We develop basic properties of a Markov process that is invariant under the action of a locally compact topological group.

1. Introduction

The invariance of probability distributions under various transformations plays an important role in the probability theory. In the classical theory, the translation invariant Markov processes in a Euclidean space $\mathbb{R}^n$ can be identified with Lévy processes, which are processes $x_t$ that have independent and stationary increments in sense that for any $s < t$, $x_t - x_s$ is independent of the process up to time $s$ and its distribution depends only on $t - s$. Lévy processes in $\mathbb{R}^n$ have been extensively studied, but they still generate enormous interests, see Applebaum [1], Bertoin [4] and Sato [35] for some of modern books on this subject. By the celebrated Lévy-Khinchin formula, a Lévy process may be represented by a triple of a drift vector, a covariance matrix and a Lévy measure, in the sense that its distribution is determined by the triple, and to any such triple, there is a Lévy process, unique in distribution. A natural extension is to study Markov processes in a group that is invariant under the left or right translations. Hunt’s 1956 pioneering work [20] provided an explicit formula for the generator of such an invariant Markov process in a Lie group, which allows us to represent such a process in distribution by a triple of a drift vector, a covariance matrix and a Lévy measure, just as for a Lévy process in $\mathbb{R}^n$.

Purpose of this paper is to develop the basic definitions and properties of invariant Markov processes under topological group actions may be considered at three levels of generality. First, we may consider Markov processes in a topological group $G$ that are invariant under the left (or right) translations. Such processes are direct extensions of classical Lévy processes in $\mathbb{R}^n$, and can be identified with processes in $G$ that have independent and stationary increments of the form $x_s^{-1}x_t$ (or $x_tx_s^{-1}$) for $s < t$. At the second level, we may consider a Markov process $x_t$ in a topological space $X$ that is invariant under the transitive action of a topological group $G$ on $X$. In this case, $X$ may be identified with a topological homogeneous space $G/K$, and $x_t$ with a Markov process in $G/K$ invariant under the natural $G$-action.

We will call a Markov process in $G$ invariant under left translations, or a Markov process in $G/K$ invariant under the natural $G$-action, a Lévy process. A homogeneous space $G/K$ does not possess a natural product structure. To study invariant Markov processes in $G/K$, we have developed a theory of a kind of product “in distribution” on $G/K$ that allows us to carry over almost all the results on the group $G$ to the homogeneous space $G/K$. For example, it is easy to show that Lévy processes in $G/K$, which are defined as $G$-invariant Markov processes, can be characterized as processes that have

2010 Mathematics Subject Classification. 60J25, 60B15.

Key words and phrases. Markov processes, topological groups.
independent and stationary increments in a suitable sense, just as their counterparts in $G$.

At the third level of generality, we may consider a Markov process $x_t$ invariant under the non-transitive action of a topological group $G$. In this case, under some suitable assumptions, $x_t$ may be decomposed into a radial part and an angular part. The radial part can be an arbitrary Markov process in a subspace that is transversal to $G$-orbits, whereas under the conditional distribution given the radial part, the angular part is a (time) inhomogeneous Markov process in a standard $G$-orbit that is invariant under the $G$-action.

This leads us to consider inhomogeneous Markov processes that are invariant under a group action. It is easy to show that inhomogeneous Markov processes $x_t$ in a group $G$ invariant under left translations, or in $G/K$ invariant under the natural $G$-action, may be identified with processes that have independent, but not necessarily stationary, increments, so such processes will be called inhomogeneous Lévy processes.

As our purpose is limited to developing the basic properties under the general framework of the action of a locally compact topological group, without involving Lie structures and Fourier analysis, so many deeper aspects of invariant Markov processes are not discussed here. We briefly mention some of these topics. Heyer [18] generalized Hunt’s generator formula on Lie groups to locally compact groups in the form of a sum of three maps, corresponding to a drift, a diffusion part, and a jump part. Using a projective basis of Lie algebras, Born [6] rewrote Heyer’s formula in a form more closely resembling Hunt’s formula. For inhomogeneous Lévy processes in Lie groups, Feinsilver [11] (1978) obtained a martingale property, generalizing a result of Stroock-Varadhan [37] (1973) for continuous processes. By this martingale property, an inhomogeneous Lévy process is determined in distribution by a time dependent triple of a deterministic path, a covariance matrix function and a Lévy measure function. A different form of martingale representation on more general locally compact groups, in terms of the abstract Fourier analysis, is obtained in Heyer-Pap [19]. The author [27] obtained an extension of Feinsilver’s result to homogeneous spaces.


Probability measures on locally compact groups have been studied extensively. For a comprehensive treatment, the reader is referred to Heyer’s 1977 classic treatise [18], which is still an invaluable reference today. Some of more recent developments can be found in Siebert [36], Pap [31], Dani-McCrudden [7, 8] and Liao [28]. A comprehensive analysis of probability measures on compact Lie groups can be found in Applebaum’s recent book [2].

We now briefly describe the content of each section. The formal definition of invariant Markov processes under group actions is given in the next section $\S 2$. Lévy processes in a topological group $G$ and in a topological homogeneous space $G/K$ are discussed respectively in sections $\S 3$ and $\S 4$. Inhomogeneous Lévy processes in $G$ and $G/K$ are
considered in §5. In §6 and §7, the decomposition of an invariant Markov process under a non-transitive action into a radial part and an angular part is introduced, and it is shown that given the radial part, the conditioned angular part is an inhomogeneous Lévy process in a standard orbit. This has been done in Liao [27] on Lie groups.

2. Invariant Markov processes under group actions

We now state some general definitions and conventions to be used in this paper, most of which are also commonly used in the literature. On a topological space $X$, let $\mathcal{B}(X)$ be its Borel $\sigma$-algebra, and let $\mathcal{B}_0(X)$ and $\mathcal{B}_+(X)$ be respectively the spaces of $\mathcal{B}(X)$-measurable functions that are bounded and nonnegative. A measure $\mu$ on $X$ is always assumed to be defined on $\mathcal{B}(X)$ unless when explicitly stated otherwise. For a measurable function $f$, the integral $\int f \, d\mu$ may be written as $\mu(f)$. If $\mu$ is a measure on a measurable space $X$ and if $F$ is a measurable map from $X$ to another measurable space $Y$, then $F\mu$ denotes the measure on $Y$ given by $F\mu(f) = \mu(f \circ F)$ for any measurable function $f \geq 0$ on $Y$. This may also be written as $F\mu(B) = \mu(F^{-1}(B))$ for measurable $B \subset X$. A kernel $K$ from $X$ to $Y$ is a family of measures $K(x, \cdot)$ on $Y$, $x \in X$, such that $x \mapsto K(x,B)$ is measurable on $X$ for any measurable $B \subset Y$. It is called a probability kernel, or a sub-probability kernel, if for all $x \in X$, $K_t(x,X) = 1$, or $K(x,X) \leq 1$.

A topological group $G$ is a group and a topological space such that both the product map $G \times G \rightarrow G, (g,h) \mapsto gh$, and the inverse map $G \rightarrow G, g \mapsto g^{-1}$, are continuous. A continuous action of $G$ on a topological space $X$ is a continuous map $G \times X \rightarrow X$ given by $(g,x) \mapsto gx$ such that $g(hx) = (gh)x$ and $ex = x$ for $g, h \in G$ and $x \in X$, where $e$ is the identity element of $G$. In the sequel, the action of a topological group $G$ on a topological space $X$ is always assumed to be continuous unless when explicitly stated otherwise. The action is called linear if $X$ is a linear space and the map $x \mapsto gx$ is linear for each $g \in G$.

The group action defined above is also called a left action. We may also consider a right action when the action map is written as $(x,g) \mapsto xg$ and satisfies $(xg)h = x(gh)$. In the sequel, all actions are assumed to be left actions unless when explicitly stated otherwise.

A function $f$ or a measure $\mu$ on $X$ is called invariant under a measurable map $g$: $X \rightarrow X$, or $g$-invariant for short, if $f \circ g = f$ or $\mu g = \mu$. An operator $T$ on $X$ with domain $D(T)$ being a set of functions on $X$ is map from $D(T)$ to a possibly different set of functions on $X$. It is called $g$-invariant if $\forall f \in D(T)$, $f \circ g \in D(T)$ and $T(f \circ g) = (Tf) \circ g$. A kernel $K$ from $X$ to itself may be regarded as an operator $Kf(x) = K(x,f)$ on $X$ with domain $D(K) = B_+(X)$, then it is $g$-invariant if $K(g(x),B) = K(x,g^{-1}(B))$ for $x \in X$ and $B \in \mathcal{B}(X)$.

A function $f$, a measure $\mu$, an operator $T$ or a kernel $K$ on $X$ is called invariant under the action of a group $G$, or $G$-invariant for short, if it is $g$-invariant for any $g \in G$.

For $g \in G$, let $l_g$, $r_g$ and $c_g$ be respectively the left translation, the right translation and the conjugation map: $G \rightarrow G$, defined by $l_gx = gx$, $r_gx = xg$ and $c_gx = gxg^{-1}$ for $x \in G$. The group $G$ acts on itself by left translation and also by conjugation, whereas the right translation is a right action of $G$ on itself. A function $f$ or a measure $\mu$ on $G$ is called left invariant if it is invariant under the action of $G$ on itself by left translations.

that is, if $f \circ l_g = f$ or $l_g\mu = \mu$ for $g \in G$. Similarly, an operator $T$ on $G$ is called left invariant if for $f \in D(T)$, $f \circ l_g \in D(T)$ and $(Tf) \circ l_g = T(f \circ l_g)$ for $g \in G$. If this holds only for $g$ contained in a subgroup $K$ of $G$, then $f$, $\mu$ or $T$ is called $K$-left invariant. The $(K)$-right invariant and $(K)$-conjugate invariant functions, measures and operators on $G$ are defined similarly. When they are both left and right (resp $K$-left and $K$-right) invariant, then they are called bi-invariant (resp. $K$-bi-invariant).
Throughout this chapter, let $X$ be a topological space and let $G$ be a topological group that acts continuously on $X$, both are equipped with lcsH (locally compact and second countable Hausdorff) topologies. We will let $C(X)$, $C_b(X)$, $C_c(X)$ and $C_0(X)$ denote respectively the spaces of continuous functions, bounded continuous functions, continuous functions with compact supports and continuous functions convergent to 0 at infinity (under the one-point compactification topology) on $X$. Note that $C_0(X)$ may also be characterised as the space of continuous functions $f$ on $X$ such that for any $\varepsilon > 0$, there is a compact $K \subset X$ with $|f| < \varepsilon$ on $K^c$ (the complement of $K$ in $X$).

A family of sub-probability kernels $P_t$ from $X$ to $X$, $t \in \mathbb{R}_+$, is called a transition semigroup on $X$ if $P_t P_s = P_{t+s}$ for $s,t \in \mathbb{R}_+$ and $P_0(x, \cdot) = \delta_x$ (the unit mass at point $x$). A Markov process $x_t$ in $X$, $t \in \mathbb{R}_+$, usually means a family of processes, one for each starting point $x \in X$, associated to a transition semigroup $P_t$ and satisfying the following simple Markov property: for $t > s \geq 0$ and $f \in \mathcal{B}_b(X)$,

$$E[f(x_t) \mid \mathcal{F}_s] = P_{t-s}f(x_s) \quad P \text{-almost surely,}$$

where $\{\mathcal{F}_t\}$ is the natural filtration of process $x_t$. If (1) holds under a larger filtration $\{\mathcal{F}_t\}$, a possibly stronger requirement, then $x_t$ is called a Markov process associated to $\{\mathcal{F}_t\}$ or an $(\mathcal{F}_t)$-Markov process. The symbol $P_x$ is used to denote the distribution of the process starting at $x$ on the canonical path space and $E_x$ is the associated expectation. Occasionally, a Markov process means a single process with a given initial distribution and this should be clear from context. We will allow a Markov process $x_t$ to have a possibly finite life time as $P_t$ is assumed to be sub-probability, not necessarily a probability kernel.

A family of sub-probability kernels $P_{s,t}$ from $X$ to $X$, $0 \leq s \leq t < \infty$, is called a two-parameter transition semigroup on $X$ if $P_{r,s}P_{s,t} = P_{r,t}$ for any $r \leq s \leq t$ and $P_{t,t}(x, \cdot) = \delta_x$. A (time) inhomogeneous Markov process $x_t$ in $X$ is a family of processes, one for each pair of starting time $s \in \mathbb{R}_+$ and starting point $x \in X$, associated to a two-parameter transition semigroup $P_{s,t}$ and satisfying the following inhomogeneous Markov property:

$$E[f(x_t) \mid \mathcal{F}_s] = P_{s,t}f(x_s), \quad P \text{-almost surely.}$$

where $\{\mathcal{F}_t\}$ is the natural filtration of process $x_t$. If the above holds for a larger filtration $\{\mathcal{F}_t\}$, then the inhomogeneous Markov process $x_t$ is said to be associated to $\{\mathcal{F}_t\}$.

A Markov process $x_t$ in $X$ is called invariant under a measurable map $g$: $X \to X$, or $g$-invariant for short, if its transition semigroup $P_t$, as an operator for each $t$, is $g$-invariant, that is, if

$$P_t(g(x), B) = P_t(x, g^{-1}(B))$$

for any $t \in \mathbb{R}_+$, $x \in X$ and $B \in \mathcal{B}(X)$. Note that (3) is equivalent to

$$E_{gx}[f(x_t)] = E_x[f(gx_t)]$$

for any $t \in \mathbb{R}_+$, $x \in X$ and $f \in \mathcal{B}_b(X)$.

**Proposition 1.** A Markov process $x_t$ in $X$ is $g$-invariant if and only if for any $x \in X$, the process $gx_t$ with $x_0 = x$ has the same distribution as the process $x_t$ with $x_0 = gx$.

**Proof.** It is clear that the same distribution implies the $g$-invariance (4). Now assume (4). Let $\{\mathcal{F}_t\}$ be the natural filtration of process $x_t$. Then for $0 < t_1 < t_2$ and $f_1, f_2 \in \mathcal{B}_b(X)$, by the simple Markov property,

$$E_{gx}[f_1(x_{t_1})f_2(x_{t_2})] = E_{gx}[f_1(x_{t_1})E_{gx}[f_2(x_{t_2}) \mid \mathcal{F}_{t_1}]] = E_{gx}[f_1(x_{t_1})P_{t_2-t_1}f_2(x_{t_1})]$$

$$= E_x[f_1(gx_{t_1})P_{t_2-t_1}f(gx_{t_1})] = E_x[f_1(gx_{t_1})P_{t_2-t_1}(f \circ g)f(x_{t_1})]$$

$$= E_x[f_1(gx_{t_1})E_x[f_2(gx_{t_2}) \mid \mathcal{F}_{t_1}]] = E_x[f_1(gx_{t_1})f_2(gx_{t_2})].$$
Inductively, it can be shown that for \( t_1 < t_2 < \cdots < t_n \),
\[
E_{x}[f_1(x_{t_1})f_2(x_{t_2}) \cdots f_n(x_{t_n})] = E_x[f_1(gx_{t_1})f_2(gx_{t_2}) \cdots f_n(gx_{t_n})].
\]
This proves the same distribution. \(\square\)

The \( g \)-invariance for an inhomogeneous Markov process \( x_t \) in \( X \) is defined in a similar fashion. The process \( x_t \) is called invariant under a measurable map \( g: X \to X \), or \( g \)-invariant, if its transition semigroup \( P_{s,t} \) is \( g \)-invariant, that is, if
\[
\forall t \geq s \geq 0, \ x \in X \text{ and } B \in \mathcal{B}(X), \quad P_{s,t}(gx,B) = P_{s,t}(x,g^{-1}(B)).
\]

The following proposition may be proved as Proposition 1.

**Proposition 2.** An inhomogeneous Markov process \( x_t \) is \( g \)-invariant if and only if for all \( s \in \mathbb{R}_+ \) and \( x \in X \), the process \( gx_t \), \( t \geq s \), with \( x_s = x \) has the same distribution as process \( x_t \), \( t \geq s \), with \( x_s = gx \).

A Markov process \( x_t \) in \( X \) is called invariant under the action of a group \( G \), or \( G \)-invariant, if its transition semigroup \( P_t \) is \( g \)-invariant for any \( g \in G \). Similarly, an inhomogeneous Markov process \( x_t \) in \( X \) is called \( G \)-invariant if its transition semigroup \( P_{s,t} \) is \( G \)-invariant.

A process is said to have rcll paths, or called a rcll process, if almost all its paths are right continuous with left limits in its state space. In literature, a rcll process is often called càdlàg (French “continue à droite, limite à gauche”). Most processes in this work will be assumed to be rcll. We will see in Remarks 5, 15, 21 and 28 that many of these processes in fact have rcll versions if they are continuous in distribution.

A Markov process \( x_t \) in \( X \) is called a Feller process if its transition semigroup \( P_t \) is Feller, that is, if for any \( f \in C_0(X) \), \( P_tf \in C_0(X) \) and \( P_tf \to f \) pointwise on \( X \) as \( t \to 0 \), noting that this in fact implies that \( P_tf \to f \) uniformly on \( X \). A Feller process has a rcll version, so it will be assumed to be rcll, and it has many other useful properties, including the strong Markov property. See for example [21, Chapter 19] for more details.

### 3. Lévy processes in topological groups

A Markov process \( x_t \) in a topological group \( G \) invariant under the action of \( G \) on itself by left translation is called left invariant. Its transition semigroup \( P_t \) is left invariant in the sense that
\[
P_t(f \circ l_g)(x) = P_t f(gx)
\]
for \( t \in \mathbb{R}_+ \), \( x \in G \) and \( f \in \mathcal{B}_b(G) \).

Let \( x_t \) be a rcll left invariant Markov process in \( G \). Recall \( e \) is the identity element of \( G \). Then \( P_t f(e) = P_t (f \circ l_e)(e) = E_e[f(xx_t)] \). By the rcll paths, one sees that \( P_t \) is a Feller transition semigroup, and hence \( x_t \) is a Feller process.

Let \( x_t \) be a process in \( G \) with an infinite life time, and let \( \{F^t\} \) be its natural filtration. It is said to have independent increments if for \( s < t \), \( x_s^{-1}x_t \) is independent of \( F_s^t \). It is said to have stationary increments if for \( s < t \), the distribution of \( x_s^{-1}x_t \) depends only on \( t-s \), that is, if \( x_s^{-1}x_t \) and \( x_0^{-1}x_t-x_s \) have the same distribution. A rcll process \( x_t \) is called a Lévy process in \( G \) if it has independent and stationary increments.

The classical example is a Lévy process \( x_t \) in the \( d \)-dim Euclidean space \( \mathbb{R}^d \) when \( \mathbb{R}^d \) is regarded as an additive group. In this case, the increment \( x_s^{-1}x_t \) is written as \( x_t-x_s \).

For a Lévy process \( x_t \) in \( G \), let
\[
x_t^e = x_0^{-1}x_t.
\]
It is clear that process \( x_t^e \) is also a Lévy process in \( G \) starting at \( e \) and is independent of \( x_0 \).
Let \( x_t \) be a Lévy process in \( G \). For \( t \in \mathbb{R}_+ \), \( x \in G \) and \( f \in \mathcal{B}_b(G) \), put

\[ P_t f(x) = E[f(x x_t^c)]. \]

It is easy to see that this defines a transition semigroup \( P_t \) on \( G \), which is conservative, that is, \( P_1 = 1 \), and is left invariant.

For \( t > s \),

\[ E[f(x_t) \mid \mathcal{F}_s] = E[f(x_s x_s^{-1} x_t) \mid \mathcal{F}_s] = E[f(x x_t^c)] |_{x=x_s} = P_{t-s} f(x_s). \]

This shows that the Lévy process \( x_t \) is a left invariant Markov process.

Now let \( x_t \) be a rcll left invariant Markov process in \( G \) with an infinite life time. Then for \( t > s \) and \( f \in \mathcal{B}_b(G) \),

\[ E[f(x_s^{-1} x_t) \mid \mathcal{F}_s] = E[f(x^{-1} x_t) \mid \mathcal{F}_s] |_{x=x_s} = P_{t-s} f(e) = P_{t-s} f(x). \]

This shows that \( x_t \) has independent and stationary increments, and hence is a Lévy process. We have proved that the class of Lévy processes in \( G \) coincides with the class of left invariant rcll Markov processes in \( G \) with infinite life times.

Let \( x_t \) be a left invariant rcll Markov process in \( G \) with a possibly finite life time. Then \( P_t 1(x) = P_t (1 \circ l_x)(e) = P_t 1(e) \), which is right continuous in \( t \) due to the right continuity of paths. By the semigroup property and the left invariance of \( P_t \),

\[ P_{t+s} 1(x) = \int P_s(e, dy) P_t 1(x) = P_t 1(e) P_{t+s} 1(e). \]

It follows that \( P_t 1(x) = e^{-\lambda t} \) for all \( x \in G \) and for some fixed \( \lambda \geq 0 \). Let \( \hat{P}_t = e^{\lambda t} P_t \). It is easy to see that \( \hat{P}_t \) is a conservative and left invariant Feller transition semigroup. The Feller process \( \hat{x}_t \) associated to \( \hat{P}_t \) is a left invariant rcll Markov process with an infinite life time, and hence is a Lévy process. Let \( \tau \) be an exponential random variable of rate \( \lambda (\tau = \infty \text{ if } \lambda = 0) \) that is independent of \( \hat{x}_t \), and let \( x_t^\prime \) be the process \( \hat{x}_t \) killed at time \( \tau \), that is, \( x_t^\prime = \hat{x}_t \) for \( t < \tau \) and \( x_t^\prime = \Delta \) for \( t \geq \tau \), where \( \Delta \) is the point at infinity (see Appendix A.3). It is easy to show that \( x_t^\prime \) is a rcll Markov process with transition semigroup \( P_t \) and hence is identical in distribution to process \( x_t \). To summarize, we have proved the following result.

**Theorem 3.** A left invariant rcll Markov process in \( G \) with an infinite life time is a Lévy process. Conversely, a Lévy process \( x_t \) in \( G \) is a left invariant rcll Markov process with an infinite life time and its transition semigroup is given by (8).

In general, a left invariant rcll Markov process \( x_t \) in \( G \) with a possibly finite life time is a Feller process and is identical in distribution to a Lévy process \( \hat{x}_t \) killed at an independent exponential time of rate \( \lambda \geq 0 \). The transition semigroup \( P_t \) of \( x_t \) and \( \hat{P}_t \) of \( \hat{x}_t \) are related as \( P_t = e^{-\lambda t} \hat{P}_t \).

**Remark 4.** Let \( x_t \) be a process in \( G \) with an infinite life time. The above proof shows that \( x_t \) is a left invariant Markov process if and only if it has independent and stationary increments.

**Remark 5.** Let \( x_t \) be a left invariant Markov process in \( G \) without assuming an infinite life time, and let \( P_t \) be its transition semigroup. Then \( P_t f(x) = P_t (f \circ l_x)(e) = \int_G f(xy) \mu_t(dy) \), where \( \mu_t = P_t(e, \cdot) \). If \( \mu_t \to \delta_e \) (the unit point mass at \( e \) weakly as \( t \to 0 \)), then \( P_t \) is Feller and hence \( x_t \) has a rcll version. Therefore, if \( x_t \) also has an infinite life time, then after a modification on a null set for each \( t \geq 0 \), \( x_t \) becomes a Lévy process in \( G \).

Given a filtration \( \{ \mathcal{F}_s \} \) on the underlying probability space \( (\Omega, \mathcal{F}, P) \). A Lévy process \( x_t \) is called associated to \( \{ \mathcal{F}_s \} \), or an \( \{ \mathcal{F}_s \} \)-Lévy process, if it is adapted to \( \{ \mathcal{F}_s \} \) and for any \( s < t \), \( x_t^{-1} x_s \) is independent of \( \mathcal{F}_s \). A Lévy process is clearly associated to its natural filtration. By the proof of Theorem 3, it is easy to see that a Lévy process is associated to a filtration if and only if it is associated to the same filtration as a Markov process.
It is easy to see that if \( x_t \) is a Lévy process associated to a filtration \( \{\mathcal{F}_t\} \) and if \( s > 0 \) is fixed, then \( x'_t = x^{-1}_s x_{t+s} \) is a Lévy process identical in distribution to the process \( x'_t \) and is independent of \( \mathcal{F}_s \). The following theorem says that \( s \) may be replaced by a stopping time.

**Theorem 6.** Let \( x_t \) be a Lévy process associated to a filtration \( \{\mathcal{F}_t\} \) in \( G \). If \( \tau \) is an \( \{\mathcal{F}_t\} \)-stopping time with \( P(\tau < \infty) > 0 \), then under the conditional probability \( P(\cdot | \tau < \infty) \), the process \( x'_t = x^{−1}_\tau x_{t+\tau} \) is a Lévy process in \( G \) that is independent of \( \mathcal{F}_\tau \). Moreover, the process \( x'_t \) under \( P(\cdot | \tau < \infty) \) has the same distribution as the process \( x'_t \) under \( P \).

**Proof.** First assume \( \tau \) takes only discrete values. Fix \( 0 < t_1 < t_2 < \ldots < t_k \), \( \phi \in C_b(G^k) \) and \( \xi \in (\mathcal{F}_\tau)_+ \). Because \( \xi_{1|\tau=t} \in (\mathcal{F}_t)_+ \), where \( 1_A \) denotes the indicator of set \( A \), we have

\[
E[\phi(x^{-1}_\tau x_{t_1}, \ldots, x^{-1}_\tau x_{t_k})\xi | \tau < \infty] = \sum_{t < \infty} E[\phi(x^{-1}_{t_1} x_{t+t_1}, \ldots, x^{-1}_{t_k} x_{t+t_k})\xi_{1|\tau=t}]/P(\tau < \infty)
\]

\[
= \sum_{t < \infty} E[\phi(x^{-1}_{t_1} x_{t+t_1}, \ldots, x^{-1}_{t_k} x_{t+t_k})] E(\xi_{1|\tau=t})/P(\tau < \infty)
\]

\[
(9) = E[\phi(x_0^{-1} x_{t_1}, \ldots, x_0^{-1} x_{t_k})] E(\xi | \tau < \infty)
\]

Setting \( \xi = 1 \) yields \( E[\phi(x^{-1}_{t_1} x_{t+t_1}, \ldots, x^{-1}_{t_k} x_{t+t_k}) | \tau < \infty] = E[\phi(x_0^{-1} x_{t_1}, \ldots, x_0^{-1} x_{t_k})] \).

Therefore, for a general \( \xi \in (\mathcal{F}_t)_+ \), the expression in (9) is equal to

\[
E[\phi(x^{-1}_{t_1} x_{t+t_1}, \ldots, x^{-1}_{t_k} x_{t+t_k}) | \tau < \infty] E(\xi | \tau < \infty)
\]

This proves the desired result for a discrete stopping time \( \tau \).

For a general stopping time \( \tau \), let \( \tau_n = (k+1)^2^{-n} \) on the set \( \{k \cdot 2^{-n} \leq \tau < (k+1)2^{-n}\} \) for \( k = 0, 1, 2, \ldots \). Then \( \tau_n \) are discrete stopping times and \( \tau_n \downarrow \tau \) as \( n \uparrow \infty \). The result for \( \tau \) follows from the discrete case and the right continuity of \( x_t \).

The convolution of two measures \( \mu \) and \( \nu \) on \( G \) is the measure \( \mu * \nu \) on \( G \) determined by

\[
(10) \mu * \nu(f) = \int f(xy) \mu(dx)\nu(dy)
\]

for \( f \in B_+(G) \), and it is a probability measure if so are \( \mu \) and \( \nu \). It is easy to check that the convolution is associative, that is, \((\mu * \nu) * \gamma = \mu * (\nu * \gamma)\), and hence, it is meaningful write a convolution product \( \mu_1 * \mu_2 * \cdots * \mu_n \) or an \( n \)-fold convolution power \( \mu * n = \mu * \mu * \cdots * \mu \).

A family of probability measures \( \mu_t \) on \( G \), \( t \geq 0 \), is called a convolution semigroup on \( G \) if \( \mu_{s+t} = \mu_s * \mu_t \). Although convolution semigroup may be defined for more general measures, in this work, a convolution semigroup will always mean a family of probability measures. It is called continuous if \( \mu_t \to \mu_0 \) weakly as \( t \to 0 \). By [18, Theorem 1.5.7., if \( \mu_t \) is continuous, then \( \mu_t \to \mu_s \) weakly as \( t \to s \) for all \( s > 0 \). An alternative proof will be given shortly.

Let \( x_t \) be a Lévy process in \( G \). It is easy to see that \( P_t(e, \cdot) \), the distributions \( \mu_t \) of \( x_t \), \( t \geq 0 \), form a continuous convolution semigroup of probability measures on \( G \) (the continuity follows from the right continuity of paths) with \( \mu_0 = \delta_e \), which will be called the convolution semigroup associated to the Lévy process \( y_t \). On the other hand, if \( \mu_t \) is a continuous convolution semigroup on \( G \) with \( \mu_0 = \delta_e \), then

\[
(11) P_t f(x) = \mu_t(f \circ l_s),
\]

for \( f \in C_b(G) \), defines a left invariant conservative Feller transition semigroup \( P_t \) on \( G \). There is a Feller process \( x_t \) associated to \( P_t \), which is left invariant, and hence is a Lévy process with \( \mu_t \) as the associated convolution semigroup.
Because \( x_t \) is a Feller process, it is stochastically continuous, that is, \( x_t = x_{t-} \) almost surely for all \( t > 0 \) (see for example ([21, Proposition 25.20])). It follows that \( \mu_t \) is in fact continuous in \( t \) under the weak convergence, as mentioned before.

To summarize, we have the following result.

**Theorem 7.** For a Lévy process \( x_t \) in \( G \), the distributions \( \mu_t \) of \( x_t^c = x_0^{-1}x_t \) form a continuous convolution semigroup on \( G \) with \( \mu_0 = \delta_e \). Conversely, if \( \mu_t \) is such a convolution semigroup on \( G \), then there is a Lévy process \( x_t \) in \( G \) with \( \mu_t \) as distribution of \( x_t^c \).

Using the Markov property, it is easy to show that the finite dimensional distributions of a Lévy process \( x_t \) in \( G \) with associated convolution semigroup \( \mu_t \) are given by

\[
E[f(x_{t_1}, x_{t_2}, \ldots, x_{t_n})] = \int f(x_0x_{t_1}, x_0x_{t_2}, \ldots, x_0x_{t_n}) \mu_0(dx_0)\mu_1(dx_1)\mu_{t_2-t_1}(dx_2)\cdots\mu_{t_n-t_{n-1}}(dx_n)
\]

for \( f \in B_0(X^n) \) and \( 0 \leq t_1 < t_2 < \cdots < t_n \), where \( \mu_0 \) is the initial distribution.

**Remark 8.** In the definition of Lévy processes, if the increments \( x_{s}^{-1}x_{t} \) for \( s < t \) are replaced by \( x_{t}x_{s}^{-1} \), this will lead to a different definition of Lévy processes, which coincide with Markov processes in \( G \) that are invariant under right translations. In [25], the Lévy processes defined earlier and defined here are called respectively left and right Lévy processes. Because the group \( G \) is in general non-commutative, left and right Lévy processes are two different classes of processes, but they are in natural duality under the map \( x_t \mapsto x_t^{-1} \). All the preceding results established for left Lévy processes hold also for right Lévy processes with suitable changes. For example, \( x_t^e = x_0^{-1}x_t \) should be changed to \( x_t^e = x_t^{-1}x_0 \) and (11) holds with \( l_x \) replaced by \( r_x \). Note that a continuous convolution semigroup \( \mu_t \) on \( G \) with \( \mu_0 = \delta_e \) can be used to generate either a left Lévy process by (11) or a right Lévy process by the counterpart of (11) with \( l_x \) replaced by \( r_x \).

In this paper, we will exclusively consider left Lévy processes, unless when explicitly stated otherwise, and we will omit the adjective “left” in its name.

4. Lévy process in topological homogeneous spaces

Let \( X \) be a topological space and let \( G \) be a topological group that acts transitively on \( X \), both are equipped with lcsCH topologies. Fix a point \( o \in X \). The isotropy subgroup \( K \) of \( G \) at \( o \), defined by \( K = \{ g \in G; go = o \} \), is a closed subgroup of \( G \).

For any closed subgroup \( K \) of \( G \), the space \( G/K \) of left cosets \( gK \), \( g \in K \), is called a homogeneous space of \( G \). It is equipped with the quotient topology, under which the natural projection

\[
\pi: G \rightarrow G/K, \quad g \mapsto gK,
\]

is continuous and open. Under this topology, the natural action of \( G \) on \( G/K \), given by \( xK \mapsto gxK \) for \( g \in G \), is continuous. Moreover, when \( K \) is the isotropy subgroup at \( o \in X \) as defined above, then under the map: \( gK \mapsto go \), \( G/K \) is homeomorphic to \( X \), and the \( G \)-action on \( X \) is just the natural action of \( G \) on \( G/K \) (see Theorem 3.2 in [16, chapter II]).

We may identify \( X \) and \( G/K \) together with the associated \( G \)-actions in this way. Under this identification, the natural projection \( \pi: G \rightarrow G/K \) is just the map: \( G \rightarrow X \) given by \( g \mapsto go \), where \( o = eK \).

Note that if \( G \) is a topological group with a lcsCH topology, and if \( K \) is a compact subgroup, then the homogeneous space \( X = G/K \), under the quotient topology, is a lcsCH space.

In the rest of this section, we will assume \( K \) is compact. We will also assume the homogeneous space \( G/K \) has a continuous local section in the sense that there are a
neighborhood $U$ of $o$ in $X$ and a continuous map $\phi: U \to G$ such that $\pi \circ \phi = \text{id}_U$ (the identity map on $U$). Because the isotropy subgroup at a different point $g o$, $g \in G$, is $gKg^{-1}$, it can be shown that these assumptions are independent of the choice for the point $o \in X$. Under these assumptions, it can be shown that the map: $(x, k) \mapsto \phi(x)k$ is a homeomorphism from $U \times K$ onto a neighborhood of $e$ of $G$. Some sufficient conditions for the existence of a continuous local section map may be found in [22, 30], and see also [22] for some non-existence examples.

It is well known and also easy to see that a continuous local section, even a smooth local section, exists if $G$ is a Lie group. By Theorems 4.6 and 4.7, and Corollary 4.8 in [23, Chapter I], the group $G$ of isometries on a connected Riemannian manifold $X$ has a Lie group structure and acts smoothly on $X$, and the isotropy subgroup $K$ of $G$ at any $o \in X$ is compact. In this case, if $G$ acts transitively on $X$, then $X = G/K$ satisfies all the assumptions stated here.

A measurable map $S: X \to G$ is called a section map on $X$ if $\pi \circ S = \text{id}_X$. By the existence of a continuous local section, one can always construct a section map on $X$. In general, it may not be continuous on $X$, but for any $x \in X$, there is a section map that is continuous on a neighborhood of $x$ (even smooth there when $G$ is a Lie group).

A measure on a lcscH space is called a Radon measure if it has a finite charge on any compact set. Note that the usual definition of Radon measures on Hausdorff spaces is called the normalized Haar measure on the compact group $K$ (applying right translations and the inverse map to a left Haar measure) that is unimodular. In this case, we will simply say Haar measures. It is easy to show (by measure on $K$) that a compact Haar measure does not agree with right Haar measures, but when they do, the group $G$ is called unimodular. In this case, we will simply say Haar measures. It is easy to show (by applying right translations and the inverse map to a left Haar measure) that a compact group $K$ is unimodular and there is a unique left invariant probability measure on $K$, which is also invariant under right translations and the inverse map on $K$. This measure is called the normalized Haar measure on the compact group $K$, and is denoted as $\rho_K$ or simply $dk$ in computation.

The convolution between two measures $\mu$ and $\nu$ on $X$ is the measure $\mu * \nu$ defined by

$$
\mu * \nu(f) = \int_{X \times X} \int_K f(S(x)ky)dk\mu(dx)\nu(dy),
$$

for any $f \in \mathcal{B}_+(X)$. This definition does not depend on the choice for the section map $S$ (because if $S'$ is another section map, then $S'(x) = S(x)s$ for some $s \in K$, and reduces to the convolution on $G$ when $K = \{e\}$. Because for any $g \in G$ and $x \in X$, $S(gx) = gS(x)k$ for some $k = k(g, x) \in K$, it is easy to show that the convolution on $X$, as on $G$, is associative, that is, $(\mu * \nu) * \gamma = \mu * (\nu * \gamma)$, so the $n$-fold convolution $\mu_1 * \mu_2 * \cdots * \mu_n$ is well defined.

Recall a measure $\mu$ on $X$ is called $K$-invariant if $k\mu = \mu$ for any $k \in K$. If $\nu$ is a $K$-invariant measure on $X$, then $\mu * \nu$ can be written a little more concisely as

$$
\mu * \nu(f) = \int_{X \times X} f(S(x)y)\mu(dx)\nu(dy)
$$

for $f \in \mathcal{B}_+(X)$, which does not depend on the choice for the section map $S$. Moreover, if $\mu$ is also $K$-invariant, then so is $\mu * \nu$.

A convolution semigroups on $X$ and its continuity are defined in the same way as on $G$ given in §3. Thus, a family of probability measures $\mu_t$ on $X$, $t \in \mathbb{R}_+$, is called a convolution semigroup on $X$ if $\mu_{t+s} = \mu_s * \mu_t$, and it is called continuous if $\mu_t \to \mu_0$ weakly as $t \to 0$. Recall that if $\mu_t$ is a continuous convolution semigroup on $G$, then
\( \mu_t \to \mu_s \) weakly as \( t \to s \) for any \( s > 0 \). The same holds for a continuous convolution semigroup \( \mu_t \) on \( X \), as an easy consequence of Proposition 12 later.

The following result provides a basic relation between invariant measures on \( G \) and on \( X = G/K \). Recall a measure \( \mu \) on \( G \) is called \( K \)-left or \( K \)-right invariant if \( k \mu = \mu \) or \( r_k \mu = \mu \) for \( k \in K \), and \( K \)-bi-invariant if it is both \( K \)-left and \( K \)-right invariant.

**Proposition 9.** The map

\[
\mu \mapsto \nu = \pi \mu
\]

is a bijection from the set of \( K \)-right invariant measures \( \mu \) on \( G \) onto the set of measures \( \nu \) on \( X \). It is also a bijection from the set of \( K \)-bi-invariant measures \( \mu \) on \( G \) onto the set of \( K \)-invariant measures \( \nu \) on \( X \). Moreover, if \( \nu \) is a measure on \( X \), then the unique \( K \)-right invariant measure \( \mu \) on \( G \) satisfying \( \nu = \pi \mu \) is given by

\[
\forall f \in B_+(G), \quad \mu(f) = \int_X \int_K f(S(x)k)dk \nu(dx),
\]

where \( S \) is any section map on \( X \). Furthermore, the map \( \mu \mapsto \pi \mu \) preserves the convolution in the sense that for any measures \( \mu_1 \) and \( \mu_2 \) on \( G \),

\[
\pi(\mu_1 * \mu_2) = (\pi \mu_1) * (\pi \mu_2),
\]

provided one of the following three conditions holds: \( \mu_1 \) is \( K \)-right invariant, or \( \mu_2 \) is \( K \)-left invariant, or \( \mu_2 \) is \( K \)-conjugate invariant.

**Proof.** For \( g \in G \), \( S \circ \pi(g) = gk \) for some \( k \in K \). If \( \mu \) is a \( K \)-right invariant measure on \( G \) with \( \pi \mu = \nu \), then for \( f \in B_+(G) \),

\[
\mu(f) = \int_K \mu(f \circ r_k)dk = \int_K \mu(f \circ r_k \circ S \circ \pi)dk = \int_K \nu(f \circ r_k \circ S)dk.
\]

This shows that \( \mu \) satisfies (15). Conversely, using the \( K \)-right invariance of \( dk \), it is easy to show that \( \mu \) given by (15) is \( K \)-right invariant with \( \pi \mu = \nu \). It is also clear that \( \mu \) is \( K \)-bi-invariant if and only if \( \nu \) is \( K \)-invariant. For two measures \( \mu_1 \) and \( \mu_2 \) on \( G \), satisfying one of the three conditions stated above, let \( \nu_1 = \pi \mu_1 \) and \( \nu_2 = \pi \mu_2 \). Then for \( f \in B_+(X) \),

\[
\pi(\mu_1 * \mu_2)(f) = \int f(\pi(g_1g_2))\mu_1(dg_1)\mu_2(dg_2)
\]

\[
= \int f(\pi(g_1k_2)) \mu_1(dg_1) \rho_k(dk) \mu_2(dg_2)
\]

\[
= \int f(g_1k \pi(g_2)) \mu_1(dg_1) \rho_k(dk) \mu_2(dg_2) = \int f(gk \pi(g_2)) \mu_1(dg) \rho_k(dk) \nu_2(dy)
\]

\[
= \int f(S(\pi(g))k'ky) \mu_1(dg) \rho_k(dk) \nu_2(dy) \quad \text{for some } k' \in K
\]

\[
= \int f(S(x)ky) \nu_1(dx) \rho_k(dk) \nu_2(dy) = \nu_1 * \nu_2(f).
\]

\[ \square \]

**Proposition 10.** Let \( \rho_G \) be a left Haar measure on \( G \).

(a) \( \rho_G \) is \( K \)-right invariant.

(b) \( \rho_X = \pi \rho_G \) is a \( G \)-invariant measure on \( X = G/K \), and any \( G \)-invariant measure on \( X \) is \( c \rho_X \) for some constant \( c \geq 0 \). In particular, if \( G \) is compact and if \( \rho_G \) is the normalized Haar measure on \( G \), then \( \rho_X \) is the unique \( G \)-invariant probability measure on \( X \).
Proof. For $k \in K$, $r_k \rho_G$ is left invariant and so $r_k \rho_G = \lambda(k) \rho_G$ for some $\lambda(k) > 0$. For $f \in C_c(G)$, $\rho_G(f \circ r_k) = \lambda(k) \rho_G(f)$. This shows $k \mapsto \lambda(k)$ is continuous. Then $k \mapsto \lambda(k^{-1})$ is a continuous group homomorphism from $K$ into the multiplicative group $(0, \infty)$. Its range as a compact subgroup is necessarily $\{1\}$, and hence $r_k \rho_G = \rho_G$. This proves (a). It is clear that $\rho_X = \pi \rho_G$ is $G$-invariant. For any $G$-invariant measure $\nu$ on $X$, the unique $K$-right invariant measure $\mu$ on $G$ with $\pi \mu = \nu$, given by (15), is clearly left invariant, and so $\mu = c \rho_G$ for some constant $c \geq 0$. This implies $\nu = c \rho_X$ and proves (b). \hfill \square

In the literature (for example, in [14]), the convolution of measures in $X = G/K$ has been defined by identifying the measures on $X$ with the $K$-right invariant measures in $G$ and then use the convolution on $G$. By Proposition 9, this definition is consistent with our definition given in (13).

Convolution of functions on $X = G/K$ have appeared in literature under various contexts, but in the present setting, they all take the following form (see for example [41]): For $f_1, f_2 \in \mathcal{B}_+(X)$, 

$$ f_1 * f_2(gK) = \int_G f_1(hK) f_2(h^{-1}gK) \rho(\text{dh}), \quad g \in G, \quad (17) $$

where $\rho$ denotes a left Haar measure on $G$. This definition is compatible with our definition of convolution of measures on $X$ by the following proposition.

Proposition 11. If $\mu_1$ and $\mu_2$ are measures on $X = G/K$ with densities $f_1$ and $f_2$ with respect to $\pi \rho$, then $\mu_1 \ast \mu_2$ has density $f_1 \ast f_2$.

Proof. By Proposition 10(a), $\rho$ is $K$-right invariant. For $f \in \mathcal{B}_+(X)$, writing $dg$ and $dh$ for $\rho(dg)$ and $\rho(dh)$ respectively,

$$ \mu_1 \ast \mu_2(f) = \int f(S(x)ky)d\mu(dx)\ast(dy) = \int f(S(hK)kgK)dk f_1(hK) f_2(gK)dhdg = \int f(gK) f_1(hK) f_2(gK)dhdg = \int f(gK)(f_1 \ast f_2)(gK)dg. $$

A probability measure $\mu$ on $G$ satisfying $\mu \ast \mu = \mu$ is called an idempotent. By [18, Theorem 1.2.10], if $\mu$ is an idempotent, then $\mu = \rho_H$ for some compact subgroup $H$ of $G$. For a convolution semigroup $\mu_t$ on $G$, $\mu_{t0} \ast \mu_0 = \mu_0$, so $\mu_0 = \rho_H$ for some compact subgroup $H$ of $G$. Then $\mu_t$ is $H$-bi-invariant because $\mu_t = \mu_0 \ast \mu_0 = \mu_t \ast \mu_0$.

Proposition 12. (a) If a convolution semigroup $\mu_t$ on $G$ is $K$-right invariant, that is, if each $\mu_t$ is $K$-right invariant, then it is $K$-bi-invariant.

(b) If $\nu_t$ is a convolution semigroup on $X = G/K$, then each $\nu_t$ is $K$-invariant, and $\nu_0 = \pi \rho_H$ for some compact subgroup $H$ of $G$ containing $K$.

(c) The map $\mu_t \mapsto \nu_t = \pi \mu_t$

is a bijection from the set of $K$-bi-invariant convolution semigroups $\mu_t$ on $G$ onto the set of convolution semigroups $\nu_t$ on $X = G/K$. Moreover, $\mu_t$ is continuous if and only if so is $\nu_t$.

Proof. By the preceding discussion, $\mu_0 = \rho_H$ and $\mu_t$ is $H$-bi-invariant for some compact subgroup $H$ of $G$. The $K$-right invariance of $\mu_0$ implies $K \subset H$. This proves (a). Let $\nu_t$ be a convolution semigroup on $X$, and let $\mu_t$ be the unique $K$-right invariant
probability measure on $G$ with $\pi \mu_t = \nu_t$. By Proposition 9 and (a), $\mu_t$ is a $K$-bi-invariant convolution semigroup on $G$, and hence $\nu_t = \pi \mu_t$ is $K$-invariant. Moreover, $\mu_0 = \rho_H$ for some compact subgroup $H$ of $G$. Because $\mu_0$ is $K$-bi-invariant, $K \subset H$. This proves (b). Now (c) follows from (b) and Proposition 9. Note that to derive the continuity of $\mu_t$ from that of $\nu_t$, (15) is used, where $\int_K f(S(x)k)\rho_K(dk)$ is continuous in $x$ if $f$ is bounded continuous on $G$. This is because the integral does not depend on $S$, and $S$ may be chosen to be continuous near any $x$ (that is, in a neighborhood of $x$).

Because a left invariant rcll Markov process in $G$ with an infinite life time is a Lévy process in $G$, a $G$-invariant rcll Markov process $x_t$ in $X = G/K$ with an infinite life time will be called a Lévy process in $X$. Let $P_t$ be its transition semigroup. By the $G$-invariance of $P_t$, it is clear that $\mu_t = P_t(o, \cdot)$, $t \in \mathbb{R}_+$, are $K$-invariant measures. We now show that they form a continuous convolution semigroup on $G$ with $\mu_0 = \delta_o$, which will be called the convolution semigroup associated to the Lévy process $x_t$. For $f \in \mathcal{B}_b(X)$ and $s, t \in \mathbb{R}_+$,

$$\mu_{s+t}(f) = P_{s+t}f(o) = P_sP_tf(o) = \int_X \mu_s(dx)P_t f(x) = \int_X \mu_s(dx)P_t f(S(x)o)$$

$$= \int_X \mu_s(dx)P_t(f \circ S(x))(o) \quad \text{(by the $G$-invariance of $P_t$)}$$

$$= \int_{X \times X} \mu_s(dx)\mu_t(dy)f(S(x)y) = \mu_s * \mu_t(f).$$

This shows that $\mu_t = P_t(o, \cdot)$, $t \in \mathbb{R}_+$, form a convolution semigroup on $X$, which is continuous by the right continuity of $x_t$.

On the other hand, given a continuous convolution semigroup $\mu_t$ on $X$ with $\mu_0 = \delta_o$, let

$$P_t f(x) = \int f(S(x)y)\mu_t(dy)$$

for $t \in \mathbb{R}_+$, $x \in X$ and $f \in \mathcal{B}_b(X)$. Because $\mu_t$ is $K$-invariant, this expression does not depend on the choice for the section map $S$, and defines a conservative $G$-invariant Feller transition semigroup $P_t$ on $X$ (recall that $S$ may be chosen to be continuous near any point in $X$). The associated Feller process $x_t$ is a $G$-invariant rcll Markov process with an infinite life time, and hence is a Lévy process in $X$ with $\mu_t$ as the associated convolution semigroup.

For a $G$-invariant rcll Markov process $x_t$ in $X$ with a possibly finite life time, it can be shown as on $G$ that its transition semigroup $P_t$ satisfies $P_t 1(x) = e^{-\lambda t}$ for some $\lambda \geq 0$, and hence $x_t$ is equal in distribution to a Lévy process killed at an independent exponential time of rate $\lambda$. To summarize, we have the following result.

**Theorem 13.** Let $x_t$ be a Lévy process in $X = G/K$, that is, a $G$-invariant rcll Markov process with an infinite life time, and let $P_t$ be its transition semigroup. Then $\mu_t = P_t(o, \cdot)$ is a continuous convolution semigroup on $X$ with $\mu_0 = \delta_o$. Conversely, if $\mu_t$ is such a convolution semigroup on $X$, then there is a Lévy process $x_t$ in $X$ with $x_0 = o$ such that $\mu_t$ is the distribution of $x_t$.

In general, a $G$-invariant rcll Markov process $x_t$ in $X$ with a possibly finite life time is identical in distribution to a Lévy process $\tilde{x}_t$ in $X$ killed at an independent exponential time of rate $\lambda \geq 0$. The transition semigroup $P_t$ of $x_t$ and $\tilde{P}_t$ of $\tilde{x}_t$ are related as $P_t = e^{-\lambda t} \tilde{P}_t$.
Although there is no natural product structure on the homogeneous space \( X = G/K \), the integrals like
\[
\int f(xy)\mu(dy) = \int f(S(x)y)\mu(dy),
\]
(19) \[
\int f(xy,yz)\mu(dy)\nu(dz) = \int f(S(x)y,S'(S(x)y)z)\mu(dy)\nu(dz)
\]
are well defined for \( K \)-invariant measures \( \mu \) and \( \nu \) on \( X \), that is, they do not depend on the choice for section maps \( S \) and \( S' \), noting that \( S'(S(x)y) = S(x)S'(y)k_{x,y} \) for some \( k_{x,y} \in K \). In this notation, the formula (12) for the finite dimensional distributions of a Lévy process in \( G \) holds also for a Lévy process \( x_t \) in \( X = G/K \). This follows from the Markov property of \( x_t \) and (18).

The following result says that a Lévy process in \( X \) may also be characterized by independent and stationary increments, just like a Lévy process in \( G \).

**Theorem 14.** Let \( x_t \) be a rcll process in \( X = G/K \) with an infinite life time and let \( \{ F^s_t \} \) be its natural filtration. If \( x_t \) is a Lévy process with associated convolution semigroup \( \mu_t \), then for any section map \( S \) and \( s < t \), \( x^{-1}_s x_t = S(x_s)^{-1}x_t \) is independent of \( F^s_t \) and has distribution \( \mu_{t-s} \). Consequently, this distribution is \( K \)-invariant, and depends only on \( t-s \) and not on the choice of \( S \). Conversely, if for some section map \( S \) and any \( s < t \), \( x^{-1}_s x_t = S(x_s)^{-1}x_t \) is independent of \( F^s_t \), and its distribution is \( K \)-invariant and depends only on \( t-s \), then \( x_t \) is a Lévy process in \( X \).

**Proof.** Let \( x_t \) be a Lévy process in \( X \) with associated convolution semigroup \( \mu_t \). Then from its finite dimensional distributions given in (12), it can be shown that \( x^{-1}_s x_t = S(x_s)^{-1}x_t \) is independent of \( F^s_t \) and its distribution is \( \mu_{t-s} \). For simplicity, we will prove that for \( r < s < t \), \( x^{-1}_r x_t \) is independent of \( x_r \) and has distribution \( \mu_{t-s} \). Fix an arbitrary section map \( S \). We will write \( x_1 x_2 \cdots x_{k-1} x_k \) for \( S(x_1)S(x_2) \cdots S(x_{k-1})x_k \), and \( x^{-1} \) for \( S(x)^{-1} \). By (12), for \( f, g \in B_+(X) \),
\[
E[f(x_r)g(x^{-1}_s x_t)] = \int \mu_0(dx_0)\mu_1(dx_1)\mu_{s-r}(dx_2)\mu_{t-s}(dx_3) f(x_0 x_1) g((x_0 x_1 x_2)^{-1} x_0 x_1 x_2 x_3) \]
\[
= \int \mu_0(dx_0)\mu_1(dx_1)\mu_{s-r}(dx_2)\mu_{t-s}(dx_3) f(x_0 x_1) g(x_3) \]
\[
= E[f(x_r)]E[g(x_{t-s})].
\]
This shows that \( S(x_s)^{-1}x_t \) is independent of \( x_r \) and has the distribution \( \mu_{t-s} \). There is no difficulty in this proof to replace \( x_r \) by \( (x_{r_1}, x_{r_2}, \ldots, x_{r_k}) \) for \( r_1 < r_2 < \cdots < r_k \leq s \), except the expressions will be much longer. Conversely, assume for some section map \( S \) and any \( s < t \), \( x^{-1}_s x_t = S(x_s)^{-1}x_t \) is independent of \( F^s_t \), and its distribution, denoted as \( \mu_{s,t} \), is \( K \)-invariant and depends only on \( t-s \). Then for \( f \in B_+(X) \),
\[
E[f(x_t) | F^s_t] = E[f(S(x_s)S(x_s)^{-1}x_t) | F^s_t] = \int_X f(S(x_s)y)\mu_{s,t}(dy)
\]
\[
= \int_X f(S(x_s)y)\mu_{0,t-s}(dy).
\]
Because \( \mu_{s,t} \) is \( K \)-invariant, by (14), \( \mu_{s,r} \ast \mu_{s,t} = \mu_{r,t} \) for \( r < s < t \). Because \( \mu_{s,t} \) depends only on \( t-s \), it then is easy to show that \( \mu_t = \mu_{0,t} \) is a continuous convolution semigroup on \( G \) and \( x_t \) is a \( G \)-invariant Markov process in \( X \) with associated convolution semigroup \( \mu_t \), and hence \( x_t \) is a Lévy process in \( X \).
We note that by Theorem 14, for a Lévy process \( x_t \) in \( X \) and a section map \( S \) on \( X \),
\[
(20) \quad x_t^0 = S(x_0)^{-1}x_t
\]
is a Lévy process in \( X \) with \( x_0^0 = 0 \) and is independent of \( x_0 \), and its distribution does not depend on the choice for \( S \). Moreover, the process \( x_t \) is equal in distribution to \( S(x_0)x_t^0 \).

**Remark 15.** Let \( x_t \) be a \( G \)-invariant Markov process in \( X = G/K \) with transition semigroup \( P_t \) and let \( \mu_t = P_t(\cdot, \cdot) \). Then \( P_t f(x) = P_t(f \circ S(x))(e) = \int_G f(S(x)y)\mu_t(dy) \). If \( \mu_t \to \delta_o \) weakly as \( t \to 0 \), then \( P_t \) is Feller and hence \( x_t \) is a rcll version. In this case, if \( x_t \) has an infinite life time, then this version of \( x_t \) is a Lévy process, and \( \mu_t \) is a continuous convolution semigroup on \( X \).

The following three results deal with some simple relations between processes in \( G \) and in \( X = G/K \).

**Proposition 16.** Let \( g_t \) be a Markov process in \( G \). Assume its transition semigroup \( P_t \) is \( K \)-right invariant, that is, \( (P_t f) \circ r_k = P_t(f \circ r_k) \) for \( f \in B_b(G) \) and \( k \in K \). Then \( x_t = \pi(g_t) = g_t o \) is a Markov process in \( X = G/K \) with transition semigroup
\[
(21) \quad Q_t f(x) = P_t(f \circ \pi)(S(x)), \quad x \in X \text{ and } f \in B_b(X),
\]
which does not depend on the choice for the section map \( S \). Moreover, if \( g_t \) is a Feller process in \( G \), then so is \( x_t \) in \( X \).

**Proof.** The \( K \)-right invariance of \( P_t \) shows that \( Q_t \), defined by (21) does not depend on the choice for \( S \). Note that (21) may be written as \( (Q_t f) \circ \pi = P_t(f \circ \pi) \). Then
\[
Q_{s+t} f(x) = P_{s+t}(f \circ \pi)(S(x)) = P_s[P_t(f \circ \pi)(S(x)) = P_s((Q_t f) \circ \pi)(S(x)) = (Q_s Q_t f)(x).
\]
This shows that \( Q_t \) is a transition semigroup. Since \( S \) may be chosen to be continuous near any fixed point \( x \) in \( X \), it is easy to see that if \( P_t \) is a Feller transition semigroup on \( G \), then so is \( Q_t \) on \( X \). The Markov property of \( x_t \) follows from \( E[f(x_{s+t}) \mid F^x_s] = E[f \circ \pi(g_{s+t}) \mid F^x_s] = P_t(f \circ \pi)(g_s) = (Q_t f \circ \pi)(g_s) = Q_t f(x_s) \). \( \square \)

**Proposition 17.** Let \( g_t \) be a right Lévy process in \( G \) as defined in Remark 8, that is, a rcll Markov process with a right invariant transition semigroup \( P_t \) and an infinite life time. Let \( \mu_t = P_t(e, \cdot) \) be the associated convolution semigroup. Then for any \( z \in X = G/K \), \( x_t = g_t z \) is a Feller process in \( X \) with transition semigroup \( Q_t \) given by (21). Moreover,
\[
(22) \quad Q_t f(x) = \int_G f(gx)\mu_t(dy), \quad x \in X \text{ and } f \in B_b(X).
\]

**Proof.** Because \( g_t = g_t^o g_0 \) and \( g_t z = g_t^o g_0 z \), replacing \( z \) by \( g_0 z \), we may assume \( g_0 = e \). If the reference point \( o \) in \( X \) is replaced by \( z \), then the natural projection \( \pi: G \to X \) given by \( g \mapsto g \) should be replaced by \( \pi_z: G \to X \) given by \( g \mapsto g z \). If \( S \) is a section map on \( X \) with respect to \( o \), then \( S_z(\cdot) = S(\cdot) S(z)^{-1} \) is a section with respect to \( z \) because \( S_z(x)z = S(x)S(z)^{-1} z = S(x)o = x \) for any \( x \in X \). Because \( P_t \) is right invariant, by Proposition 16, \( x_t = g_t z \) is a Feller process in \( X \) with transition semigroup
\[
Q_t f(x) = P_t(f \circ \pi_z)(S(x)S(z)^{-1}) = P_t(f \circ \pi_z \circ r_{S(x)S(z)^{-1}}(e)) \]
\[
= \int_G P_t(e, dx) f(gS(x)S(z)^{-1} z) = \int_G \mu_t(dy) f(gS(x)o) = \int_G \mu_t(dy) f(gx).
\]
This proves (22). From above, \( Q_t f(x) = \int_G \mu_t(dy) f(gS(x)o) = \int_G P_t(e, dy) f(gS(x)o) = \int_G P_t(S(x), dy) f(go) = P_t(f \circ \pi)(S(x)). \) This is (21). \( \square \)
Recall that for \( g \in G \), \( c_g : G \to G \) is the conjugation map \( x \mapsto gxg^{-1} \). A Lévy process \( g_t \) in \( G \) is called \( K \)-conjugate invariant if its transition semigroup is \( K \)-conjugate invariant, that is, if \( P_t(f \circ c_k) = (P_t f) \circ c_k \) for \( f \in \mathcal{B}_b(G) \) and \( k \in K \). By the left invariance of \( P_t \), this is equivalent to the \( K \)-right invariance of \( P_t \). In terms of the process, this is the same as saying that for any \( t \in \mathbb{R}_+ \) and \( k \in K \), \( kg_tk^{-1} = g_t \) in distribution.

**Theorem 18.** Let \( g_t \) be a \( K \)-conjugate invariant Lévy process in \( G \). Then \( x_t = g_t o \) is a Lévy process in \( X = G/K \) with transition semigroup \( Q_t \) given by (21).

**Note:** It can be shown that any Lévy process in \( X = G/K \) may be obtained from an \( K \)-conjugate invariant Lévy process in \( G \) as in Theorem 18 when \( G \) is a Lie group, see [25, Theorem 2.2].

**Proof of Theorem 18.** By Proposition 16, \( x_t = g_t o \) is a rcll Markov process in \( X \) with transition semigroup \( Q_t \) and infinite life time. It remains to show that \( Q_t \) is \( G \)-invariant. This follows from the left invariance and \( K \)-right invariance of \( P_t \), because for \( f \in \mathcal{B}_b(X) \) and \( g \in G \), \( Q_t(f \circ g)(x) = P_t(f \circ g)(S(x)) = P_t(f \circ \pi \circ l_g)(S(x)) = P_t(f \circ \pi)(gS(x)) = P_t(f \circ \pi)(S(gx)k) \) (for some \( k \in K \)) = \( P_t(f \circ \pi \circ r_k)(S(gx)) = P_t(f \circ \pi)(S(gx)) = Q_t f(gx) \).

\[ \square \]

5. **Inhomogeneous Lévy processes**

As before, let \( G \) be a topological group equipped with a lsc\( H \) topology. An inhomogeneous Markov process \( x_t \) in \( G \) with transition semigroup \( P_{s,t} \) is called left invariant if it is invariant under the action of left translations, that is, if \( P_{s,t}(f \circ l_g) = (P_{s,t} f) \circ l_g \) for \( g \in G \) and \( f \in \mathcal{B}_b(G) \).

Assume \( x_t \) has an infinite life time. By the simple Markov property and the left invariance, for \( s < t \) and \( f \in \mathcal{B}_b(G) \),

\[ E[f(x_s^{-1}x_t) \mid \mathcal{F}_s^x] = P_{s,t}(f \circ l_t^{-1})(x_s) = P_{s,t} f(e). \]

This shows that \( x_s^{-1}x_t \) is independent of \( \mathcal{F}_s^x \) and has the distribution \( P_{s,t}(e, \cdot) \). Thus, like a left invariant homogeneous Markov process considered in \( \S 3 \), \( x_t \) has independent increments. However, these increments are in general not stationary.

A rcll process \( x_t \) in \( G \), with an infinite life time and independent increments, will be called an inhomogeneous Lévy process. In contrast, the Lévy processes defined in \( \S 3 \) may be called homogeneous. Note that the class of inhomogeneous Lévy processes includes homogeneous ones as special cases. As for homogeneous Lévy processes, it is easy to show that an inhomogeneous Lévy process \( x_t \) is a left invariant inhomogeneous Markov process. To summarize, we have the following result.

**Theorem 19.** The class of left invariant inhomogeneous Markov processes in \( G \), that have rcll paths and infinite life times, coincides with the class of inhomogeneous Lévy processes in \( G \). For such a process \( x_t \), with transition semigroup \( P_{s,t} \), \( P_{s,t}(e, \cdot) \) is the distribution of the increment \( x_s^{-1}x_t, 0 \leq s \leq t \).

A family of probability measures \( \mu_{s,t} \) on \( G \), \( 0 \leq s \leq t \), is called a two-parameter convolution semigroup on \( G \) if \( \mu_{r,s} = \mu_{r,s} * \mu_{s,t} \) for \( r \leq s \leq t \). It is called right continuous if \( \mu_{s,t} \to \mu_{u,v} \) weakly as \( s \downarrow u \) and \( t \downarrow v \). It is called continuous if \( \mu_{s,t} \) is continuous in \( (s, t) \) under the weak convergence. Note that if \( \mu_t \) is a continuous convolution semigroup, then \( \mu_{s,t} = \mu_{t-s} \) is a continuous two-parameter convolution semigroup. For simplicity, a two-parameter convolution semigroup may also be called a convolution semigroup.

It is easy to see that if \( x_t \) is an inhomogeneous Lévy process in \( G \), then by the rcll paths, the distributions \( \mu_{s,t} \) of its increments \( x_s^{-1}x_t, s \leq t \), form a right continuous convolution semigroup with \( \mu_{t,t} = \delta_e \) for all \( t \geq 0 \), which is called the convolution
semigroup associated to process \( x_t \). Moreover, \( \mu_{s,t} \) is continuous if and only if \( x_t \) is stochastically continuous.

Now let \( \mu_{s,t} \) be a continuous (two-parameter) convolution semigroup on \( G \) with \( \mu_{t,t} = \delta_e \). For \( u, t \in \mathbb{R}_+ \), \( x \in G \) and \( f \in C_c(\mathbb{R}_+ \times G) \), let

\[
R_tf(u,x) = \int_G f(u + t, xy)\mu_{u,u+t}(dy).
\]

It is easy to show that this defines a conservative Feller transition semigroup \( R_t \) on the product space \( \mathbb{R}_+ \times G \). Let \( z_t \) be the associated Feller process in \( \mathbb{R}_+ \times G \). Then \( z_t \) has an infinite life time, and given \( z_0 = (u,x) \), \( z_t = (u + t, y_t) \) is such that for \( f \in C_c(G) \),

\[
E[(y_t)] = \int_G f(xy)\mu_{u,u+t}(dy).
\]

Then \( x_t \) is a left invariant inhomogeneous Markov process with an infinite life time, and its transition semigroup \( P_{s,t} \) is given by \( P_{s,t}(\epsilon, \cdot) = \mu_{s,t} \), and hence \( x_t \) is an inhomogeneous Lévy process associated to the convolution semigroup \( \mu_{s,t} \). As a component of a Feller process, \( x_t \) is stochastically continuous.

To summarize, we have established the following result.

**Theorem 20.** Let \( \mu_{s,t}, 0 \leq s \leq t \), be a two-parameter convolution semigroup on \( G \) with \( \mu_{t,t} = \delta_e \). Then \( \mu_{s,t} \) is associated to a stochastically continuous inhomogeneous Lévy process in \( G \) if and only if \( \mu_{s,t} \) is continuous.

By the Markov property, it can be shown that the finite dimensional distributions of an inhomogeneous Lévy process \( x_t \) in \( G \) with initial distribution \( \mu_0 \) and associated convolution semigroup \( \mu_{s,t} \) are given by

\[
E[f(x_{t_1}, x_{t_2}, \ldots, x_{t_n})] = \int f(x_{0}x_{1}, x_{0}x_{1}x_{2}, \ldots, x_{0}x_{1}x_{2} \ldots x_{n})\mu_0(dx_0)\mu_{t_1,t_2}(dx_1)\mu_{t_2,t_3}(dx_2)\ldots\mu_{t_{n-1},t_n}(dx_n)
\]

for \( 0 \leq t_1 < t_2 < \cdots < t_n \) and \( f \in B_b(G^n) \).

**Remark 21.** Let \( x_t \) be a process in \( G \) with an infinite life time. By the proof of Theorem 19, \( x_t \) is a left invariant inhomogeneous Markov process if and only if it has independent increments. In this case, if for any \( s \geq 0 \), \( x_{s}^{-1}x_t \to e \) in distribution as \( t \to s \), then the distributions \( \mu_{s,t} \) of \( x_{s}^{-1}x_t \), \( s \leq t \), form a continuous convolution semigroup with \( \mu_{t,t} = \delta_e \) for \( t \geq 0 \). It follows that the transition semigroup \( R_t \) of the process \( z_t = (u + t, x_{u+t}) \), given in (23), is Feller, and hence \( x_t \) has a rcll version that is an inhomogeneous Lévy process in \( G \).

More generally, let \( x_t \) be a left invariant inhomogeneous Markov process in \( G \) without assuming an infinite life time, and let \( P_{s,t} \) be its transition semigroup. Then \( P_{s,t}f(x) = P_{s,t}(f \circ \alpha_t)(e) = \int_G f(xy)\mu_{s,t}(dy) \), where \( \mu_{s,t} = P_{s,t}(\epsilon, \cdot) \). Assume \( \mu_{s,t} \) is continuous in \((s, t)\) under the weak convergence. Although \( \mu_{s,t} \) is a sub-probability, \( R_t \) defined by (23) is still a Feller transition semigroup on \( \mathbb{R}_+ \times G \), and hence \( x_t \) has a rcll version.

Recall that a Radon measure has a finite charge on any compact set. A measure is called diffuse if it does not charge points. The reader is referred to [21, Chapter 12] for the standard definition of a Poisson random measure.

**Theorem 22.** Let \( x_t \) be a left invariant inhomogeneous Markov process in \( G \) with a possibly finite life time and transition semigroup \( P_{s,t} \). Assume \( x_t \) has rcll paths and is stochastically continuous. Then there are a diffuse Radon measure \( \lambda \) on \( \mathbb{R}_+ \) and a stochastically continuous inhomogeneous Lévy process \( \hat{x}_t \) in \( G \) with transition semigroup \( \hat{P}_{s,t} \) such that

\[
P_{s,t} = e^{-\lambda(s,t)}\hat{P}_{s,t}.
\]
This means that process \( x_t \) is equal in distribution to \( \hat{x}_t \) killed at the random time point of an independent Poisson random measure \( \xi \) on \( \mathbb{R}_+ \) with intensity measure \( \lambda \). More precisely, if for any starting time \( s \), let \( \tau \) be the first point of \( \xi \) on \( (s, \infty) \), and let \( x'_t = \hat{x}_t \) for \( s \leq t < \tau \) and \( x'_t = \Delta \) for \( t \geq \tau \), then \( x'_t \) is an inhomogeneous Markov process with transition semigroup \( P_{s,t} \).

Proof. To prove (25), note that by the left invariance and the semigroup property of transition semigroup, \( P_{s,t} 1 = P_{s,t} 1(e) \) and for \( r < s < t \), \( P_{r,s} 1(e) = \int P_{r,t}(e, dx) P_{s,t} 1(x) = P_{r,s} 1(e) P_{s,t} 1(e) \). By the right continuity of \( P_{s,t} 1(e) \) in \( t \), it follows that there is a Radon measure \( \lambda \) on \( \mathbb{R}_+ \) such that \( P_{s,t} 1(e) = e^{-\lambda((s,t])} \). Let \( \hat{P}_{s,t} = e^{\lambda((s,t])} P_{s,t} \). Then \( P_{s,t} \) is a conservative two-parameter transition semigroup and \( \hat{\mu}_{s,t} = \hat{P}_{s,t}(e, \cdot) \) is a two-parameter convolution semigroup of probability measures on \( G \). By the stochastic continuity of \( x_t \), \( \lambda \) is a diffuse measure and hence \( \hat{\mu}_{s,t} \) is continuous. By Theorem 20, there is a stochastically continuous inhomogeneous Lévy process \( \hat{x}_t \) in \( G \) associated to \( \hat{\mu}_{s,t} \). This proves (25). \( \square \)

As for a homogeneous Lévy process, we define an inhomogeneous Lévy process \( x_t \) to be associated to a filtration \( \{ \mathcal{F}_t \} \) if \( x_t \) is adapted to \( \{ \mathcal{F}_t \} \) and for \( s < t \), \( x_{t-1}^t \) is independent of \( \mathcal{F}_s \). This is equivalent to saying that \( x_t \) is associated to the filtration \( \{ \mathcal{F}_t \} \) as an inhomogeneous Markov process.

Let \( x_t \) be an inhomogeneous Lévy process \( x_t \) in \( G \) with associated convolution semigroup \( \mu_{s,t} \). If it is associated to a filtration \( \{ \mathcal{F}_t \} \), then for fixed \( r \in \mathbb{R}_+ \), the process \( x'_t = x^{-1}_t x_{t+1}^t \), \( t \in \mathbb{R}_+ \), is an inhomogeneous Lévy process with associated convolution semigroup \( \nu_{s,t} = \mu_{r_t,s,t+\tau} \), and is independent of \( \mathcal{F}_r \). In a certain sense, this holds when \( r \) is replaced by a stopping time. The following result is an inhomogeneous analog of Theorem 6.

**Theorem 23.** Let \( x_t \) be an inhomogeneous Lévy process in \( G \) with associated convolution semigroup \( \mu_{s,t} \). Assume it is associated to a filtration \( \{ \mathcal{F}_t \} \). If \( \tau \) is an \( \{ \mathcal{F}_t \} \)-stopping time, then for \( t_1 < t_2 < \cdots < t_n \) and \( f \in \mathcal{B}_b(G^n) \),

\[
E[f(x^{-1}_{t_1} x_{\tau+t_1}, x^{-1}_{t_2} x_{\tau+t_2}, \ldots, x^{-1}_{t_n} x_{\tau+t_n}) 1_{\tau < \infty} \mid \mathcal{F}_\tau] = \int f(x_1, x_1 x_2, \ldots, x_1 x_2 \cdots x_n) \mu_{\tau, \tau+t_1}(dx_1) \mu_{\tau+t_1, \tau+t_2}(dx_2) \cdots \mu_{\tau+t_n, \tau+t_n}(dx_n) 1_{\tau < \infty}.
\]

This implies that under the conditional distribution given \( \tau \), \( x'_t = x^{-1}_t x_{\tau+t} \) is an inhomogeneous Lévy process in \( G \) with associated convolution semigroup \( \nu_{s,t} = \mu_{\tau+s, \tau+t} \), and is independent of \( \mathcal{F}_\tau \). More precisely, this means that for any bounded \( \mathcal{F}_\tau \)-measurable \( H \),

\[
E[H f(x^{-1}_{t_1} x_{\tau+t_1}, x^{-1}_{t_2} x_{\tau+t_2}, \ldots, x^{-1}_{t_n} x_{\tau+t_n}) 1_{\tau < \infty} \mid \sigma(\tau)] = E[H \mid \sigma(\tau)] \int f(x_1, \ldots, x_1 x_2 \cdots x_n) \mu_{\tau, \tau+t_1}(dx_1) \cdots \mu_{\tau+t_n, \tau+t_n}(dx_n) 1_{\tau < \infty}.
\]
Proof. We may assume $f \in C_b(G^n)$. As in the proof of Theorem 6, first assume $\tau$ has discrete values. Then
\[
E[f(x^{-1}_r x^{-1}_{r+t_1} x^{-1}_{r+t_2} \cdots x^{-1}_{r+t_n}) 1_{[\tau<\infty]} \mid F_\tau] = \sum_{t<\infty} E[f(x^{-1}_r x^{-1}_{r+t_1} x^{-1}_{r+t_2} \cdots x^{-1}_{r+t_n}) 1_{[\tau=t]} | F_t] = \sum_{t<\infty} \int f(x_1, x_1 x_2, \ldots, x_1 x_2 \cdots x_n) \\
\times \{ \mu_{t,t+t_1}(dx_1) \mu_{t+t_1,t+t_2}(dx_2) \cdots \mu_{t+t_{n-1},t+t_n}(dx_n) \} 1_{[\tau=t]} = \int f(x_1, x_1 x_2, \ldots, x_1 x_2 \cdots x_n) \\
\times \{ \mu_{\tau,\tau+t_1}(dx_1) \mu_{\tau+t_1,\tau+t_2}(dx_2) \cdots \mu_{\tau+t_{n-1},\tau+t_n}(dx_n) \} 1_{[\tau<\infty]}.
\]

For a general stopping time $\tau$, choose discrete stopping times $\tau_m \downarrow \tau$. The result follows after taking the limit of the above expression with $\tau = \tau_m$ as $m \to \infty$ and using the right continuity of $\mu_{s,t}$.

The basic theory of inhomogeneous Lévy processes in a homogeneous space $X = G/K$ may be developed parallel to the case of Lévy processes in $G/K$ discussed in §4 with suitable modifications, assuming $K$ is compact and $G/K$ has a continuous local section. Thus, a $G$-invariant inhomogeneous Markov process $x_t$ in $X$, that has rcll paths and an infinite life time, will be called an inhomogeneous Lévy process in $X$. Let $P_{s,t}$ be its transition semigroup and let $\mu_{s,t} = P_{s,t}(\cdot, \cdot)$. Then $\mu_{s,t}$ are $K$-invariant, and by the Markov property and the $G$-invariance of $P_{s,t}$, it can be shown that the finite dimensional distributions of $x_t$ are given in (24) if the integral like $\int f(x,y,z)\mu(dy)v(dz)$ is understood in the sense of (19) with a choice of a section map $S$ (but not dependent on the choice).

By (24), it can be shown that an inhomogeneous Lévy process in $X$ can be characterized by independent increments just like an inhomogeneous Lévy process in $G$. This is similar to the homogeneous case, see the proof of Theorem 14. The result is summarized below.

**Theorem 24.** Let $x_t$ be a rcll process in $X = G/K$ with an infinite life time and let $\{F^x_t\}$ be its natural filtration. If $x_t$ is an inhomogeneous Lévy process with associated transition semigroup $P_{s,t}$, then for any section map $S$ and $s < t$, $x^{-1}_s x_t = S(x_s)^{-1} x_t$ is independent of $F^x_s$, and its distribution is $\mu_{s,t} = P_{s,t}(\cdot, \cdot)$, so is $K$-invariant and does not depend on the choice for $S$. Conversely, if for some section map $S$ and any $s < t$, $x^{-1}_s x_t = S(x_s)^{-1} x_t$ is independent of $F^x_s$ and its distribution is $K$-invariant, then $x_t$ is an inhomogeneous Lévy process in $X$.

A (two-parameter) convolution semigroup $\mu_{s,t}$ on $X = G/K$, $0 \leq s \leq t$, and its continuity and right continuity, are defined just as on $G$. For a convolution semigroup $\mu_{s,t}$ on $G$, because for any $t$, $\mu_{t,s} * \mu_{s,t} = \mu_{t,t}$, it follows that $\mu_{t,t} = \rho_{H}$ for some compact subgroup $H$ of $G$ (which may depend on $t$). Because $\mu_{s,t} = \mu_{s,s} * \mu_{s,t} = \mu_{s,t} * \mu_{t,t}$, it follows that if each $\mu_{s,t}$ is $K$-right invariant, then each $\mu_{s,t}$ is $K$-bi-invariant. The following simple result may be derived from Proposition 9 in the same way as Proposition 12.

**Proposition 25.** (a) If $\mu_{s,t}$ is a convolution semigroup on $G$ such that each measure $\mu_{s,t}$ is $K$-right invariant, then each $\mu_{s,t}$ is $K$-bi-invariant.

(b) If $\nu_{s,t}$ is a convolution semigroup on $X = G/K$, then each $\nu_{s,t}$ is $K$-invariant.

(c) The map $\mu_{s,t} \mapsto \nu_{s,t} = \pi \mu_{s,t}$.
is a bijection from the set of $K$-bi-invariant convolution semigroups $\mu_{s,t}$ on $G$ onto the set of convolution semigroups $\nu_{s,t}$ on $X = G/K$. Moreover, $\mu_{s,t}$ is continuous (resp. right continuous) if and only if so is $\nu_{s,t}$.

For an inhomogeneous Lévy process $x_t$ in $X$, the distribution $\mu_{s,t}$ of its increments $x_t^{-1}x_s = S(x_t)^{-1}x_s$, $s \leq t$, form a right continuous convolution semigroup on $X$ with $\mu_{s,t} = \delta_0$ for all $t \geq 0$ (not dependent on the choice for the section map $S$), which is called the convolution semigroup associated to process $x_t$. The right continuity requires a short proof given below.

Because a lcscH space is paracompact, there is a partition of unity $\{\psi_j\}$ on $X$. This is a collection of functions $\psi_j \in C_c(X)$ with $0 \leq \psi_j \leq 1$ such that $\sum_j \psi_j = 1$ is a locally finite sum, that is, any point of $X$ has a neighborhood on which only finitely many terms $\psi_j$ are nonzero. We may also assume that for each $j$, there is a section map $S_j$ on $X$ that is continuous on the support of $\psi_j$. Then by (24), for $s < t$, $f \in C_b(X)$ and a section map $S$ on $X$,

$$\mu_{s,t}(f) = E[f(S(z_s)^{-1}z_t)] = \sum_j E[\psi_j(z_s)f(S(z_s)^{-1}z_t)]$$

(26)

$$= \sum_j E[\psi_j(z_s)f(S_j(z_s)^{-1}z_t)] = E[\sum_j \psi_j(z_s)f(S_j(z_s)^{-1}z_t)]$$

This shows that $\mu_{s,t}(f)$ is right continuous in $s$ and $t$.

The following two results may be proved as for Theorems 20 and 22, respectively.

**Theorem 26.** Let $\mu_{s,t}$ be a convolution semigroup on $X = G/K$ with $\mu_{s,t} = \delta_0$. Then $\mu_{s,t}$ is associated to a stochastically continuous inhomogeneous Lévy process in $X$ if and only if $\mu_{s,t}$ is continuous.

**Theorem 27.** A stochastically continuous $G$-invariant inhomogeneous Markov process $x_t$ in $X = G/K$, with rcll paths and a possibly finite life time, is identical in distribution to a stochastically continuous inhomogeneous Lévy process $\hat{x}_t$ killed at the random time point of an independent Poisson random measure on $\mathbb{R}_+$ with a Radon intensity measure $\lambda$. The transition semigroups $P_{s,t}$ of $x_t$ and $\hat{P}_{s,t}$ of $\hat{x}_t$ are related by (25).

**Remark 28.** Let $x_t$ be a $G$-invariant inhomogeneous Markov process in $X = G/K$. It can be shown as in Remark 21 that if $\mu_{s,t} = P_{s,t}(\cdot, \cdot)$ is continuous in $(s, t)$ under the weak convergence, then $x_t$ has a rcll version. If $x_t$ also has an infinite life time, then this version is an inhomogeneous Lévy process in $X$.

As for a Lévy process, an inhomogeneous Lévy process $g_t$ in $G$ is called $K$-conjugate invariant if its transition semigroup $P_{s,t}$ is $K$-conjugate invariant, that is, if $P_{s,t}(f \circ \check{k}) = (P_{s,t}f) \circ \check{k}$ for $f \in B_b(G)$ and $k \in K$. This is equivalent to saying that for any $t > s \geq 0$ and $k \in K$, $k(g_t^{-1}g_s)k^{-1} \overset{d}{=} g_t^{-1}g_s$ (equal in distribution). For $g_0 = e$, this is also equivalent to saying that for any $k \in K$, the two processes $g_t$ and $kg_t$ are equal in distribution. The following result can be easily proved as for Theorem 18, and its converse holds on a homogeneous space of a Lie group, see [27, Theorem 33].

**Theorem 29.** Let $g_t$ be a $K$-conjugate invariant inhomogeneous Lévy process in $G$. Then $x_t = g_t o$ is an inhomogeneous Lévy process in $X = G/K$. Moreover, the transition semigroups $P_{s,t}$ of $g_t$ and $Q_{s,t}$ of $x_t$ are related by $(Q_{s,t}f) \circ \pi = P_{s,t}(f \circ \pi)$ for $f \in B_b(X)$.

6. **Markov processes under a non-transitive action**

Let $X$ be a topological space under the continuous action of a topological group $G$, both are assumed to be lcscH, and let $x_t$ be a $G$-invariant rcll Markov process in $X$ with transition semigroup $P_t$. Suppose the $G$-action on $X$ is non-transitive. Then $X$ is a
collection of disjoint $G$-orbits. In this case, under some suitable regularity condition, we may obtain a decomposition of the process into two components, one transversal to the $G$-orbits and the other along an orbit, with the former preserving the Markov property and the latter preserving the $G$-invariance.

We first derive a simple result under a general setting. Let $X/G$ be the space of $G$-orbits and let $J: X \to (X/G)$ be the projection map: $x \mapsto Gx$. Equip $X/G$ with the quotient topology induced by $J$ so that $J$ is continuous and open. Note that if $G$ is compact, then $X/G$ is lscH.

**Theorem 30.** $J(x_t)$ is a rcll Markov process in $X/G$ with transition semigroup $Q_t$ given by

$$Q_t f(y) = P_t(f \circ J)(x), \quad y = J(x) \in X/G \quad \text{and} \quad f \in B_b(X/G)$$

($Q_t$ does not depend on the choice of $x$ in $Gy$ due to the $G$-invariance of $P_t$). Moreover, when $G$ is compact, if $x_t$ is a Feller process, then so is $J(x_t)$.

**Proof.** For $f \in B_b(X/G)$ and $y \in (X/G)$,

$$E[f \circ J(x_{t+s}) | F^x_t] = P_s(f \circ J)(x_t) = Q_s f(J(x_t)).$$

This proves that $J(x_t)$ is a Markov process in $X/G$ with transition semigroup $Q_t$. When $G$ is compact, if $f \in C_b(X/G)$, then $f \circ J \in C_b(X)$ and $(Q_t f) \circ J = P_t(f \circ J) \in C_b(X)$, which implies $Q_t f \in C_b(X/G)$. The Feller property of $J(x_t)$ follows from that of $x_t$. \qed

Let $Y$ be a topological subspace of $X$ that is transversal to the action of $G$ in the sense that it intersects each $G$-orbit at exactly one point, that is,

$$\forall y \in Y, \quad (Gy) \cap Y = \{y\} \quad \text{and} \quad X = \cup_{y \in Y} Gy.$$

Then $Y$ will be called a transversal subspace of $X$ (under the $G$-action). Let $J_1: X \to Y$ be the projection map $J_1(x) = y$ for $x \in Gy$. Note that $Y$ is naturally identified with the orbit space $X/G$ via the map $y \mapsto Gy$ and $J_1$ identified with $J$, but the subspace topology on $Y$ may not be the same as the quotient topology on $X/G$ induced by $J$. However, when $G$ is compact and $Y$ is closed in $X$, then the two topologies on $Y$ agree and $J_1$ is continuous.

In the rest of this section, we will assume the two topologies agree even when $G$ is not compact. Then $J_1$ is continuous. The following result is an immediate consequence of Theorem 30.

**Theorem 31.** $y_t = J_1(x_t)$ is a rcll Markov process in the transversal subspace $Y$ with transition semigroup $Q_t$ given by

$$Q_t f(y) = P_t(f \circ J_1)(y), \quad y \in Y \quad \text{and} \quad f \in B_b(Y).$$

Moreover, if $G$ is compact and if $x_t$ is a Feller process in $X$, then $y_t$ is a Feller process in $Y$.

Now assume the isotropy subgroup of $G$ at every point $y \in Y$ is the same compact subgroup $K$ of $G$. This assumption is often satisfied if the transversal subspace $Y$ is properly chosen. For example, consider $\mathbb{R}^d$ without the origin under the action of the group $O(d)$ of orthogonal transformations on $\mathbb{R}^d$. The $O(d)$-orbits are spheres centered at the origin, and any curve from the origin to infinity is transversal if the curve intersects each of these spheres only once. When the curve is straight, that is, if it is a half line, all its points will have the same isotropy subgroup of $O(d)$.

Let $Z = G/K$. For $y \in Y$ and $z \in Z$, the product $zy = gy$, where $z = gK$, is well defined. For $y \in Y$, $zy$ traces out the $G$-orbit through $y$ as $z$ varies over $Z$, and hence $Z$ may be regarded as the standard $G$-orbit. Note that map $F: (Y \times Z) \to X$ given by $(y, z) \mapsto zy$ is bijective. Because the restricted action map $G \times Y \to X, (g, y) \mapsto gy$, is...
continuous and the projection map \( G \times Y \to Y \times Z \), \((g,y) \mapsto (y,gK)\), is open, it follows that \( F \) is continuous. If \( G \) is compact, then it is easy to show that \( F \) has a continuous inverse and hence \( F: Y \times Z \to X \) is a homeomorphism.

We will now take this to be part of our assumption. By slightly changing the notation, our general setup may be stated as follows. Let \( X = Y \times Z \) be a topological product with \( Z = G/K \), where \( G \) is a topological group and \( K \) is a compact subgroup. Assume both \( Y \) and \( G \) are lscH. Let \( G \) act on \( X \) by its natural action on \( G/K \), that is, \( g(y,z) = (y,gz) \) for \( g \in G \), \( y \in Y \) and \( z \in Z \). Then \( Y \times \{o\} \) is transversal to the \( G \)-action and \( K \) is the common isotropy subgroup of \( G \) at every point of \( Y \times \{o\} \). Let

\[
J_1: X \to Y, \quad \text{and} \quad J_2: X \to Z
\]

be the natural projections \((y,z) \mapsto y \) and \((y,z) \mapsto z\), respectively.

The processes \( y_t = J_1(x_t) \) and \( z_t = J_2(x_t) \) are called respectively the radial and angular parts of process \( x_t \). By Theorem 31, the radial part \( y_t \) is a Markov process in \( Y \). As the angular process \( z_t \) lives in the standard \( G \)-orbit \( Z \) which is invariant under the \( G \)-action, it is natural to expect that it should inherit the \( G \)-invariance of \( x_t \) in some sense. Before discussing the properties of the angular process \( z_t \), we present some examples of spaces under group actions for which the topological assumptions made here are satisfied.

Example 1: Let \( X = \mathbb{R}^{n+m} = Y \times Z \) with \( Y = \mathbb{R}^m \) and \( Z = \mathbb{R}^n \). In this case, \( G = \mathbb{R}^n \) (additive group), and \( K \) consists of only the origin in \( \mathbb{R}^n \).

Example 2: Consider the action of the orthogonal group \( G = O(d) \) on \( X = \mathbb{R}^d \) \((d \geq 2)\). Any half line \( Y \) from the origin is a transversal subspace. Except the origin, all the points in \( Y \) have the same isotropy subgroup \( K \) of \( O(d) \), which may be identified with the orthogonal group \( O(d-1) \) on a \((d - 1)\)-dimensional linear subspace of \( \mathbb{R}^d \). The \( O(d)\)-orbits are spheres in \( \mathbb{R}^d \) centered at origin.

Let \( X' = \mathbb{R}^d \) without the origin. Then \( X' = Y' \times Z = (0, \infty), \) and \( Z \) is the unit sphere \( S^{d-1} = O(d)/O(d-1) \), via the map \((r,z) \mapsto rz \) from \( Y' \times Z \) to \( X' \).

Example 3: Consider the space \( X \) of \( n \times n \) real symmetric matrices \((n \geq 2)\) under the action of \( G = O(n) \) by conjugation: \((g,x) \mapsto gxg^{-1} \) for \( g \in O(n) \) and \( x \in X \). Because symmetric matrices in \( X \) with the same set of eigenvalues are \( O(n)\)-conjugate to each other, the set \( Y \) of all \( n \times n \) diagonal matrices with non-ascending diagonal elements is a transversal subspace.

Let \( Y' \) be the subset of \( Y \) consisting of diagonal matrices with strictly descending diagonals. Then all \( y \in Y' \) have the same isotropy subgroup \( K \) of \( G = O(n) \), which is the finite subgroup of \( O(n) \) consisting of diagonal matrices with \( \pm 1 \) along diagonal. Note that an element of \( Y' \) not in \( Y \) has a larger isotropy subgroup. Let \( X' \) be the subset of \( X \) consisting of symmetric matrices with distinct eigenvalues. Then \( X' \) may be identified with \( Y' \times (G/K) \) via the map \((y,gK) \mapsto gyg^{-1} \) from \( Y' \times (G/K) \) to \( X' \).

Example 4: For readers who know symmetric spaces, we mention one more example. Let \( X = G/K \) be a symmetric space of noncompact type, with a fixed Weyl chamber \( a_+ \) and the centralizer \( M \) of \( a_+ \) in \( K \), see [16]. Let \( A_+ \subset G \) be the image of \( a_+ \) under the Lie group exponential map from the Lie algebra of \( G \) to \( G \). The set \( X' \) of regular points in \( X \) is diffeomorphic to \( A_+ \times (K/M) \) under the map: \( A_+ \times (K/M) \to \ker K \), so \( X' \) under the \( K \)-action satisfies our assumption with \( Y = A_+ \) and \( Z = K/M \). A special case is the space \( X \) of \( n \times n \) real positive definite symmetric matrices of unit determinant, and \( X' \) is the space of those matrices with distinct eigenvalues. This is a subset of the space in Example 3.
It may occur as in Examples 2, 3 and 4 that not all the points of the transversal subspace \( Y \) share the same isotropy subgroup of \( G \), but by restricting to a slightly smaller open subset \( Y' \) of \( Y \) and the \( G \)-invariant open subset \( X' \) of \( X \) that is the union of \( G \)-orbits through \( Y' \), our assumptions are satisfied. In this case, if \( x_t \) is a \( G \)-invariant rcll Markov process in \( X \), then its restriction to \( X' \), defined below, is a \( G \)-invariant rcll Markov process in \( X' \).

The restriction of the process \( x_t \) to \( X' \) is the process \( \hat{x}_t \) in \( X' \) defined by
\[
\hat{x}_t = x_t \quad \text{for} \quad t < \zeta \quad \text{and} \quad \hat{x}_t = \Delta \quad \text{for} \quad t \geq \zeta,
\]
where \( \zeta = \inf\{ t \geq 0; x_t \notin X' \text{ or } x_{t^-} \notin X' \} \) (\( \inf \emptyset = \infty \) by convention) and \( \Delta \) is the point at infinity (see Appendix A.3). Then \( \hat{x}_t \) is a rcll process in \( X' \). Moreover, \( \zeta \) is a stopping time under the natural filtration \( \{ \mathcal{F}_t \} \) of process \( x_t \) because \([ \zeta = 0 ] \subseteq [ x_0 \notin X' ]\) and for any \( t > 0 \),
\[
[t < \zeta] = \bigcup_{n > 1} \{ [x_t \in X'_n] \cap [x_r \in X'_n \text{ for any } r \in \mathbb{Q} \text{ with } 0 < r < t] \} \in \mathcal{F}_t,
\]
where \( X'_n \) are open subsets of \( X \) with closures \( X'_n \) contained in \( X' \) and \( X'_n \uparrow X' \) as \( n \uparrow \infty \), and \( \mathbb{Q} \) is the set of rational numbers.

**Proposition 32.** If \( x_t \) is a \( G \)-invariant rcll Markov process in \( X \), then \( \hat{x}_t \) is a \( G \)-invariant rcll Markov process in \( X' \) with transition semigroup \( \hat{P}_t \) given by
\[
\hat{P}_t f(x) = E_x[f(x_t); t < \zeta] \quad \text{for} \quad f \in \mathcal{B}_+(X') \quad \text{and} \quad x \in X'.
\]

**Proof.** Let \( \theta_t \) be the time shift (see Appendix A.3). For \( s < t \), \( 1_{[t < \zeta]} \cdot 1_{[s < \zeta]}(1_{[t - s < \zeta]} \circ \theta_s) \)
and
\[
E_x[f(\hat{x}_t) | \mathcal{F}_s] = E_x[f(x_t)1_{[t < \zeta]} | \mathcal{F}_s] = E_x\{[f(x_{t-s})1_{[t-s < \zeta]}] \circ \theta_s | \mathcal{F}_s\}1_{[s < \zeta]} = E_{x_s}[f(x_{t-s})1_{[t-s < \zeta]}]1_{[s < \zeta]} = \hat{P}_{t-s} f(\hat{x}_s),
\]
under the convention that any function on \( X \) or \( X' \) vanishes at \( \Delta \). This shows that \( \hat{x}_t \) is a Markov process in \( X' \) with transition semigroup \( \hat{P}_t \). By Proposition 1, the \( G \)-invariance of process \( x_t \) implies the \( G \)-invariance of \( \hat{P}_t \).

**Remark 33.** We may also define the restriction of process \( x_t \) in \( X' \) by stopping \( x_t \) at the first time \( \tau \) when it exits \( X' \), where \( \tau = \inf\{ t > 0; x_t \notin X' \} \). Thus, let \( x'_\tau \) be defined by (31) with \( \hat{x}_t \) and \( \zeta \) replaced by \( x'_\tau \) and \( \tau \). However, \( \tau \) in general is not a stopping time under \( \mathcal{F}_t \) and \( x'_\tau \) may not be rcll in \( X' \). By the standard stochastic analysis, \( \tau \) is a stopping times under the filtration \( \{ \mathcal{F}_t \} \) that is the completion of \( \{ \mathcal{F}_t \} \), and if the original process \( x_t \) is quasi-left-continuous, that is, if \( x_{\sigma^-} = \lim_{\sigma \to \tau} x_{\sigma^-}, \) almost surely on \( [\sigma < \infty] \) for any \( \{ \mathcal{F}_t \} \)-stopping times \( \sigma \uparrow \tau \), then the two processes \( x'_\tau \) and \( \hat{x}_t \) are identical almost surely. To show this, let \( X'_{\tau} \) be the open subsets of \( X' \) with \( X'_{\tau} \cup X' \) as defined earlier, and let \( \tau_{\zeta} \) be the first times when \( x_t \) exits \( X'_{\tau} \). It is easy to see that \( \tau_{\zeta} \uparrow \zeta \), and so by the quasi-left-continuity, \( x_{\zeta} = \lim_{\tau_{\zeta}} x_{\tau_{\zeta}} \) almost surely. Because \( X'_{\tau} \) is open, \( x_{\tau_{\zeta}} \notin X'_{\tau} \), and hence \( x_{\zeta} \notin X' \) almost surely. This shows \( \zeta = \tau \) almost surely. It is well known that a Feller process is quasi-left-continuous (see [21, Proposition 25.20]).

### 7. Angular part

As discussed in the previous section §6, when discussing an invariant Markov process under a non-transitive action, by suitably restricting the process, it may be possible to work under the following setup, which will be assumed throughout this section. Let \( X = Y \times Z \) be a topological product with \( Z = G/K \), where \( G \) is a topological group with identity element \( e \) and \( K \) is a compact subgroup. Assume both \( Y \) and \( G \) are lscH. Let \( G \) act on \( X \) by its natural action on \( G/K \). Then with \( o = eK, Y \times \{ o \} \) is transversal to the \( G \)-action and \( K \) is the common isotropy subgroup of \( G \) at every point of \( Y \times \{ o \} \).
We will also assume \( G/K \) has a continuous local section (as defined in §4). As in §6, let 
\[ J_1 : X \to Y \text{ and } J_2 : X \to Z \]
be the natural projections.

We mention an important case when this setup holds. When \( X \) is a manifold under the (smooth) action of a Lie group \( G \), and \( Y \) is a submanifold of \( X \) transversal to the \( G \)-action, if the tangent space of \( X \) at any \( y \in Y \) is the direct sum of the tangent space of \( Y \) at \( y \) and that of the orbit through \( y \), that is, if

\[
\forall y \in Y, \quad T_yX = T_y(Gy) \oplus T_yY \quad \text{(a direct sum)}.
\]

then by [17, II, Lemma 3.3], for any \( y \in Y \), there is a neighborhood \( U \) of \( y \) in \( Y \), and a neighborhood \( V \) of \( o = eK \) in \( Z = G/K \), where \( e \) is the identity element of \( G \) as before, such that the map \((u, v) \mapsto vu \) is a diffeomorphism from \( U \times V \) onto a neighborhood of \( y \) in \( X \). From this, it is easy to show that \( F : Y \times Z \to X \) is a diffeomorphism. The existence of a continuous local section holds automatically in the Lie group case.

Let \( x_t \) be a \( G \)-invariant Markov process in \( X \). By Theorem 31, the radial process 
\[ y_t = J_1(x_t) \]
is a Markov process in \( Y \). The purpose of this section is to study the angular process 
\[ z_t = J_2(x_t) \]
under the conditional distribution given the radial process \( y_t \). We now begin with some preparation.

Let \( (S, \rho) \) be a complete and separable metric space, and let \( D(S) \) be the space of all \( S \)-paths in \( S \), that is, the space of \( S \)-paths: \( \mathbb{R}_+ \to S \). Under the Skorohod metric, \( D(S) \) is a complete and separable metric space (see [10, chapter 3]). By Proposition 5.3(c) in [10, chapter 3], the convergence \( x_n \to x \) in \( D(S) \) means that up to a time change that is asymptotically an identity, \( x_n(t) \) converge to \( x(t) \) uniformly for bounded \( t \). Precisely, this means that for any \( T > 0 \), there are bijections \( \lambda_n : \mathbb{R}_+ \to \mathbb{R}_+ \) (necessarily continuous with \( \lambda_n(0) = 0 \)) such that \( |\lambda_n(t) - t| \to 0 \) and \( \rho(x_n(t), x(\lambda_n(t))) \to 0 \) as \( n \to \infty \) uniformly for \( t \in [0, T] \). By Proposition 7.1 in [10, chapter 3], the Borel \( \sigma \)-algebra on \( D(S) \) is generated by the coordinate maps: \( D(S) \to S \) given by \( x \mapsto x(t) \) for \( t \in \mathbb{R}_+ \). It is easy to show that the topology on \( D(S) \) induced by the Skorohod metric is determined completely by the topology of \( (S, \rho) \), and does not depend on the choice of \( \rho \).

Let \( X_\Delta = X \cup \{ \Delta \} \) be the one-point compactification of \( X \). Because \( X \) is a \( \text{lcscH} \) space, \( X_\Delta \) is metrizable (see Theorem 12.12 in [5, chapter I]), and as being compact, \( X_\Delta \) may be equipped with a complete and separable metric. Let \( D'(X) \) be the space of all \( S \)-paths in \( X \) with possibly finite life times. These are \( S \)-paths in \( X_\Delta \) such that each path \( x(\cdot) \) is associated to a life time \( \zeta \in [0, \infty] \) with \( x(t) \in X \) for \( t < \zeta \) and \( x(t) = \Delta \) for \( t \geq \zeta \). By Proposition 34 below, \( D'(X) \) is a Borel subset of \( D(X_\Delta) \), and hence \( D'(X) \) is a Borel space (see Appendix A.7 for the definition of Borel spaces).

**Proposition 34.** \( D'(X) \) is a Borel subset of \( D(X_\Delta) \), and hence \( D'(X) \) is a Borel space.

**Proof.** Let \( X_n \) be open subsets of \( X \) such that \( X_n \subset X_{n+1} \) and \( X_n \uparrow X \). Then \( D'(X) = \bigcap_{m \geq 0} A(m) \), where \( A(m) = \bigcup_{n=1}^\infty \bigcup_{r \in \mathbb{Q}, r>0} A(m, n, r) \) and \( A(m, n, r) = \{ x \in D(X_\Delta) : x(s) = \Delta \text{ for } s \in \mathbb{Q} \cap [r, \infty] \text{ and } x(s) \in X_n \text{ for } s \in \mathbb{Q} \cap [0, r-1/m] \} \). □

For \( b > 0 \), let \( D'_b(X) \) be the subset of \( D'(X) \) consisting of all paths with a common constant life time \( b \). Then \( D'_b(X) \) is a closed in \( D'(X) \) under the Skorohod metric, and \( D'_b(X) = D(X) \).

Recall \( J_1 : X \to Y \) and \( J_2 : X \to Z \) are the natural projections \( Y \times Z \to Y \) and \( Y \times Z \to Z \). We will also use \( J_1 \) and \( J_2 \) to denote the maps 
\[ J_1 : D'(X) \to D'(Y), \quad x(\cdot) \mapsto y(\cdot), \]
and 
\[ J_2 : D'(X) \to D'(Z), \quad x(\cdot) \mapsto z(\cdot), \]
respectively, given by the decomposition \( x(\cdot) = (y(\cdot), z(\cdot)) \).

Recall \( x_t \) is a \( G \)-invariant \( S \)-Markov process in \( X \). It may be regarded as the coordinate process on the canonical path space \( D'(X) \), that is, \( x_t(\omega) = \omega(t) \) for \( \omega \in \Omega \). The radial part \( y_t = J_1(x_t) \) and the angular part \( z_t = J_2(x_t) \) are thus the coordinate processes on \( D'(Y) \) and \( D'(Z) \), respectively. For \( t > s \geq 0 \), let \( \mathcal{F}^Y_{s,t} = \sigma\{y_u; \, u \in [s, t]\} \),
the $\sigma$-algebra on $D'(Y)$ generated by the radial process on the time interval $[s, t]$, and set $\mathcal{F}_t = \mathcal{F}_{1,t}$, and $\mathcal{F}_\infty = \sigma\{\cup_{t>0} \mathcal{F}_t\}$. The distribution $P_x$ of the process $x_t$ with $x_0 = x$ is a probability kernel from $X$ to $\Omega$. By the standard probability theory, if the underlying probability space $(\Omega, \mathcal{F}, P)$ is a Borel space, then for any $\sigma$-algebra $\mathcal{G} \subset \mathcal{F}$, there is a regular conditional distribution $Q(\omega, B)$ of $P$ given $\mathcal{G}$, which is a probability kernel from $(\Omega, \mathcal{G})$ to $(\Omega, \mathcal{F})$. For any $y \in Y$, the transition semigroup $P_x(\cdot, \cdot)$ on $Y$ is a probability kernel from $(\Omega, \mathcal{G})$ to $(\Omega, \mathcal{F})$. This result can be extended to our probability kernel $P_x$ (by suitably adapting the standard proof as in [21, Theorem 6.3]). Let $P_x'(\cdot, \cdot)$ be a regular conditional distribution of $P_x$ given the radial process $y_t = J_1 x_t$, that is, given $J_1^{-1}(\mathcal{F}_{\infty})$. This is a probability kernel from $(X \times \Omega, \mathcal{B}(X) \times J_1^{-1}(\mathcal{F}_{\infty}))$ to $(\Omega, \mathcal{F})$. Because $\mathcal{B}(X) \times J_1^{-1}(\mathcal{F}_{\infty})$ is a measurable set, there is a probability kernel $P_x'(\cdot, \cdot)$ constant on the set $J_1^{-1}([y])$. Let $P_x'(y(\cdot), \cdot) = P_x'(y(\cdot), o(\cdot), \cdot)$, where $o(\cdot) \in D'(Z)$ is the constant path $o(t) = o$ for all $t \geq 0$.

For $x \in X$ with decomposition $x = (y, z)$, let
\begin{equation}
Q_y = J_1 P_z = P_x(J_1^{-1}(:)).
\end{equation}
This is the distribution of the radial process $y_t = J_1 x_t$ with $y_0 = y$, and it does not depend on $x$ in the orbit $G y$. For $z \in Z$, $y(\cdot) \in D'(Y)$ and $x = (y(0), z)$, let
\begin{equation}
R_x^{y(\cdot)}(\cdot) = D_x^{y(\cdot)}(\cdot, J_1^{-1}(:)).
\end{equation}
This is a probability kernel from $Z \times D'(Y)$ to $D'(Z)$ such that for any $y \in Y$, $z \in Z$ measurable $F \subset D'(Z)$, with $x = (y, z)$,
\begin{equation}
R_x^{y(\cdot)}(F) = P_x[J_2^{-1}(F) \cap J_1^{-1}(\mathcal{F}_{\infty})]
\end{equation}
for $Q_y$-almost all $y(\cdot)$.

Thus, $R_x^{y(\cdot)}$ is the conditional distribution of the angular process $z_t = J_2 (x_t)$ given the radial process $y(\cdot)$ under $P_x$ with $x = (y(0), z)$.

The life time $\zeta$ of process $x_t$ is equal to that of the radial process $y_t$, and hence is $J_1^{-1}(\mathcal{F}_{\infty})$-measurable. Thus, given $J_1^{-1}(\mathcal{F}_{\infty})$ or under $R_x^{y(\cdot)}$, $\zeta$ is a constant. For $z \in Z$ and $Q_x$-almost all $y(\cdot) \in D'(Y)$, the measure $R_x^{y(\cdot)}$ is supported $D'(X)$.

For $t \in \mathbb{R}_+$, the time shift $\theta_t^y$ on $D'(Y)$ is defined as usual by $\theta_t^y (y(\cdot)) = y(\cdot + t)$. Then $y_s \circ \theta_t^y = y_{s+t}$ for the coordinate process $y_s$ on $D'(Y)$.

Recall Lévy processes in Lie groups and homogeneous spaces, including inhomogeneous ones, are defined to have infinite life times, but these definitions may be easily modified to include processes defined on a finite time interval $[0, T]$, and have some constant $T > 0$. The following result says that almost surely, given the radial process $y_t$, the conditioned angular process $z_t$ is an inhomogeneous Lévy process in $Z = G/K$ for $0 \leq t < \zeta$.

**Theorem 35.** For $y \in Y$ and $z \in Z$, and $Q_y$-almost all $y(\cdot)$ in $D'(Y)$, the coordinate process $z_t = J_2 (x_t)$ on $D'(Z)$ is an inhomogeneous Lévy process under $R_x^{y(\cdot)}$. The associated convolution semigroup $\mu_{s,t}$, setting $\mu_{s,t} = 0$ for $t \geq \zeta$, is $\mathcal{F}_t$-measurable for any $s \leq t$ (that is, $\mu_{s,t}(B)$ is $\mathcal{F}_t$-measurable for $B \in \mathcal{B}(Z)$) and has the time shift property
\begin{equation}
\mu_{s,t} = \mu_{t-s,0} \circ \theta_t^y.
\end{equation}
Moreover, the transition semigroup $P_t$ of the Markov process $x_t$ is given by
\begin{equation}
\forall f \in B_+(X) \text{ and } x = (y, z) \in X, \quad P_t f(x) = Q_y \left[ \mu_{0,t} (dz) f(y, z z_1) 1_{[\zeta > t]} \right].
\end{equation}

**Proof.** Recall $Q_t$ is the transition semigroup of the radial process $y_t$. By the existence of a regular conditional distribution, there is a probability kernel $R_t(y, y_1, \cdot)$ from $Y^2 = Y \times Y$ to $Z$ such that for $y \in Y$, $P_t((y, o), dy) \times dz_1 = Q_t(y, dy_1) R_t(y, y_1, dz_1)$. Then let $R_t(y, y_1, \cdot) = J_1 R'_t(y, (y_1, o), \cdot)$, noting $R'_t(y, (y_1, z), \cdot)$ is constant in $z \in Z$ and supported by $J_1^{-1}([y_1])$. 

The $G$-invariance of $P_t$ implies that $P_t((y, o), \cdot)$ is $K$-invariant, and hence the measure $R_t(y, y_1, \cdot)$ is $K$-invariant for $Q_t(y, \cdot)$-almost all $y_1$. Modifying $R_t$ on the exceptional set $(\{y, y_1\}; R_t(y, y_1, \cdot)$ is not $K$-invariant), its $y$-section has zero $Q_t(y, \cdot)$-measure for all $y \in Y$, we may assume $R_t(y, y_1, \cdot)$ is $K$-invariant for all $y, y_1 \in Y$. Therefore, for $z \in Z$, it is meaningful to write $R_t(y, y_1, z^{-1}dz_1) = R_t(y, y_1, S(z^{-1})dz_1)$ and $\int f(s_1)R_t(y, y_1, dz_1) = \int f(S(z_1)R_t(y, y_1, dz_1)$ because they are independent of choice of section map $S$. We then have, for $x = (y, z) \in X$,

\[
(39) \quad \forall y \in Y \text{ and } z \in Z, \quad P_t(x, dz_1 \times dy_1) = Q_t(y, dy_1)R_t(y, y_1, z^{-1}dz_1).
\]

For $0 < s_1 < s_2 < \cdots < s_k < \infty, y \in Y, z \in Z, h \in C_0(Y^k)$ and $f \in C_0(Z^k)$,

\[
E_x[h(y_{s_1}, \ldots, y_{s_k})f(z_{s_1}, \ldots, z_{s_k})1_{\{\zeta > s_k\}}] \quad (40)
\]

This implies that $P_x$-almost surely on $[\zeta > s_k],

\[
E_x[f(z_{s_1}, \ldots, z_{s_k}) | y_{s_1}, \ldots, y_{s_k}] = \int R_{s_k}(y_{s_1}, \ldots, y_{s_k})R_{s_{k-1}}(y_{s_1}, \ldots, y_{s_{k-1}})dz_1 \cdots dz_{s_k}(41)
\]

By (40), $K$-invariance of $P_t((y, o), y_1, \cdot)$ and the measurability of $\mu_{s,t}^{(m)}$, $y_{s_1}, \ldots, y_{s_k}$,

\[
\mu_{s,t}^{(m)}(f) = E_x[f(z_{s_1}^{-1}z_t) | y_{s_1}, \ldots, y_{s_k}] = E_x[f(z_{s_1}^{-1}z_t) | y_{s_1}, \ldots, y_{s_k}]
\]

$P_x$-almost surely on $[\zeta > T]$ for $f \in C_0(Z)$, which is independent of the choice for the section map $S$ to represent $z_{s_1}^{-1}z_t = S(z_{s_1}^{-1}z_t)$.

By the right continuity of $y_{s_t}$, as $m \to \infty, \sigma y_{s_1}, \ldots, y_{s_k} \uparrow \mathcal{F}_{z_t}^Y$, and $\sigma y_{s_1}, \ldots, y_{s_k} \uparrow \mathcal{F}_{s_t}^Y$, it follows that as $m \to \infty, \mu_{s,t}^{(m)}(f) \to E_x[f(z_{s_1}^{-1}z_t) | \mathcal{F}_{s_t}^Y] = E_x[f(z_{s_1}^{-1}z_t) | \mathcal{F}_{s_t}^Y]$. $P_x$-almost surely on $[\zeta > T]$, the exceptional set may be chosen simultaneously for countably many $f \in C_0(Z)$ and hence for all $f \in C_0(Z)$. It follows that $P_x$-almost surely on $[\zeta > T]$, there is an $K$-invariant probability measure $\mu_{s,t}$ such that $\mu_{s,t}^{(m)} \to \mu_{s,t}$ weakly and for $f \in C_0(Z),

\[
\mu_{s,t}(f) = E_x[f(z_{s_1}^{-1}z_t) | \mathcal{F}_{Y,s}^t] = E_x[f(z_{s_1}^{-1}z_t) | \mathcal{F}_{Y,s}^t] \quad P_x$-almost surely on $[\zeta > T]$.
\]

Note that $\mu_{s,t}^{(m)}$ is independent of choice $T \in \Gamma_m$ with $T \geq t$, and hence $\mu_{s,t}$ is defined for any $s, t \in \Gamma$ with $s < t < \zeta$. Set $\mu_{s,t} = \delta_0$ and $\mu_{s,t} = 0$ for $t \geq \zeta$. By (41), $\mu_{s,t}$ is an $\mathcal{F}_{Y,s}^t$-measurable random measure independent of starting point $x$ and has the time shift property (37) for $s, t \in \Gamma$. Moreover, $\mu_{r,s} \ast \mu_{s,t} = \mu_{r,t}$ for $r \leq s \leq t$ in $\Gamma$. Because $\Gamma$ is
countable, the exceptional set of zero $P_{x}$-measure for the weak convergence $\mu_{s,t}^{m} \to \mu_{s,t}$ may be chosen simultaneously for all $s \leq t$ and $T$ in $\Gamma$.

For $0 \leq t_{1} < t_{2} < \cdots < t_{n} \leq T$ in $\Gamma$, it can be shown from (40) and by choosing a partition $s_{1} < s_{2} < \cdots < s_{k}$ of $[0, T]$ from $\Gamma$ containing all $t_{i}$, spaced by $1/2^m$, that almost surely on $[\zeta > T]$, for $f \in C_{b}(Z^n),$

$$E_x[f(z_1, \ldots, z_n) | F_T^x] = \lim_{m \to \infty} E_x[f(z_1, \ldots, z_n) | y_{s_1}, \ldots, y_{s_k}]$$

$$= \lim_{m \to \infty} \int f(z_{s_1}, z_{s_2}, \ldots, z_{s_k}) \mu_{0,t_{1}}(dz_{1}) \mu_{t_{1},t_{2}}(dz_{2}) \cdots \mu_{t_{n-1},t_{n}}(dz_{n}).$$

This implies that for $0 \leq t_{1} < \cdots < t_{n} \leq T$ in $\Gamma$, $y \in Y$ and $z \in Z$, and $Q_{y}$-almost all $y(\cdot) \in [\zeta > T],$

$$R_{z}^{(y)}[f(z_{t_{1}}, \ldots, z_{t_{n}})] = E_x[f(z_{t_{1}}, \ldots, z_{t_{n}}) | F_T^x]$$

(44) $$= \int f(z_{s_1}, z_{s_2}, \ldots, z_{s_k}) \mu_{0,t_{1}}(dz_{1}) \mu_{t_{1},t_{2}}(dz_{2}) \cdots \mu_{t_{n-1},t_{n}}(dz_{n}).$$

Then the finite dimensional distribution of the conditioned angular process $z_t$ under $R_{z}^{(y)}$, when restricted to time points in $\Gamma \cap [0, T]$, has the form consistent with an inhomogeneous Lévy process in $Z$. To prove that the conditioned process $z_t$ is an inhomogeneous Lévy processes in $Z$, restricted to time interval $[0, T]$, it remains to extend $\mu_{s,t}$ in (43) to all real $s \leq t$ in $[0, T]$ and prove (44) for real times $t_{1} < t_{2} < \cdots < t_{n} \leq T$ in $[0, T]$.

By a computation similar to the one leading to (26), it can be show that for $s < t \leq T$ in $\Gamma$ and $f \in C_{b}(Z),$

$$\mu_{s,t}(f) = E_x \sum_{j} \psi_{j}(z_{s}) f(S_{j}(z_{s})^{-1}z_{t}) | F_T^x \quad P_{x} \text{-almost surely on } [\zeta > T],$$

where $\{\psi_{j}\}$ is a partition of unity on $Z$ and for each $j$, $S_{j}$ is a section map on $Z = G/K$ that is continuous on the support of $\psi_{j}$. The above expression for $\mu_{s,t}(f)$ extends to real times $s < t$, and it is clearly right continuous in $s$ and $t$. Moreover, considering $P_{x}(\cdot | F_T^x)$ as a regular conditional distribution of $P_{x}$ given $F_T^x$, no additional exceptional set is produced. Because for real $s < t$, $\mu_{s,t}$ is the weak limit of $\mu_{p,q}$ for $p, q \in \Gamma$ as $p \downarrow s$ and $q \downarrow t$, taking limit in (44) shows that it holds on real time points. It is also clear that $\mu_{s,t}$ is $F_T^{\infty}$-measurable and has the time shift property (37).

It remains to prove (38) which follows from

$$P_{t}f(x) = E_x[f((y_1, z_1)1_{[\zeta > t]}) = E_x\{R_{z}^{(y)}[f((y', z_1)1_{[\zeta > t]}])\}$$

$$= Q_{y}[\int \mu_{0,t}(dz_1)f((y_1, z_1)1_{[\zeta > t]}),$$

where the last equality is due to (44).

The following result provides a converse to Theorem 35.

**Theorem 36.** Let $y_t$ be a reg Markov process in $Y$ with life time $\zeta$ and for any $y \in Y$, let $Q_{y}$ be its distribution on $D'(Y)$ with $Q_{y}(0) = 1$. Assume for any $y \in Y$, $z \in Z = G/K$ and $Q_{y}$-almost all $y(\cdot)$, there is a probability measure $R_{z}^{(y)}$ on $D'(Y)$ such that under $R_{z}^{(y)}$, the coordinate process $z_t$ on $D'(Z)$ is an inhomogeneous Lévy process $z_t$ in $Z$ for $t < \zeta$, with $z_0 = z$, and the associated convolution semigroup $\mu_{s,t}$ (setting $\mu_{s,t} = 0$ for $t \geq \zeta$) is $F_T^{\infty}$-measurable and has the time shift property (37) as in Theorem 35. Then $x_t = (y_t, z_t)$ is a $G$-invariant reg Markov process in $X$ with transition semigroup $P_{t}$ given by (38).
Proof. For $z \in Z$ and $y(\cdot) \in D'(Y)$, let $R^y_z$ be the distribution on $D'(Z)$ of the inhomogeneous Lévy process $z_t$ in $Z$ for $t < \zeta$ with $z_0 = z$, associated to convolution semigroup $\mu_{s,t}$. Then for $0 \leq t_1 < t_2 < \cdots < t_n < \zeta$ and $f \in C_0(Z^n),
\begin{align*}
R^y_z[f(z_{t_1}, z_{t_2}, \ldots, z_{t_n})] &= \int f(z_{t_1}, z_{t_2}, \ldots, z_{t_n}) \\
&\quad \mu_{0,t_1}(dz_1)\mu_{t_1,t_2}(dz_2)\cdots\mu_{t_{n-1},t_n}(dz_n).
\end{align*}
Because $\mu_{s,t}$ is $F^y_z$ measurable, $R^y_z[f(z_{t_1}, z_{t_2}, \ldots, z_{t_n})]$ is measurable on $Z \times D'(Y)$.
Then a simple monotone class argument shows that $R^y_z$ is a kernel from $Z \times D'(Y)$ to $D'(Z)$.
For $x \in X$ with $x = (y,z)$, let $P_x$ be the measure on $D'(Y) \times D'(Z)$ defined by
\[ P_x(F) = Q_y\{R^y_z[f(y(\cdot),\cdot)]\}. \]
Then $P_x$ is a probability measure supported by the closed subset $H$ of $D'(Y) \times D'(Z)$ consisting of $(y(\cdot), z(\cdot))$ with the same life time for $y(\cdot)$ and $z(\cdot)$. The map: $x(\cdot) \rightarrow (J_1 x(\cdot), J_2 x(\cdot))$ is continuous and bijective from $D'(X)$ onto $H$, and its inverse is also continuous under the Skorohod metric. We will identify $D'(X)$ with $H$ via this map, and then $P_x$ may be regarded as a probability measure on $D'(X)$.
For $0 \leq s_1 < s_2 < \cdots < s_n = s < t, h \in C_b(X^n)$ and $f \in C_b(X),
\begin{align*}
E_x[h(x_{s_1}, x_{s_2}, \ldots, x_{s_n})f(x_t)] &= Q_y\{R^y_z[h(z_{s_1}, y_{s_1}, y_{s_2}, \ldots, y_{s_n} y_{s_n})f(z_{y(\cdot)}(y_{s_1}), y_{s_2}, \ldots, y_{s_n})] \mid \theta^y_z \}
\end{align*}
This shows that under $P_x$, $x_t$ is a Markov process in $X$ with transition semigroup $P_t$ given by (38). It follows directly from (38) that $P_t$ is $G$-invariant. \qed

References


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