1-a, 1]. \end{cases} \tag{28}$$

$$\text{If } 1/2 \le a \le 1, \qquad R^a(h) := \begin{cases} 1-a-h & \text{if } \ h \in [0,1-a] \\ 0 & \text{if } \ h \in [1-a, a] \\ h-a & \text{if } \ h \in [a, 1]. \end{cases} \tag{29}$$

Let us now study the time-Markov field property of this process, and the structure of its bridges.

We only analyze the case $a \le 1/2$ (that is $a \le 1-a$) since the study of the case $a \ge 1/2$ is similar (replace a by $1-a$).

Proposition 3.2. *Suppose $a \le 1/2$. The process $t \mapsto Z(a,t)$ considered on the time interval $[0,1]$ is not a time-Markov field, but a concatenation of such ones on each time intervals $[0,a]$, $[a,1-a]$ and $[1-a,1]$. Indeed the bridges of the process $\frac{1}{\sqrt{2}}Z(a,\cdot)$ between times 0 and a (resp. between a and $1-a$, resp. between $1-a$ and 1) are Brownian bridges. Therefore the conditional law of $(\frac{1}{\sqrt{2}}Z(a,t), t \in [0,1])$ given the four values $Z(a,0), Z(a,a), Z(a,1-a), Z(a,1)$ is equal to the law of a Brownian motion pinned at the four instants $0, a, 1-a, 1$. With other words, the process $(\frac{1}{\sqrt{2}}Z(a,t), t \in [0,1])$ is a mixture of concatenation of Brownian bridges.*

Proof. Consider a stationary Gaussian process with unit variance on the time interval $]0,T[$ and an affine covariance function R. Following [1] Théorème 2.2, (iii) (which improves and corrects a result presented by Jamison in [6]), one knows that this Gaussian process is a Markov field on the time interval $]0,T[$ if and only if, on this interval, R is of the form

$$R(h) = 1 - c\,h \quad \text{ with } 0 \le c \le \frac{2}{T}\ . \tag{30}$$

- Thus, on the time interval $]0,a[$, the stationary Gaussian process $Z(a,\cdot)$ is a Markov field since the normed Gaussian process

$$\tilde{Z}(t) := \frac{1}{\sqrt{1-a}} Z(a,t) = \frac{1}{\sqrt{1-a}}(B_{t+1-a} - B_t), t \in]0,a[,$$

satisfies the condition (30): $c = \dfrac{1}{1-a} \le \dfrac{2}{a}$. See also the remark of Slepian in [18].

Let us compute its bridge between x at time 0 and y at time a (the pinned values are then $x = B_{1-a}$ and $y = B_1 - B_{1-a}$). We decompose $Z(a,\cdot)$ as follows:

$$Z(a,t) = Z^1(t) + Z^2(t) \text{ where } \left\{ \begin{array}{l} Z^1(t) := B_{t+1-a} - B_{1-a} \\ Z^2(t) := B_{1-a} - B_t. \end{array} \right. \tag{31}$$

Notice that the process $\big(Z^1(t), t \in [0,a]\big)$ is a Brownian motion which is independent from $\big(Z^2(t), t \in [0,a]\big)$.

Let $\big(B^{xy}_{oa}(t), t \in [0,a]\big)$ be the $x \hookrightarrow y$ Brownian bridge which starts at x and ends at y at time a. Recall :

$$\mathcal{L}\big(B^{xy}_{oa}(t), t \in [0,a]\big) = \mathcal{L}\big(x + \frac{t}{a}(y-x) + B^{oo}_{oa}(t), t \in [0,a]\big) \tag{32}$$

and

$$\mathcal{L}\big(B^{oo}_{oa}(t) + \widetilde{B}^{oo}_{oa}(t), t \in [0,a]\big) = \mathcal{L}\big(\sqrt{2}B^{oo}_{oa}(t), t \in [0,a]\big) \tag{33}$$

where B^{oo}_{oa} is a Brownian loop on the time interval $[0,a]$ and $\widetilde{B}^{oo}_{oa}$ is an independent copy of B^{oo}_{oa}.

By construction,

$$\begin{aligned} \mathcal{L}\big(Z^1(t), t \in [0,a] \,\big|\, Z^1(a) = z\big) &= \mathcal{L}\big(B_t, t \in [0,a] \,\big|\, B_0 = 0, B_a = z\big) \\ &= \mathcal{L}\big(\frac{t}{a}z + B^{oo}_{oa}(t), t \in [0,a]\big). \end{aligned} \tag{34}$$

On the other side, $\mathcal{L}\big(Z^2(t), t \in [0,a]\big) = \mathcal{L}\big(B_{1-a-t}, t \in [0,a]\big)$. Thus

$$\begin{aligned} &\mathcal{L}\Big(Z^2(t); t \in [0,a] \,\big|Z^2(0) = x, Z^2(a) = \mathfrak{z}\Big) \\ &\quad = \mathcal{L}\Big(B_{1-a-t}, t \in [0,a] \,\big|B_{1-a} = x, B_{1-2a} = \mathfrak{z}\Big) \\ &\quad = \mathcal{L}\Big(x + \frac{t}{a}(\mathfrak{z} - x) + B^{oo}_{oa}(t), t \in [0,a]\Big). \end{aligned}$$

Consequently,

$$\begin{aligned} &\mathcal{L}\Big(Z(t), t \in [0,a]\big|Z^2(0) = x, Z^1(a) = z, Z^2(a) = \mathfrak{z}\Big) \\ &\quad = \mathcal{L}\Big(x + \frac{t}{a}(z + \mathfrak{z} - x) + \sqrt{2}B^{a}_{0\hookrightarrow 0}(t); t \in [0,a]\Big). \end{aligned}$$

Finally, since $Z(0) = Z^2(0)$ and $Z(a) = Z^1(a) + Z^2(a)$, one obtains:

$$\begin{aligned} &\mathcal{L}\Big(Z(t), t \in [0,a]\big|Z(0) = x, Z(a) = y\Big) \\ &\quad = \mathcal{L}\Big(x + \frac{t}{a}(y-x) + \sqrt{2}B^{oo}_{oa}(t); t \in [0,a]\Big) \\ &\quad = \mathcal{L}\Big(\sqrt{2}B^{\frac{x}{\sqrt{2}}\,\frac{y}{\sqrt{2}}}_{oa}(t), t \in [0,a]\Big). \end{aligned}$$

- On the time interval $]a, 1-a[$, the study of the process $Z(a,\cdot)$ can be reduced to the study of $\big(Z(a, s+a), s \in]0, 1-2a[\big)$; therefore it satisfies the Markov field property if and only if $1 - 2a < a \Leftrightarrow a > 1/3$. In that case, the condition (30) is always satisfied: $c = \dfrac{1}{1-a}$ is always smaller than $\dfrac{2}{1-2a}$.
  Furthermore, we are able to compute explicitly the bridge of $\big(Z(a, s+a), s \in]0, 1-2a[\big)$ thanks a decomposition as above. Indeed

$$Z(a, a+s) = B_s + B_1 - B_{a+s} = B_s + W_{1-a-s} \tag{35}$$

where W is a Brownian motion, independent to B. By a similar argumentation as above, we conclude that the bridges of $Z(a,\cdot+a)$ on $]0,1-2a[$ are Brownian bridges.

- On the time interval $]1-a,1[$, the study of the process $Z(a,\cdot)$ can be reduced to the study of $\big(Z(a,s+1-a), s\in]0,a[\big)$, which disintegrates as

$$Z(a,1-a+s) = B_{1-2a+s} + B_1 - B_{1-a+s} = B_{1-2a+s} + \widetilde{W}_{a-s} \tag{36}$$

where $\widetilde{W}$ is a Brownian motion, independent to B. By a similar argumentation as above, its bridges are Brownian bridges.

□

Recall that $\varphi\mapsto X^\varphi$ is a linear map. In case the function φ enjoys some mild regularity, one gets the following useful path representation of $X^\varphi(t)$ as $d\varphi$-mixture of the processes $Z(a,t)$.

Proposition 3.3. *Suppose that φ is a right-continuous map with bounded variation over $[0,1]$. Then*

$$X_t^\varphi = \varphi(0)B_1 + \int_0^1 Z(r,t)\, d\varphi(r), \quad \forall\, t\in[0,1]. \tag{37}$$

Proof. Suppose first that φ is of class C^1, then $d\varphi(r) = \varphi'(r)dr$ and φ' is a continuous function. We have

$$\begin{aligned}\int_0^t (B_{t-r}-B_t)\varphi'(r)dr &= \int_0^t \varphi'(t-u)B_u du - B_t\big(\varphi(t)-\varphi(0)\big)\\ &= -\varphi(0)B_t + \int_0^t \varphi(t-u)dB_u - B_t\big(\varphi(t)-\varphi(0)\big)\\ &= -\varphi(t)B_t + \int_0^t \varphi(t-u)dB_u.\end{aligned}$$

We proceed similarly with the second integral:

$$\begin{aligned}\int_t^1 (B_{1-r+t}-B_t)\varphi'(r)dr &= \int_t^1 \varphi'(1+t-u)B_u du - B_t\big(\varphi(1)-\varphi(t)\big)\\ &= \varphi(t)B_t - \varphi(t)B_1 + \int_t^1 \varphi(1+t-u)dB_u.\end{aligned}$$

By (17), we deduce

$$\int_0^t (B_{t-r}-B_t)\varphi'(r)dr + \int_t^1 (B_{1-r+t}-B_t)\varphi'(r)dr = -\varphi(t)B_1 + X_t^\varphi.$$

Since $\varphi(t) = \varphi(0) + \displaystyle\int_0^t \varphi'(r)dr$, we get (37).

When the function φ is no more of class C^1, the representation remains valid as long as φ is of bounded variation over $[0,1]$, via limiting procedure and the continuity of $\varphi\mapsto X_t^\varphi$. □

With other words the map $\varphi\mapsto X^\varphi$ admits the following decomposition:

$$X^\varphi = X^{\varphi(0)} + X^{\varphi-\varphi(0)} = < B_1\delta_0 - (Z(\cdot,t))', \varphi >$$

where the derivative should be understood in the sense of distributions and $<\mu,\varphi>$ means that the distribution μ acts on the function φ.

3.2. **Comparison between the processes X^{φ} and $X^{\varphi'}$.** It is of interest to relate both processes X^{φ} and $X^{\varphi'}$ when φ is differentiable in $L^2\big(0,1\big)$.

Proposition 3.4. *Suppose that the function φ belongs to the Cameron-Martin space. Then*

$$X_t^{\varphi} = X_0^{\varphi} + \big(\varphi(0) - \varphi(1)\big)B_t + \int_0^t X_s^{\varphi'}\, ds, \quad 0 \le t \le 1. \tag{38}$$

Remark 3.2. This proposition will be useful to study the trigonometric and monomial processes, cf Section 3.3 and Example 3.2 respectively.

Proof of Proposition 3.4.

- Suppose first that φ is of class C^2.

  Then we use as tool the following stochastic Fubini theorem (cf Exercise 5.17, Chapter IV in [16]). Let ψ be in $L^2\big([0,1]^2\big)$, then

$$\int_0^1 \Big(\int_0^1 \psi(u,s)dB_s\Big)du = \int_0^1 \Big(\int_0^1 \psi(u,s)du\Big)dB_s. \tag{39}$$

  Using (17), we have $\int_0^t X_u^{\varphi'} du = A_1(t) + A_2(t)$ where

$$A_1(t) := \int_0^t \Big(\int_0^u \varphi'(u-s)dB_s\Big)du \text{ and } A_2(t) := \int_0^t \Big(\int_u^1 \varphi'(1+u-s)dB_s\Big)du.$$

  Thanks (39),

$$A_1(t) = \int_0^t \big(\varphi(t-s) - \varphi(0)\big)dB_s = -\varphi(0)B_t + \int_0^t \varphi(t-s)dB_s.$$

  We proceed similarly with $A_2(t)$.

$$\begin{aligned} A_2(t) &= \int_0^1 \Big(\int_0^{s\wedge t} \varphi'(1+u-s)du\Big)dB_s \\ &= \int_0^t \big(\varphi(1) - \varphi(1-s)\big)dB_s + \int_t^1 \big(\varphi(1+t-s) - \varphi(1-s)\big)dB_s \\ &= \varphi(1)B_t - \int_0^1 \varphi(1-s)dB_s + \int_t^1 \varphi(1+t-s)dB_s. \end{aligned}$$

  Consequently,

$$\begin{aligned} \int_0^t X_u^{\varphi'} du &= \big(\varphi(1) - \varphi(0)\big)B_t - \int_0^1 \varphi(1-s)dB_s + \int_0^t \varphi(t-s)dB_s \\ &\quad + \int_t^1 \varphi(1+t-s)dB_s. \end{aligned}$$

  The result follows from (17) and (20).

- Suppose now that φ belongs to the Cameron-Martin space, that is φ is differentiable and the two functions φ and φ' are elements of $L^2\big(0,1\big)$. Let $(\psi_n)_{n\ge1}$ be a sequence of functions of class C^1 defined on $[0,1]$ and converging to φ' in $L^2\big(0,1\big)$. Define $\varphi_n(x) := \varphi(0) + \int_0^x \psi_n(u)du, \quad \forall\, x \in [0,1]$. Since φ' is integrable, $\varphi(x) := \varphi(0) + \int_0^x \varphi'(u)du$. Consequently,

$$\sup_{0\le x\le 1} \big|\varphi_n(x) - \varphi(x)\big| \le \int_0^1 |\psi_n(u) - \varphi'(u)|du.$$

  Since φ_n is of class C^2, then

$$X_t^{\varphi_n} = X_0^{\varphi_n} + \big(\varphi_n(0) - \varphi_n(1)\big)B_t + \int_0^t X_s^{\varphi_n'} ds, \quad 0 \le t \le 1.$$

By assertion 4. of Proposition 3.1, each term converges in $L^2(\Omega)$ as n grows, which implies (38).

□

Example 3.2. (1) Take $\varphi(u) = \dfrac{1}{1-e^{-\lambda}} e^{-\lambda u}$ as in Section 2. Then $\varphi' = -\lambda\varphi$. According to Proposition 3.4, since $\varphi(0) - \varphi(1) = 1$, the process $Y = X^{\varphi}$ satisfies

$$Y_t = Y_0 + B_t - \lambda \int_0^t Y_u \, du$$

and we recover the equation (3). Reciprocally, suppose that the process X^{φ} satisfies $X^{\varphi'} = -\lambda X^{\varphi}$ for some regular function φ. This identity is equivalent to $X^{\varphi' + \lambda\varphi} = 0$. Then, the isometry property proved in Proposition 3.1 implies that the function φ itself solves the differential equation $\varphi' + \lambda\varphi = 0$, which means that it is proportional to $u \mapsto e^{-\lambda u}$.

This case is the unique one where the integral equation (38) on X^{φ} is indeed autonomous, due to the proportionality between φ and φ'.

(2) With the notation introduced in Example 3.1, the convoluted process $X^{\sharp k}$ associated with the monomial of degree k satisfies the non-autonomous integral equation

$$X_t^{\sharp k} = X_0^{\sharp k} - B_t + k \int_0^t X_s^{\sharp (k-1)} \, ds. \tag{40}$$

To obtain an autonomous equation, one has to consider the $\mathbb{R}^{k+1}$-valued process $\mathbf{X}^{\sharp k}$ whose coordinates are $X^{\sharp k}, \cdots, X^{\sharp 1}, X^{\sharp 0}$, which then satisfies the linear integral system

$$\mathbf{X}_t^{\sharp k} = \mathbf{X}_0^{\sharp k} - B_t \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 0 \end{bmatrix} + \int_0^t A \, \mathbf{X}_s^{\sharp k} \, ds, \tag{41}$$

where the $(k+1) \times (k+1)$ matrix A is given by $\begin{bmatrix} 0 & k & 0 & \dots & 0 \\ 0 & 0 & k-1 & \dots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & \dots & 0 & 1 \\ 0 & & \dots & & 0 \end{bmatrix}$.

This more general vector-valued framework will be studied in Section 4.

(3) The random variable $(1 - e^{-\lambda})\, Y_t$ defined in the first example can be obtained as limit in $L^2(\Omega)$ of the sequence $\displaystyle\sum_{k=0}^{n} \frac{(-\lambda)^k}{k!} X_t^{\sharp k}$ when n tends to infinity. It is a consequence of Proposition 3.1 and the fact that $\displaystyle\sum_{k=0}^{n} \frac{(-\lambda)^k}{k!} X^{\sharp k} = X^{\psi_n}$ where $\psi_n(x) := \displaystyle\sum_{k=0}^{n} \frac{(-\lambda x)^k}{k!}$.

(4) In Section 3.3 below, we analyse the trigonometric convoluted Brownian motion, which is of particular interest.

3.3. **The trigonometric convoluted Brownian motion.** Take now for function φ a trigonometric one, either $x \mapsto \cos(\lambda x)$ or $x \mapsto \sin(\lambda x)$, where λ is a real number.

$$\begin{cases} X_t^{\cos} &= \int_0^t \cos\big(\lambda(t-s)\big)\, dB_s + \int_t^1 \cos\big(\lambda(1+t-s)\big)\, dB_s, \\ X_t^{\sin} &= \int_0^t \sin\big(\lambda(t-s)\big)\, dB_s + \int_t^1 \sin\big(\lambda(1+t-s)\big)\, dB_s. \end{cases} \tag{42}$$

In a more elegant way, one considers the complex-valued process $\mathbf{X}_t^\lambda := X_t^{\cos} + i\, X_t^{\sin}$ which satisfies

$$\begin{aligned} \mathbf{X}_t^\lambda &:= \int_0^t \exp\big(i\lambda(t-s)\big)\, dB_s + \int_t^1 \exp\big(i\lambda(1+t-s)\big)\, dB_s \\ &= \int_0^1 \exp\big(i\lambda(t-s)\big)\, dB_s + \big(\exp(i\lambda)-1\big) \int_t^1 \exp\big(i\lambda(t-s)\big)\, dB_s. \end{aligned}$$

Let us first analyze one special case.

1- The periodic case, $\lambda \in 2\pi\mathbb{Z}$.

For $\lambda = 2k\pi, k \in \mathbb{Z}^*$, $\mathbf{X}^\lambda$ admits a simple representation.

$$\begin{aligned} \mathbf{X}_t^{2k\pi} &= \int_0^1 \exp\big(i2k\pi(t-s)\big)\, dB_s \\ &= \exp(i2k\pi t) \int_0^1 \exp(-i2k\pi s)\, dB_s = \exp(i2k\pi t)\, \mathbf{X}_0^{2k\pi}. \end{aligned}$$

This process is degenerated - as the product of a determinist time function by a fixed random variable - and it is $1/k$-periodic: $\mathbf{X}_{t+\frac{1}{k}}^{2k\pi} = \mathbf{X}_t^{2k\pi}$. Therefore the stationary centered Gaussian process $X^{\cos}$, real part of $\mathbf{X}^{2k\pi}$ (resp. $X^{\sin}$ the imaginary part of $\mathbf{X}^{2k\pi}$), disintegrates as a mixture of two Gaussian random variables:

$$X_t^{\cos} = \cos(2k\pi t) \int_0^1 \cos(2k\pi s)\, dB_s + \sin(2k\pi t) \int_0^1 \sin(2k\pi s)\, dB_s.$$

Moreover the above two stochastic integrals are independent. Thus following [5], p.524 and [6], Theorem p.1627, $X^{\cos}$ (resp. $X^{\sin}$) is a Markov field on the time interval $[0, \frac{1}{2k}[$. Nevertheless it is not a Markov field on the full time interval $[0,1]$.

2- The general case, $\lambda \notin 2\pi\mathbb{Z}$.

When the function φ is trigonometric, there is no proportionality between φ and φ' but there is proportionality between φ and φ''. Indeed, following (38), the pair of processes $(X^{\cos}, X^{\sin})$ satisfies the autonomous system of equations:

$$\begin{cases} X_t^{\cos} &= \int_0^1 \cos\big(\lambda(1-s)\big) dB_s + \big(1-\cos\lambda\big) B_t - \lambda \int_0^t X_s^{\sin}\, ds, \\ X_t^{\sin} &= \int_0^1 \sin\big(\lambda(1-s)\big) dB_s - \sin\lambda\, B_t + \lambda \int_0^t X_s^{\cos}\, ds, \end{cases} \tag{43}$$

or, equivalently, the complex-valued process $\mathbf{X}_t^\lambda$ satisfies the equation:

$$\mathbf{X}_t^\lambda = \int_0^1 e^{i\lambda(1-s)} dB_s + \big(1-e^{i\,\lambda}\big) B_t + \lambda i \int_0^t \mathbf{X}_s^\lambda\, ds. \tag{44}$$

Notice that $\dfrac{1}{1-e^{i\,\lambda}} \mathbf{X}_t^\lambda$ satisfies a similar equation to (3), where the parameter λ is replaced by $-i\lambda$. We have proved the following.

Proposition 3.5. *The processes $X^{\cos}$ and $X^{\sin}$ are centered, stationary, periodic Gaussian processes. Their covariance functions satisfy*

(45)

$$\begin{cases} R^{\cos}(h) &= \frac{\sin(\lambda/2)}{2\lambda}\Big(\cos\big(\lambda(\frac{3}{2}-h)\big)+\cos\big(\lambda(\frac{1}{2}+h)\big)\Big)+h\sin\big(\frac{\lambda}{2}\big)\sin\big(\lambda(h-\frac{1}{2})\big) \\ & \quad +\frac{\cos(\lambda h)}{2} \\ R^{\sin}(h) &= -\frac{\sin(\lambda/2)}{2\lambda}\Big(\cos\big(\lambda(\frac{3}{2}-h)\big)+\cos\big(\lambda(\frac{1}{2}+h)\big)\Big)+h\sin\big(\frac{\lambda}{2}\big)\sin\big(\lambda(h-\frac{1}{2})\big) \\ & \quad +\frac{\cos(\lambda h)}{2}. \end{cases}$$

Proof. By (18)

$$\begin{aligned} R^{\cos}(h) &= \int_0^h \cos\big(\lambda(1-u)\big)\cos\big(\lambda(h-u)\big)du+\int_0^{1-h}\cos(\lambda u)\cos\big(\lambda(h+u)\big)du \\ &=: I_1(h)+I_2(h). \end{aligned}$$

Using the identity $\cos a\,\cos b=\dfrac{\cos(a+b)+\cos(a-b)}{2}$ one gets

$$I_1(h)=\frac{1}{4\lambda}\Big(\sin\big(\lambda(1+h)\big)-\sin\big(\lambda(1-h)\big)\Big)+\frac{h}{2}\cos\big(\lambda(1-h)\big).$$

In a similar way

$$I_2(h)=\frac{1}{4\lambda}\Big(\sin\big(\lambda(2-h)\big)-\sin(\lambda h)\Big)+\frac{1-h}{2}\cos(\lambda h).$$

Therefore

$$\begin{aligned} R^{\cos}(h) &= \frac{1}{4\lambda}\Big(\sin\big(\lambda(2-h)\big)-\sin\big(\lambda(1-h)\big)+\sin\big(\lambda(1+h)\big)-\sin(\lambda h)\Big) \\ &\quad +\frac{h}{2}\Big(\cos\big(\lambda(1-h)\big)-\cos(\lambda h)\Big)+\frac{\cos(\lambda h)}{2}. \end{aligned}$$

Using the identities

$$\sin a-\sin b=2\sin\frac{a-b}{2}\cos\frac{a+b}{2},\qquad \cos a-\cos b=-2\sin\frac{a-b}{2}\sin\frac{a+b}{2},$$

one obtains the first equality of (45). Let us prove the second equality of (45).

$$\begin{aligned} R^{\sin}(h) &= \int_0^h \sin\big(\lambda(1-u)\big)\sin\big(\lambda(h-u)\big)du+\int_0^{1-h}\sin(\lambda u)\sin\big(\lambda(h+u)\big)du \\ &=: J_1(h)+J_2(h) \end{aligned}$$

Using the identity $\sin a\,\sin b=\dfrac{\cos(a-b)-\cos(a+b)}{2}$ one gets

$$J_1(h)=\frac{h}{2}\cos\big(\lambda(1-h)\big)+\frac{1}{4\lambda}\Big(\sin\big(\lambda(1-h)\big)-\sin\big(\lambda(1+h)\big)\Big)$$

and

$$J_2(h)=\frac{1-h}{2}\cos(\lambda h)+\frac{1}{4\lambda}\Big(\sin(\lambda h)-\sin\big(\lambda(2-h)\big)\big)\Big).$$

Therefore

$$\begin{aligned} R^{\sin}(h) &= \frac{1}{4\lambda}\Big(\sin\big(\lambda(1-h)\big)-\sin\big(\lambda(2-h)\big)+\sin(\lambda h)-\sin\big(\lambda(1+h)\big)\Big) \\ &\quad +\frac{h}{2}\Big(\cos\big(\lambda(1-h)\big)-\cos(\lambda h)\Big)+\frac{\cos(\lambda h)}{2} \\ &= -\frac{\sin(\lambda/2)}{2\lambda}\Big(\cos\big(\lambda(\frac{3}{2}-h)\big)+\cos\big(\lambda(\frac{1}{2}+h)\big)\Big) \\ &\quad +h\,\sin\big(\frac{\lambda}{2}\big)\sin\big(\lambda(h-\frac{1}{2})\big)+\frac{\cos(\lambda h)}{2}. \end{aligned}$$

□

As in Section 2, we can prove that the process $(X^{\cos}, X^{\sin})$ admits a semimartingale decomposition in a filtration enlarged by its initial condition, and therefore that it is a Markov field. However, this vector-valued process is a particular case of the convoluted processes we will treat in the next section. In particular, we will analyse in more generality their semimartingale property and their Markovianity.

4. Vector-valued convoluted processes

We extend here the definition (1) of convoluted Brownian motion to the multidimensional case. Let A be a $n \times n$ matrix and ϕ a vector in $\mathbb{R}^n$. We introduce the $\mathbb{R}^n$-valued process

$$\mathbf{X}_t^{A,\phi} := \int_0^t e^{(t-s)A}\phi\, dB_s + \int_t^1 e^{(1+t-s)A}\phi\, dB_s, \quad t \in [0,1], \tag{46}$$

where $\big(B_t, t \in [0,1]\big)$ is as before a standard real-valued Brownian motion.

The vector-valued process $(\mathbf{X}_t^{A,\phi})_{t\in[0,1]}$ is centered and Gaussian and we compute in Proposition 4.2 its covariance matrix. We study in detail the two particular cases of $(X^{\cos}, X^{\sin})$ and $\mathbf{X}^{\sharp k}$, see Section 4.2. The process $\mathbf{X}^{A,\phi}$ is not Markov, but its bridges are Markov. The proof of this property is based on two steps. First, we prove in Proposition 4.4 that, under conditions $(\mathfrak{H}_1)$, $(\mathfrak{H}_2)$, $\mathbf{X}^{A,\phi}$ is a semimartingale in an enlarged filtration. Second, using its explicit canonical decomposition, we are able to show that conditionally on $\mathbf{X}_0^{A,\phi}$, $\mathbf{X}^{A,\phi}$ is Markov, see Section 4.4. However the monomial convoluted Brownian motion $\mathbf{X}^{\sharp k}$ does not satisfies $(\mathfrak{H}_1)$. We then prove that adding a component permits to recover a Markov field property, see Section 4.5.

4.1. **General properties.**

As in (4), the process $\mathbf{X}^{A,\phi}$ admits indeed another representation and solves a stochastic linear integral system.

Proposition 4.1. *The process $\mathbf{X}_t^{A,\phi}, t \in [0,1]$, admits the following representation:*

$$\mathbf{X}_t^{A,\phi} = e^A \int_0^1 e^{(t-s)A}\phi\, dB_s + (Id - e^A)\int_0^t e^{(t-s)A}\phi\, dB_s. \tag{47}$$

In particular it is 1-periodic and

$$\mathbf{X}_0^{A,\phi} = \mathbf{X}_1^{A,\phi} = \int_0^1 e^{(1-s)A}\phi\, dB_s. \tag{48}$$

Reciprocally, the unique solution of the integral system

$$Z_t = \int_0^1 e^{(1-s)A}\phi\, dB_s \; + (Id - e^A)\phi B_t + \int_0^t AZ_s\, ds\ , \quad t \in [0,1]. \tag{49}$$

is the process $Z \equiv \mathbf{X}^{A,\phi}$.

Proof. Identities (47) and (48) are consequences of (46). The proof of the second assertion is omitted since it is a direct generalization of the one presented in Section 2.4. □

Remark 4.1. (1) Equation (49) is not a classical stochastic integral system since the r.v. $\displaystyle\int_0^1 e^{(1+t-s)A}\phi\, dB_s$ is not $\mathcal{F}_0$-measurable.

(2) We recover the two-dimensional process $(X^{\cos}, X^{\sin})^*$ defined by (42) setting $\phi = (1,0)^*$ and $A = \lambda \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Indeed, since

$$e^{tA} = \begin{pmatrix} \cos(\lambda t) & -\sin(\lambda t) \\ \sin(\lambda t) & \cos(\lambda t) \end{pmatrix} \tag{50}$$

then equation (46) and equation (42) are identical.

Clearly the vector-valued process $(\mathbf{X}_t^{A,\phi})_{t\in[0,1]}$ is centered and Gaussian. It is therefore characterized by its covariance matrix done in the proposition below, which will permit us to develop several examples in the next Section 4.2.

Recall that if $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are two $\mathbb{R}^n$-valued random vectors, their covariance matrix is defined by $\mathbf{Cov}(\mathbf{Z}_1,\mathbf{Z}_2) := \mathbb{E}\big[\mathbf{Z}_1\mathbf{Z}_2^*\big]$, or equivalently, by

$$\xi_1^*\,\mathbf{Cov}(\mathbf{Z}_1,\mathbf{Z}_2)\,\xi_2 = \mathrm{Cov}\big(\xi_1^*\mathbf{Z}_1,\xi_2^*\mathbf{Z}_2\big),\quad \forall\xi_1,\xi_2\in\mathbb{R}^n. \tag{51}$$

To simplify the notations, we define the function ρ from $[0,1]$ into the set of $n\times n$-matrices by

$$\rho(t) := \int_0^t e^{uA}\phi\phi^* e^{uA^*}du,\quad t\in[0,1]. \tag{52}$$

Proposition 4.2. *The process $(\mathbf{X}_t^{A,\phi})_{t\in[0,1]}$ is Gaussian and stationary. Moreover, for any $0\le s\le s+h\le 1$, we have:*

$$R^{A,\phi}(h) := \mathbf{Cov}\big(\mathbf{X}_s^{A,\phi},\mathbf{X}_{s+h}^{A,\phi}\big) = e^{hA}\rho(1-h)+\rho(h)e^{(1-h)A^*}. \tag{53}$$

Proof. Let $\xi_i\in\mathbb{R}^n, i\in\{1,2\}$. We deduce from (46) that

$$\xi_i^*\mathbf{X}_t^{A,\phi} = \int_0^t \xi_i e^{(t-u)A}\phi\, dB_u + \int_t^1 \xi_i e^{(1+t-u)A}\phi\, dB_u,\quad 0\le t\le 1.$$

Therefore

$$\xi_i^*\mathbf{X}^{A,\phi} = X^{\varphi_i},\ \text{where}\quad \varphi_i(t) := \xi_i^* e^{tA}\phi,\quad t\in[0,1]. \tag{54}$$

Relation (51) implies, for $0\le s\le s+h\le 1$,

$$\xi_1^*\,\mathbf{Cov}\big(\mathbf{X}_s^{A,\phi},\mathbf{X}_{s+h}^{A,\phi}\big)\,\xi_2 = \mathrm{Cov}\big(X_s^{\varphi_1},X_{s+h}^{\varphi_2}\big).$$

We now apply (19) with $\varphi=\varphi_1$ and $\psi=\varphi_2$:

$$\begin{aligned}\mathrm{Cov}\big(X_s^{\varphi_1},X_{s+h}^{\varphi_2}\big) &= \int_0^s \varphi_2(s-u)\varphi_1(s+h-u)du\\ &\quad+\int_s^{s+h}\varphi_2(1+s-u)\varphi_1(s+h-u)du\\ &\quad+\int_{s+h}^1\varphi_2(1+s-u)\varphi_1(1+s+h-u)du.\end{aligned}$$

Proceeding as in the proof of Proposition 3.1, we get:

$$\mathrm{Cov}\big(X_s^{\varphi_1},X_{s+h}^{\varphi_2}\big) = \int_0^{1-h}\varphi_2(r)\varphi_1(h+r)dr + \int_0^h \varphi_2(1-r)\varphi_1(h-r)dr.$$

Using (54) and (51) leads to:

$$\mathbf{Cov}\big(\mathbf{X}_s^{A,\phi},\mathbf{X}_{s+h}^{A,\phi}\big) = \int_0^{1-h} e^{(h+r)A}\phi\phi^* e^{rA^*}dr + \int_0^h e^{(h-r)A}\phi\phi^* e^{(1-r)A^*}dr.$$

The change of variable $u:=h-r$ in the second integral gives:

$$\begin{aligned}\mathbf{Cov}\big(\mathbf{X}_s^{A,\phi},\mathbf{X}_{s+h}^{A,\phi}\big) &= e^{hA}\Big(\int_0^{1-h} e^{rA}\phi\phi^* e^{rA^*}dr\Big)\\ &\quad+\Big(\int_0^h e^{rA}\phi\phi^* e^{rA^*}dr\Big)e^{(1-h)A^*}.\end{aligned}$$

□

Remark 4.2. (1) In the case $n=1$, $A=-\lambda$ and $\phi=\dfrac{1}{1-e^{-\lambda}}$, it is easy to check that Identity (53) corresponds to (6).

(2) In the particular case $h = 0$, then (53) leads to the covariance matrix $K^{A,\phi}$ (which does not depend on t) of the vector $\mathbf{X}_t^{A,\phi}$:

$$K^{A,\phi} = \int_0^1 e^{uA}\phi\phi^* e^{uA^*} du.$$

(3) The map $R^{A,\phi}(\cdot)$ has the following remarkable structure:

$$R^{A,\phi}(h) = \sigma(h) + \sigma(1-h)^*, \tag{55}$$

where the matrix-valued map σ is defined by

$$\sigma(h) := e^{hA}\rho(1-h), \quad h \in [0,1]. \tag{56}$$

4.2. **Some illustrating examples.**

4.2.1. *The trigonometric convoluted Brownian motion.* We begin with the two-dimensional convoluted process $(X^{\cos}, X^{\sin})^*$ defined in (42) through the trigonometric functions sin and cos. We already observed in Remark 4.1 that

$$(X^{\cos}, X^{\sin})^* = \mathbf{X}^{A,e_1} \text{ with } e_1 := (1,0)^* \text{ and } A := \lambda \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We also computed in Proposition 3.5 the covariance terms $\mathrm{Cov}\big(X_s^{\cos}, X_t^{\cos}\big)$ and $\mathrm{Cov}\big(X_s^{\sin}, X_t^{\sin}\big)$. Anyway the formula (53) permits to go further computing the mixed covariance terms of the form $\mathrm{Cov}\big(X_s^{\cos}, X_t^{\sin}\big)$. Indeed, using (50), one obtains for the explicit computation of the matrix-valued map ρ defined in (52):

$$\rho(t) = \begin{pmatrix} \dfrac{t}{2} + \dfrac{\sin(2\lambda t)}{4\lambda} & \dfrac{1-\cos(2\lambda t)}{4\lambda} \\[2ex] \dfrac{1-\cos(2\lambda t)}{4\lambda} & \dfrac{t}{2} - \dfrac{\sin(2\lambda t)}{4\lambda} \end{pmatrix}.$$

Then, the matrix $\sigma(h)$ defined by (56) has the form

$$\sigma(h) = \begin{pmatrix} \sigma_{11}(h) & \sigma_{12}(h) \\ \sigma_{21}(h) & \sigma_{22}(h) \end{pmatrix}$$

where

$$\sigma_{12}(h) := -\frac{1-h}{2}\sin(\lambda h) - \frac{1}{4\lambda}\cos\big(\lambda(2-h)\big),$$

$$\sigma_{21}(h) := \frac{1-h}{2}\sin(\lambda h) - \frac{1}{4\lambda}\cos\big(\lambda(2-h)\big) + \frac{1}{4\lambda}\cos(\lambda h).$$

We thus deduce:

$$\mathrm{Cov}\big(X_s^{\cos}, X_{s+h}^{\sin}\big) = \begin{cases} \sigma_{12}(h) + \sigma_{21}(1-h) & \text{for } h \geq 0 \\ \sigma_{12}(1-h) + \sigma_{21}(h) & \text{for } h \leq 0. \end{cases} \tag{57}$$

4.2.2. *The monomial convoluted Brownian motion.* We now analyze in more detail the $(k+1)$-dimensional convoluted process $\mathbf{X}^{\sharp k} := (X^{\sharp k}, \cdots, X^{\sharp 1}, X^{\sharp 0})^*$ defined in (26) as convolution with monomials of degree lower than k (or, equivalently, defined by the linear system (41)). We set

$$e_{k+1} := (0, \cdots, 0, 1)^* \text{ and } A := \begin{bmatrix} 0 & k & 0 & \dots & 0 \\ 0 & 0 & k-1 & \dots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & \dots & 0 & 1 \\ 0 & & \dots & & 0 \end{bmatrix}. \tag{58}$$

Proposition 4.3. *The process $\mathbf{X}^{\sharp k}$ coincides with the vector-valued convoluted process $\mathbf{X}^{A,e_{k+1}}$ where A and e_{k+1} are defined by (58). Let $\rho(t)$ be the associated $(k+1)\times(k+1)$-matrix defined by (52). The entries of $\rho(t)$ are monomials in t and satisfy:*

$$\rho_{i,j}(t) = \frac{1}{2k+3-(i+j)}\, t^{2k+3-(i+j)}, \quad 1 \le i,j \le k+1,\ 0 \le t \le 1. \tag{59}$$

The covariance matrix $R^{\sharp k}$ of $\mathbf{X}^{\sharp k}$, defined in (53), has as (i,j)-entry the following polynomial in h of degree $2k+3-(i+j)$:

$$R^{\sharp k}_{ij}(h) = \int_0^h s^{k+1-i}(1-h+s)^{k+1-j}ds + \int_h^1 (h+1-s)^{k+1-i}(1-s)^{k+1-j}ds. \tag{60}$$

Proof. We first prove that $\mathbf{X}^{\sharp k}$ satisfies (49) with A and e_{k+1} as above. Then Proposition 4.1 will imply $\mathbf{X}^{\sharp k} \equiv \mathbf{X}^{A,e_{k+1}}$.

According to (41), it remains to prove that

$$\mathbf{X}^{\sharp k}_1 = e^A \int_0^1 e^{(1-s)A} e_{k+1}\, dB_s \quad \text{and} \quad (e^A - Id)e_{k+1} = (1,\cdots,1,0)^*. \tag{61}$$

a) Since A is nilpotent with index of nilpotency $k+1$, the matrix e^{tA} is a polynomial in t of degree k.

For any $i,j \in \{1,\cdots,k+1\}$, it is convenient to introduce the unit matrix $E^{i,j}$ defined by $E^{i,j}_{kl} = 1\!\!1_{(i,j)}(k,l)$. We claim that:

$$e^{tA} = \sum_{i=0}^{k}\sum_{j=0}^{k-i}\binom{k-i}{j} t^j E^{i+1,i+j+1}. \tag{62}$$

Calling $\Gamma(t)$ the right hand-side of (62), it is clear that $\Gamma(0) = \sum_{i=1}^{k+1} E^{i,i} = Id$.

Since $A = \sum_{l=0}^{k-1}(k-l)E^{l+1,l+2}$, relation (62) implies:

$$\begin{aligned} A\Gamma(t) &= \sum_{i=0}^{k}\sum_{j=0}^{k-i}\sum_{l=0}^{k-1}\binom{k-i}{j}(k-l)t^j E^{l+1,l+2}E^{i+1,i+j+1} \\ &= \sum_{i'=0}^{k-1}\sum_{j'=1}^{k-i'}\binom{k-i'}{j'} j' t^{j'-1} E^{i'+1,i'+j'+1}. \end{aligned}$$

On the other side,

$$\frac{d\Gamma}{dt}(t) = \sum_{i=0}^{k}\sum_{j=1}^{k-i}\binom{k-i}{j} j t^{j-1} E^{i+1,i+j+1}.$$

Finally, $\dfrac{d\Gamma}{dt}(t) = A\Gamma(t)$ and $\Gamma(0) = Id$, therefore (62) holds.

b) Let $e_i := (0,\cdots,0,1,0,\cdots,0)^*$ be the i-th basis vector of $\mathbb{R}^{k+1}$. Since $E^{i,l}e_{k+1} = 1\!\!1_{\{l=k+1\}}e_i$, then

$$e^{tA}e_{k+1} = \sum_{i=1}^{k+1} t^{k+1-i} e_i. \tag{63}$$

In particular, $e^A e_{k+1} = \sum_{i=1}^{k+1} e_i = (1,\cdots,1,0)^* + e_{k+1}$, which implies (61).

c) We now prove that the entries of the matrix $\rho(t)$ are monomials in t and satisfy (59). By (63),

$$e_{k+1}^* e^{uA^*} = \sum_{i=1}^{k+1} u^{k+1-i} e_i^*,$$

which implies

$$e^{uA} e_{k+1} e_{k+1}^* e^{uA^*} = \sum_{1\le i,j\le k+1} u^{2k+2-i-j} e_i^* e_j^*.$$

Since $e_i e_j^* = E^{i,j}$, then $e^{uA} e_{k+1} e_{k+1}^* e^{uA^*} = \displaystyle\sum_{1\le i,j\le k+1} u^{2k+2-(i+j)} E^{i,j}$.

Integrating this identity in u over the interval $[0,t]$ and using (52) gives (59).

We now prove (60). Using (56), (62) and (59), we have:

$$\sigma_{ij}(h) = \sum_{l=i}^{k+1} \binom{k-i+1}{l-i} \frac{1}{2k+3-(l+j)}\, h^{l-i}\,(1-h)^{2k+3-(l+j)}.$$

For i, j, h fixed, we define the polynomial function g_h by

$$g_h(x) := \sum_{l=i}^{k+1} \binom{k-i+1}{l-i} \frac{1}{2k+3-l-j}\, h^{l-i}\, x^{2k+3-(l+j)}.$$

Then

$$g_h'(x) = \left[\sum_{m=0}^{k+1-i} \binom{k-i+1}{m} h^m x^{k+1-i-m}\right] x^{k+1-j} = (h+x)^{k+1-i} x^{k+1-j}.$$

Note that $g_h(0) = 0$, since, for any $j \le k+1$, $2k+3-(k+1+j) \ge 1$. Therefore $g_h(x) = \displaystyle\int_0^x (h+s)^{k+1-i} s^{k+1-j} ds$ and

$$\sigma_{ij}(h) = g_h(1-h) = \int_0^{1-h} (h+s)^{k+1-i} s^{k+1-j} ds.$$

Finally (60) follows from (55). □

4.3. **The semimartingale representation of $\mathbf{X}^{A,\phi}$.** According to (48), the initial value of the process $\mathbf{X}^{A,\phi}$ is given by

$$\mathbf{X}_0^{A,\phi} = \int_0^1 h(s)dB(s) \quad \text{where} \quad h(s) := e^{(1-s)A}\phi. \tag{64}$$

Since this $\mathbb{R}^n$-valued random vector is not $\mathcal{F}_0$-measurable, generalizing the approach developed in Section 2.2, we propose to enlarge the filtration $(\mathcal{F}_t)_t$ with $\mathbf{X}_0^{A,\phi}$ to obtain a semimartingale representation of $X^{A,\phi}$. Our approach is based on Théorème II.1 in [2], whose statement is recalled below.

Lemma 4.1. *Consider the random vector $\xi = (\xi_1, \cdots, \xi_n)^* \in \mathbb{R}^n$ whose coordinates satisfy $\xi_i := \displaystyle\int_0^1 h_i(s)\, dB_s$, $h_i \in L^2(0,1), i \in \{1, \cdots, n\}$ and h the $\mathbb{R}^n$-valued map $t \mapsto h(t) := \big(h_1(t), \cdots, h_n(t)\big)^*$. Let $(\mathcal{G}_t)_t$ be the initial enlargement of the filtration $(\mathcal{F}_t)_t$ by ξ. Suppose that, for any $t \in [0,1[$, the matrix $H(t) := \displaystyle\int_t^1 h(u)h(u)^* du$ is invertible. Then, defining the matrix-valued map G by*

$$G(t,u) := h(t)^* H(t)^{-1} h(u) \mathbb{1}_{\{0\le t\le u\le 1\}}, \tag{65}$$

one gets that

$$W_t := -B_t + \int_0^t \int_s^1 G(s,u)dB_u\, ds, \quad 0 \le t \le 1, \tag{66}$$

is a $(\mathcal{G}_t)_t$-Brownian motion which is independent of ξ.

Proposition 4.4. *Suppose that the matrix A and the vector ϕ satisfy the following assumptions:*
$(\mathfrak{H}_1)$ *The matrix $e^A - Id$ is invertible*
$(\mathfrak{H}_2)$ $\mathrm{Span}(A^k\phi, k \in \mathbb{N}) = \mathbb{R}^n$.

Let $(\mathcal{G}_t)_t$ be the filtration obtained from the initial enlargement of the Brownian filtration by the random vector $\mathbf{X}_0^{A,\phi}$.

(1) *Define a real-valued bounded variation process V by*

$$V_t := \int_0^t \phi^* e^{(1-s)A^*} H(s)^{-1}(e^A - Id)^{-1}\Big[e^{(1-s)A}\mathbf{X}_s^{A,\phi} - \mathbf{X}_0^{A,\phi}\Big]ds.$$

Then the real-valued process

$$\tilde{B}_t := -B_t + V_t \tag{67}$$

is a $(\mathcal{G}_t)_t$-Brownian motion independent from $\mathbf{X}_0^{A,\phi}$.

(2) *The vector-valued process $\mathbf{X}_t^{A,\phi}$ admits the following semimartingale decomposition:*

$$\mathbf{X}_t^{A,\phi} = \mathbf{X}_0^{A,\phi} - (Id - e^A)\phi\, \tilde{B}_t + \int_0^t A\mathbf{X}_s^{A,\phi}\, ds + (Id - e^A)\phi\, V_t\ , \quad t \in [0,1]. \tag{68}$$

We begin with a preliminary result.

Lemma 4.2. *Under assumption $(\mathfrak{H}_2)$, the matrix*

$$H(t) := \int_t^1 e^{(1-s)A}\phi\phi^* e^{(1-s)A^*} ds, \quad t \in [0,1[\tag{69}$$

is invertible for any $t \in [0,1[$.

Proof. We prove in fact that $H(t)$ is invertible if and only if $\mathrm{Span}(A^k\phi, k \in \mathbb{N}) = \mathbb{R}^n$. Let $u \in \mathbb{R}^n$. Then, $u^*H(t)u = \int_t^1 \big(u^* e^{(1-s)A}\phi\big)^2 ds$. Note that $u^*H(t)u = 0$ if and only if $u^* e^{sA}\phi = 0, \quad \forall\, s \in [0, 1-t[$. Since $u^* e^{sA}\phi = \sum_{k\ge 0} u^* A^k \phi\, \frac{s^k}{k!}$, this is equivalent to

$$u^* A^k \phi = 0, \quad \forall\, k \in \mathbb{N}. \tag{70}$$

It is clear that (70) holds true if and only if $\mathrm{Span}(A^k\phi, k \ge 0) \subset u^\perp$, which completes the proof. $\square$

Proof of Proposition 4.4. First, according to Lemma 4.2, the matrix $H(t)$ is invertible.

Then, since the random variable $\mathbf{X}_0^{A,\phi}$ satisfies (48), its coordinates belong to the first chaos of B, and we can apply Lemma 4.1: the process $W_t := -B_t + \int_0^t v_s\, ds$ is a $(\mathcal{G}_t)$-Brownian motion which is independent of $\mathbf{X}_0^{A,\phi}$, where $v_s := \int_s^1 G(s,u)\, dB_u$, $0 \le s \le 1$. According to (65), $G(s,u) = \phi^* e^{(1-s)A^*} H(s)^{-1} e^{(1-u)A}\phi$ and then

$$v_s = \phi^* e^{(1-s)A^*} H(s)^{-1} \int_s^1 e^{(1-u)A}\phi\, dB_u. \tag{71}$$

We decompose the above stochastic integral as:

$$\int_s^1 e^{(1-u)A}\phi\, dB_u = \mathbf{X}_0^{A,\phi} - \int_0^s e^{(1-u)A}\phi\, dB_u, \tag{72}$$

so that identity (47) can be re-written as:

$$\mathbf{X}_s^{A,\phi} = e^{sA}\mathbf{X}_0^{A,\phi} + (Id - e^A)e^{(s-1)A}\int_0^s e^{(1-u)A}\phi\, dB_u. \tag{73}$$

Under the assumption $(\mathfrak{H}_1)$,

$$\int_0^s e^{(1-u)A}\phi\, dB_u = \big(e^A - Id\big)^{-1}\Big[e^A\mathbf{X}_0^{A,\phi} - e^{(1-s)A}\mathbf{X}_s^{A,\phi}\Big],$$

which implies:

$$\int_s^1 e^{(1-u)A}\phi\, dB_u = (e^A - Id)^{-1}\Big(e^{(1-s)A}\mathbf{X}_s^{A,\phi} - \mathbf{X}_0^{A,\phi}\Big)$$

and completes, with (71), the proof of the first assertion.

Back to (49), replacing the Brownian motion B by $\tilde{B} + V$ leads to (68). □

4.4. **The bridges of the process $\mathbf{X}^{A,\phi}$.** As in the one-dimensional case, we are interested in the disintegration of $\mathbf{X}^{A,\phi}$ along its initial (and final) time marginal, which leads to its time-Markov field property.

To that aim, we prove that conditionally on $\mathbf{X}_0^{A,\phi} = \mathbf{x}$, the process $\big(\mathbf{X}_t^{A,\phi}, t \in [0,1]\big)$ is Markov. More precisely, let us define, for any $t \in [0,1]$, the $n \times n$ matrices

$$\Lambda_t^0 := (e^A - Id)\phi\phi^* e^{(1-t)A^*} H(t)^{-1}(e^A - Id)^{-1} \tag{74}$$

$$\Lambda_t^1 := A - \Lambda_t^0 e^{(1-t)A}. \tag{75}$$

In the next theorem we identify the $\mathbf{x} \hookrightarrow \mathbf{x}$ bridge of $\mathbf{X}^{A,\phi}$ as a Markov process solution of an explicit linear stochastic differential system.

Theorem 4.1. *Suppose Assumptions $(\mathfrak{H}_1)$ and $(\mathfrak{H}_2)$ are satisfied and denote by ν the Gaussian law of the random vector $\mathbf{X}_0^{A,\phi}$. Then, $\mathbf{X}^{A,\phi}$ is a ν-mixture of its bridges, where the $\mathbf{x} \hookrightarrow \mathbf{x}$ bridge solves the affine SDE in $\mathbb{R}^n$*

$$\begin{cases} dZ_t &= (e^A - Id)\phi\, d\tilde{B}_t + (\Lambda_t^0\,\mathbf{x} + \Lambda_t^1 Z_t)\, dt, \quad t \in [0,1[, \\ Z_0 &= \mathbf{x}. \end{cases} \tag{76}$$

Proof. It is a consequence of the definitions (74) and (75) and of identity (68). □

We thus obtained, under some additional assumptions, a multidimensional generalization of Proposition 2.3.

Application to the trigonometric convoluted Brownian motion

According to Remark 4.1, the trigonometric convoluted case corresponds to $n = 2$, $\phi = (1,0)^*$ and $A = \lambda \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

We now verify that Assumptions $(\mathfrak{H}_1)$ and $(\mathfrak{H}_2)$ are satisfied. Indeed

$$(Id - e^A)^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -\cot(\lambda/2) \\ \cot(\lambda/2) & 1 \end{pmatrix}$$

and $\mathrm{Span}(\phi, A\phi) = \mathrm{Span}((1,0)^*, \lambda(0,1)^*) = \mathbb{R}^2$. Consequently Theorem 4.1 applies. All the entries of Λ_t^0, Λ_t^1 are calculable but we do not go further because the explicit formulas are complicated.

4.5. **More on the monomial convoluted Brownian motion $\mathbf{X}^{\sharp k}$.** The $(k+1)$-dimensional monomial convoluted Brownian motion $\mathbf{X}^{\sharp k}$, whose covariance was calculated in Section 4.2.2, does not satisfy Assumption $(\mathfrak{H}_1)$: $Id - e^A$ is not invertible, when the matrix A is given by (58). Therefore one can not derive its semimartingale representation (resp. the structure of its bridges) as a direct application of Section 4.3 (resp. Section 4.4). Nevertheless we recover some Markovianity considering this process jointly with an additional coordinate $\bar{X}$, constructed as a weighted primitive.

4.5.1. *A Markovian enhancement of* $\mathbf{X}^{\sharp k}$. In all this section, A denotes the matrix given by (58).

Proposition 4.5. *Consider* $\bar{X}_t := \int_0^t (1-u)^{k-1} X_u^{\sharp 1}\, du \in \mathbb{R}$ *and define the process* $\mathbf{Z}_t =: \begin{pmatrix} \mathbf{X}_t^{\sharp k} \\ \bar{X}_t \end{pmatrix} \in \mathbb{R}^{k+2}, t \in [0,1]$. *Conditionally on* $\mathbf{Z}_0 = \begin{pmatrix} \mathbf{x} \\ 0 \end{pmatrix}$, $\mathbf{Z}$ *is a Markov process which solves the affine SDE:*

$$\left\{ \begin{array}{rl} d\mathbf{X}_t^{\sharp k} & = (e^A - Id)e_{k+1}\, d\tilde{B}_t + \big(A\mathbf{X}_t^{\sharp k} + \tilde{\Lambda}_t^0\, \mathbf{x} + \tilde{\Lambda}_t^1 \mathbf{Z}_t\big)\, dt, \\ d\bar{X}_t & = (1-t)^{k-1} X_t^{\sharp 1}\, dt, \\ \mathbf{Z}_0 & = \ (\mathbf{x}, 0) \end{array} \right. \tag{77}$$

where $\tilde{\Lambda}_t^0$ *and* $\tilde{\Lambda}_t^1$ *are the matrices defined by* (82).

Before proving Proposition 4.5, we begin with three preliminary results, Lemmas 4.3-4.5. In the first one, we prove that $Id - \exp A$ can be, in some sense, weakly inverted.

Lemma 4.3. *The matrix* C *with entries* $C_{ij} := \binom{k-i+1}{j-i+1} \mathbb{1}_{\{j \ge i\}}, 1 \le i, j \le k$, *is invertible. Now, fix an element* $\mathbf{y} = (y_1, \cdots, y_{k+1})^*$ *in* $\mathbb{R}^{k+1}$. *Then, the equation*

$$(e^A - Id)\ \mathbf{x} = \mathbf{y}$$

admits a solution in $\mathbb{R}^{k+1}$ *if and only if* $y_{k+1} = 0$. *In that case the set of solutions is the* 1*-dimensional vector subspace* $\mathbb{R} \times C^{-1}(y_1, \cdots, y_k)^*$.

Proof. C is a triangular matrix whose diagonal entries are $k - i + 1, 1 \le i \le k$. They do not vanish, therefore C is invertible.

We keep the notations introduced in the proof of Proposition 4.3. By (62) we have:

$$e^A - Id = \sum_{i'=1}^{k} \sum_{j'=i'}^{k} \binom{k-(i'-1)}{j'-(i'-1)} E^{i',j'+1}.$$

Consequently: $(e^A - Id)\ \mathbf{x} = \sum_{i'=1}^{k} \Big(\sum_{j'=i'}^{k} C_{i'j'} x_{j'+1} \Big) e_{i'}$, where $\mathbf{x} = (x_1, \cdots, x_{k+1})^*$. The last assertion follows immediately.

□

Lemma 4.4. *The first component of the vector* $\int_0^t e^{(1-u)A} e_{k+1}\ dB_u$ *is given by the scalar stochastic integral* $\int_0^t (1-u)^k dB_u$.

Proof. We know that $e^{tA} e_{k+1} = \sum_{i=1}^{k+1} t^{k-(i-1)} e_i$. Consequently, the first component of $\int_0^t e^{(1-u)A} e_{k+1}\ dB_u$ is $\int_0^t (1-u)^k dB_u$. □

Lemma 4.5. *For $t \in [0,1]$, the stochastic integral $\int_0^t (1-u)^k dB_u$ decomposes as follows:*

$$\int_0^t (1-u)^k dB_u = -(1-t)^k X_t^{\sharp 1} - k\bar{X}_t + X_0^{\sharp 1} + \frac{1}{k+1}\big(1-(1-t)^{k+1}\big) X_0^{\sharp 0}. \tag{78}$$

Proof. Integrating by part:

$$\int_0^t (1-u)^k dB_u = (1-t)^k B_t + k\int_0^t (1-u)^{k-1} B_u du. \tag{79}$$

Then, using Example 3.1 and (40) with $k=1$, we write B_t as a linear combination of $X_t^{\sharp 1}, X_0^{\sharp 1}$ and $X_0^{\sharp 0}$: $B_t = X_0^{\sharp 1} + tX_0^{\sharp 0} - X_t^{\sharp 1}$. After easy calculations, we get (78). □

Proof of Proposition 4.5. We revisit the proof of Proposition 4.4, using now Lemma 4.3 instead of Assumption $(\mathfrak{H}_1)$. Relation (73) reads in our framework:

$$(e^A - Id)\int_0^s e^{(1-u)A} e_{k+1}\, dB_u = e^A\, \mathbf{X}_0^{\sharp k} - e^{(1-s)A}\, \mathbf{X}_s^{\sharp k}. \tag{80}$$

Applying Lemma 4.3, the i-th component of $\int_0^s e^{(1-u)A} e_{k+1}\, dB_u$ is given by

$$\Big(\int_0^s e^{(1-u)A} e_{k+1}\, dB_u\Big)_i = \Big(C^{-1}\big(e^A\, \mathbf{X}_0^{\sharp k} - e^{(1-s)A}\, \mathbf{X}_s^{\sharp k}\big)\Big)_{i-1}, \quad i = 2, \cdots, k+1.$$

Note that (80) does not determine the first component of $\int_0^s e^{(1-u)A} e_{k+1}\, dB_u$. But, by Lemmas 4.4 and 4.5,

$$\Big(\int_0^s e^{(1-u)A} e_{k+1}\, dB_u\Big)_1 = -(1-s)^k X_s^{\sharp 1} - k\bar{X}_s + X_0^{\sharp 1} + \frac{1}{k+1}\big(1-(1-s)^{k+1}\big) X_0^{\sharp 0}.$$

Both identities imply:

$$\int_0^s e^{(1-u)A} e_{k+1}\, dB_u = \Gamma_s^0\, \mathbf{X}_0^{\sharp k} + \Gamma_s^1\, \mathbf{Z}_s$$

where Γ_s^0 (resp. Γ_s^1) is a deterministic suitable $(k+1)\times(k+1)$ matrix (resp. $(k+1)\times(k+2)$ matrix).

Now, remark that Assumption $(\mathfrak{H}_2)$ is satisfied since

$$\mathrm{Span}(A^i e_{k+1}, i = 0, \cdots, k) = \mathrm{Span}(e_{k+1}, e_k, \cdots, e_1) = \mathbb{R}^{k+1}.$$

Therefore Lemma 4.2 implies that the matrix $H(s)$ is invertible, and the strategy used in the proof of Proposition 4.4 can be developed in our context. Using (71) and (72), we deduce that

$$B_t = -\tilde{B}_t - \int_0^t e_{k+1}^* e^{(1-s)A^*} H(s)^{-1}\Big[(-Id + \Gamma_s^0)\mathbf{X}_0^{\sharp k} + \Gamma_s^1 \mathbf{Z}_s\Big] ds \tag{81}$$

where the process $\tilde{B}$ is a $(\mathcal{G}_t)$-Brownian motion independent from $\mathbf{X}_0^{\sharp k}$ and $(\mathcal{G}_t)_t$ is the filtration obtained with the initial enlargement of the Brownian filtration with the random vector $\mathbf{X}_0^{\sharp k}$.

Back to (68), replacing B by the right hand side of (81), one obtains for all $t \in [0,1]$,

$$\begin{aligned}
\mathbf{X}_t^{\sharp k} \;=\; & \mathbf{X}_0^{\sharp k} + (e^A - Id) e_{k+1}\, \tilde{B}_t + \int_0^t A\mathbf{X}_s^{\sharp k}\, ds \\
& + \int_0^t (e^A - Id) e_{k+1} e_{k+1}^* e^{(1-s)A^*} H(s)^{-1} (-Id + \Gamma_s^0)\, \mathbf{X}_0^{\sharp k}\, ds \\
& + \int_0^t (e^A - Id) e_{k+1} e_{k+1}^* e^{(1-s)A^*} H(s)^{-1} \Gamma_s^1\, \mathbf{Z}_s\, ds.
\end{aligned}$$

Therefore, defining three matrices by

$$\begin{array}{rcl}\tilde{\Gamma}_t & := & (e^A - Id)e_{k+1}e_{k+1}^* e^{(1-s)A^*} H(s)^{-1},\\ \tilde{\Lambda}_t^0 & := & \tilde{\Gamma}_t(-Id + \Gamma_t^0),\\ \tilde{\Lambda}_t^1 & := & \tilde{\Gamma}_t \Gamma_t^1,\end{array} \tag{82}$$

one gets the affine stochastic differential system (77) satisfied by the process $\mathbf{Z} = (\mathbf{X}^{\sharp k}, \bar{X})$ pinned at time 0 in $(\mathbf{x}, 0)$. □

4.5.2. *The special case of* $\mathbf{X}^{\sharp 1}$. In this section, we treat the case $k = 1$ explicitly. The $\mathbb{R}^2$-valued process $\mathbf{X}^{\sharp 1} = (X^{\sharp 1}, X^{\sharp 0})$ is associated with the matrix $A := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Since its second component is not time-dependent but is equal to the constant r.v. B_1 (see Example 3.1), we are principally interested in the dynamics of $X^{\sharp 1}$. Indeed, the process $X^{\sharp 1}$ admits the following representation:

$$X_t^{\sharp 1} = \int_0^1 B_s\, ds + tB_1 - B_t, \quad t \in [0,1]. \tag{83}$$

Thus it is the (non independent) sum of its initial condition $X_0^{\sharp 1} = \displaystyle\int_0^1 B_s\, ds$ and a $0 \hookrightarrow 0$-Brownian bridge. Indeed, identity (37) applies with $\varphi(x) \equiv x$ and (27) gives:

$$\begin{array}{rcl} X_t^{\sharp 1} & = & \displaystyle\int_0^t \big(B_{t-r} - B_t + B_1\big)dr + \int_t^1 \big(B_{1-r+t} - B_t\big)dr \\ & = & \displaystyle\int_0^1 B_r dr + tB_1 - B_t. \end{array}$$

Therefore $X^{\sharp 1}$ is not Markov. Enlarging the filtration with its initial condition $\int_0^1 B_r dr$ - as we did in Section 2.2 for the PerOU process - is nevertheless not enough to recover the Markovianity. However, as seen in the latter section 4.5.1, the right enlargement is obtained with the 2-dimensional initial random vector $\mathbf{X}_0^{\sharp 1} = (X_0^{\sharp 1}, B_1) = \big(\int_0^1 B_r dr, B_1\big)$. Proposition 4.5 shows that once we complete $X^{\sharp 1}$ with its primitive process $\bar{X}$, we recover the Markov property. We now determine explicitly the SDE satisfied by the enhancement of $\mathbf{X}^{\sharp 1}$, that is by the (pinned) $\mathbb{R}^3$-valued process $(\mathbf{X}^{\sharp 1}, \bar{X})$.

Proposition 4.6. *The process* $(X^{\sharp 1}, X^{\sharp 0}, \bar{X})$*, pinned at time 0 in* $(x, y, 0)$*, solves the following affine stochastic differential system:*

$$\left\{\begin{array}{rl} dX_t^{\sharp 1} & = d\tilde{B}_t + \beta(t, X_t^{\sharp 1})\, dt + \gamma(t, \bar{X}_t)\, dt, \\ d\bar{X}_t & = X_t^{\sharp 1}\, dt, \quad t \in [0,1], \end{array}\right. \tag{84}$$

where $\beta(t,z) := -\dfrac{2x}{1-t} + \dfrac{3y}{(1-t)^2} - \dfrac{4}{1-t}\, z$ *and* $\gamma(t,z) := -\dfrac{6}{(1-t)^2}\, z$.
Here is $\tilde{B}$ *a* $(\mathcal{G}_t)$*-Brownian motion independent of* $\mathbf{X}_0^{\sharp 1}$ *where* $(\mathcal{G}_t)_t$ *is the filtration obtained by the initial enlargement of* $(\mathcal{F}_t)_t$ *with the random vector* $\mathbf{X}_0^{\sharp 1}$.

Proof. First remark that (83) implies that $X_0^{\sharp 0} = 2\bar{X}_1 = B_1$.

We now apply Lemma 4.1 to $(\xi_1, \xi_2)^* = (X_0^{\sharp 0}, X_0^{\sharp 1})^*$ taking $h_1(t) \equiv 1$ and $h_2(t) := 1-t$. Therefore the matrix H is given by:

$$H_{11}(t) = 1 - t, \quad H_{12}(t) = H_{21}(t) = \frac{(1-t)^2}{2}, \quad H_{22}(t) = \frac{(1-t)^3}{3},$$

and the matrix G satisfies:

$$\begin{aligned} G(t,u) &= \frac{12}{(1-t)^4}\begin{pmatrix} 1 & 1-t \end{pmatrix}\begin{pmatrix} \frac{(1-t)^3}{3} & -\frac{(1-t)^2}{2} \\ -\frac{(1-t)^2}{2} & 1-t \end{pmatrix}\begin{pmatrix} 1 \\ 1-u \end{pmatrix} \\ &= \frac{1}{(1-t)^2}\big(6(1-u)-2(1-t)\big). \end{aligned}$$

Thus

$$\int_s^1 G(s,u)\,dB_u = \frac{1}{(1-s)^2}\Big[6\int_s^1 (1-u)dB_u - 2(1-s)(B_1-B_s)\Big].$$

We reformulate it using first the integration by parts:

$$\int_s^1 (1-u)dB_u = -(1-s)B_s + \int_0^1 B_u du - \int_0^s B_u du,$$

together with the boundary conditions $X_0^{\sharp 1} = \int_0^1 B_s ds = x,\ X_0^{\sharp 0} = B_1 = y$. We obtain

$$\int_s^1 G(s,u)\,dB_u = \frac{-2(y+2B_s)}{1-s} + \frac{6x}{(1-s)^2} - \frac{6}{(1-s)^2}\int_0^s B_u du.$$

Using (83), we get $B_t = yt + x - X_t^{\sharp 1}$ and then $\displaystyle\int_0^s B_t\,dt = s^2y/2 + sx - \bar{X}_s$. Therefore

$$\int_s^1 G(s,u)\,dB_u = \frac{2x}{1-s} + \Big(1 - \frac{3}{(1-s)^2}\Big)y + \frac{4X_s^{\sharp 1}}{1-s} + \frac{6}{(1-s)^2}\bar{X}_s. \tag{85}$$

We can now conclude, using (66), (85) and :

$$X_t^{\sharp 1} = x + \tilde{B}_t + \int_0^t \big(y - \int_s^1 G(s,u)\,dB_u\big)ds.$$

□

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