Theory of Stochastic Processes
Vol. 21 (37), no. 2, 2016, pp. 29–57

MING LIAO

MARKOV PROCESSES AND GROUP ACTIONS

We develop basic properties of a Markov process that is invariant under the action of a locally compact topological group.

1. Introduction

The invariance of probability distributions under various transformations plays an important role in the probability theory. In the classical theory, the translation invariant Markov processes in a Euclidean space $\mathbb{R}^n$ can be identified with Lévy processes, which are processes x_t that have independent and stationary increments in sense that for any $s < t$, $x_t - x_s$ is independent of the process up to time s and its distribution depends only on $t - s$. Lévy processes in $\mathbb{R}^n$ have been extensively studied, but they still generate enormous interests, see Applebaum [1], Bertoin [4] and Sato [35] for some of modern books on this subject. By the celebrated Lévy -Khinchin formula, a Lévy process may be represented by a triple of a drift vector, a covariance matrix and a Lévy measure, in the sense that its distribution is determined by the triple, and to any such triple, there is a Lévy process, unique in distribution. A natural extension is to study Markov processes in a group that is invariant under the left or right translations. Hunt's 1956 pioneering work [20] provided an explicit formula for the generator of such an invariant Markov process in a Lie group, which allows us to represent such a process in distribution by a triple of a drift vector, a covariance matrix and a Lévy measure, just as for a Lévy process in $\mathbb{R}^n$.

Purpose of this paper is to develop the basic definitions and properties of invariant Markov processes under the more general framework of a topological group acting continuously on a space, both are assumed to be locally compact. Some of these results may be known in various contexts, but this is the first time they are put together in a unified and cohesive fashion.

Invariant Markov processes under topological group actions may be considered at three levels of generality. First, we may consider Markov processes in a topological group G that are invariant under the left (or right) translations. Such processes are direct extensions of classical Lévy processes in $\mathbb{R}^n$, and can be identified with processes in G that have independent and stationary increments of the form $x_s^{-1}x_t$ (or $x_t x_s^{-1}$) for $s < t$. At the second level, we may consider a Markov process x_t in a topological space X that is invariant under the transitive action of a topological group G on X. In this case, X may be identified with a topological homogeneous space G/K, and x_t with a Markov process in G/K invariant under the natural G-action.

We will call a Markov process in G invariant under left translations, or a Markov process in G/K invariant under the natural G-action, a Lévy process. A homogeneous space G/K does not possess a natural product structure. To study invariant Markov processes in G/K, we have developed a theory of a kind of product "in distribution" on G/K that allows us to carry over almost all the results on the group G to the homogeneous space G/K. For example, it is easy to show that Lévy processes in G/K, which are defined as G-invariant Markov processes, can be characterized as processes that have

2010 *Mathematics Subject Classification.* 60J25, 60B15.
Key words and phrases. Markov processes, topological groups.

independent and stationary increments in a suitable sense, just as their counterparts in G.

At the third level of generality, we may consider a Markov process x_t invariant under the non-transitive action of a topological group G. In this case, under some suitable assumptions, x_t may be decomposed into a radial part and an angular part. The radial part can be an arbitrary Markov process in a subspace that is transversal to G-orbits, whereas under the conditional distribution given the radial part, the angular part is a (time) inhomogeneous Markov process in a standard G-orbit that is invariant under the G-action.

This leads us to consider inhomogeneous Markov processes that are invariant under a group action. It is easy to show that inhomogeneous Markov processes x_t in a group G invariant under left translations, or in G/K invariant under the natural G-action, may be identified with processes that have independent, but not necessarily stationary, increments, so such processes will be called inhomogeneous Lévy processes.

As our purpose is limited to developing the basic properties under the general framework of the action of a locally compact topological group, without involving Lie structures and Fourier analysis, so many deeper aspects of invariant Markov processes are not discussed here. We briefly mention some of these topics. Heyer [18] generalized Hunt's generator formula on Lie groups to locally compact groups in the form of a sum of three maps, corresponding to a drift, a diffusion part and a jump part. Using a projective basis of Lie algebras, Born [6] rewrote Heyer's formula in a form more closely resembling Hunt's formula. For inhomogeneous Lévy processes in Lie groups, Feinsilver [11] (1978) obtained a martingale property, generalizing a result of Stroock-Varadhan [37] (1973) for continuous processes. By this martingale property, an inhomogeneous Lévy process is determined in distribution by a time dependent triple of a deterministic path, a covariance matrix function and a Lévy measure function. A different form of martingale representation on more general locally compact groups, in terms of the abstract Fourier analysis, is obtained in Heyer-Pap [19]. The author [27] obtained an extension of Feinsilver's result to homogeneous spaces.

Lévy processes in noncompact semisimgple Lie groups and symmetric spaces exhibit interesting path limiting properties which are not present for their counterparts in Euclidean spaces. Limiting properties of products of random matrices and random walks in semisimple Lie groups, which may be regarded as discrete time Lévy processes, were studied Furstenberg-Kesten [12] (1960), Furstenberg [13] (1963), Tutubalin [40] (1965), Virtser [42] (1970), Guivarc'h-Raugi [15] (1985), and Raugi [34] (1997). The study of Limiting properties of Brownian motion in semisimple Lie groups and symmetric spaces can be traced to Dynkin [9] (1961), Orihara [33] (1970) and Malliavin-Malliavin [29] (1972), and continued in Norris-Rogers-Williams [32] (1986), Taylor [38, 39] (1988, 1991), Babillot [3] (1991) and Liao [24] (1998). Author's monograph [25] (2004) provides an account of limiting and dynamical properties of Lévy processes in noncompact semisimple Lie groups and symmetric spaces.

Probability measures on locally compact groups have been studied extensively. For a comprehensive treatment, the reader is referred to Heyer's 1977 classic treatise [18], which is still an invaluable reference today. Some of more recent developments can be found in Siebert [36], Pap [31], Dani-McCrudden [7, 8] and Liao [28]. A comprehensive analysis of probability measures on compact Lie groups can be found in Applebaum's recent book [2].

We now briefly describe the content of each section. The formal definition of invariant Markov processes under group actions is given in the next section §2. Lévy processes in a topological group G and in a topological homogeneous space G/K are discussed respectively in sections §3 and §4. Inhomogeneous Lévy processes in G and G/K are considered in §5. In §6 and §7, the decomposition of an invariant Markov process under a non-transitive action into a radial part and an angular part is introduced, and it is shown that given the radial part, the conditioned angular part is an inhomogeneous Lévy process in a standard orbit. This has been done in Liao [27] on Lie groups.

2. Invariant Markov processes under group actions

We now state some general definitions and conventions to be used in this paper, most of which are also commonly used in the literature. On a topological space X, let $\mathcal{B}(X)$ be its Borel σ-algebra, and let $\mathcal{B}_b(X)$ and $\mathcal{B}_+(X)$ be respectively the spaces of $\mathcal{B}(X)$-measurable functions that are bounded and nonnegative. A measure μ on X is always assumed to be defined on $\mathcal{B}(X)$ unless when explicitly stated otherwise. For a measurable function f, the integral $\int f d\mu$ may be written as $\mu(f)$. If μ is a measure on a measurable space X and if F is a measurable map from X to another measurable space Y, then $F\mu$ denotes the measure on Y given by $F\mu(f) = \mu(f \circ F)$ for any measurable function $f \geq 0$ on Y. This may also be written as $F\mu(B) = \mu(F^{-1}(B))$ for measurable $B \subset X$. A kernel K from X to Y is a family of measures $K(x, \cdot)$ on Y, $x \in X$, such that $x \mapsto K(x, B)$ is measurable on X for any measurable $B \subset Y$. It is called a probability kernel, or a sub-probability kernel, if for all $x \in X$, $K_t(x, X) = 1$, or $K(x, X) \leq 1$.

A topological group G is a group and a topological space such that both the product map $G \times G \to G$, $(g, h) \mapsto gh$, and the inverse map $G \to G$, $g \mapsto g^{-1}$, are continuous. A continuous action of G on a topological space X is a continuous map $G \times X \to X$ given by $(g, x) \mapsto gx$ such that $g(hx) = (gh)x$ and $ex = x$ for $g, h \in G$ and $x \in X$, where e is the identity element of G. In the sequel, the action of a topological group G on a topological space X is always assumed to be continuous unless when explicitly stated otherwise. The action is called linear if X is a linear space and the map $x \mapsto gx$ is linear for each $g \in G$.

The group action defined above is also called a left action. We may also consider a right action when the action map is written as $(x, g) \mapsto xg$ and satisfies $(xg)h = x(gh)$. In the sequel, all actions are assumed to be left actions unless when explicitly stated otherwise.

A function f or a measure μ on X is called invariant under a measurable map g: $X \to X$, or g-invariant for short, if $f \circ g = f$ or $g\mu = \mu$. An operator T on X with domain $D(T)$ being a set of functions on X is map from $D(T)$ to a possibly different set of functions on X. It is called g-invariant if $\forall f \in D(T)$, $f \circ g \in D(T)$ and $T(f \circ g) = (Tf) \circ g$. A kernel K from X to itself may be regarded as an operator $Kf(x) = K(x, f)$ on X with domain $D(K) = \mathcal{B}_+(X)$, then it is g-invariant if $K(g(x), B) = K(x, g^{-1}(B))$ for $x \in X$ and $B \in \mathcal{B}(X)$.

A function f, a measure μ, an operator T or a kernel K on X is called invariant under the action of a group G, or G-invariant for short, if it is g-invariant for any $g \in G$.

For $g \in G$, let l_g, r_g and c_g be respectively the left translation, the right translation and the conjugation map: $G \to G$, defined by $l_g x = gx$, $r_g x = xg$ and $c_g x = gxg^{-1}$ for $x \in G$. The group G acts on itself by left translation and also by conjugation, whereas the right translation is a right action of G on itself. A function f or a measure μ on G is called left invariant if it is invariant under the action of G on itself by left translations. that is, if $f \circ l_g = f$ or $l_g \mu = \mu$ for $g \in G$. Similarly, an operator T on G is called left invariant if for $f \in D(T)$, $f \circ l_g \in D(T)$ and $(Tf) \circ l_g = T(f \circ l_g)$ for $g \in G$. If this holds only for g contained in a subgroup K of G, then f, μ or T is called K-left invariant. The (K-) right invariant and (K-) conjugate invariant functions, measures and operators on G are defined similarly. When they are both left and right (resp K-left and K-right) invariant, then they are called bi-invariant (resp. K-bi-invariant).

Throughout this chapter, let X be a topological space and let G be a topological group that acts continuously on X, both are equipped with lcscH (locally compact and second countable Hausdorff) topologies. We will let $C(X)$, $C_b(X)$, $C_c(X)$ and $C_0(X)$ denote respectively the spaces of continuous functions, bounded continuous functions, continuous functions with compact supports and continuous functions convergent to 0 at infinity (under the one-point compactification topology) on X. Note that $C_0(X)$ may also be characterised as the space of continuous functions f on X such that for any $\varepsilon > 0$, there is a compact $K \subset X$ with $|f| < \varepsilon$ on K^c (the complement of K in X).

A family of sub-probability kernels P_t from X to X, $t \in \mathbb{R}_+ = [0, \infty)$, is called a transition semigroup on X if $P_t P_s = P_{t+s}$ for $s, t \in \mathbb{R}_+$ and $P_0(x, \cdot) = \delta_x$ (the unit mass at point x). A Markov process x_t in X, $t \in \mathbb{R}_+$, usually means a family of processes, one for each starting point $x \in X$, associated to a transition semigroup P_t and satisfying the following simple Markov property: for $t > s \geq 0$ and $f \in \mathcal{B}_b(X)$,

$$E[f(x_t) \mid \mathcal{F}_s] = P_{t-s} f(x_s) \quad P \text{ almost surely,} \tag{1}$$

where $\{\mathcal{F}_t\}$ is the natural filtration of process x_t. If (1) holds under a larger filtration $\{\mathcal{F}_t\}$, a possibly stronger requirement, then x_t is called a Markov process associated to $\{\mathcal{F}_t\}$ or an $\{\mathcal{F}_t\}$-Markov process. The symbol P_x is used to denote the distribution of the process starting at x on the canonical path space and E_x is the associated expectation. Occasionally, a Markov process means a single process with a given initial distribution and this should be clear from context. We will allow a Markov process x_t to have a possibly finite life time as P_t is assumed to be sub-probability, not necessarily a probability kernel.

A family of sub-probability kernels $P_{s,t}$ from X to X, $0 \leq s \leq t < \infty$, is called a two-parameter transition semigroup on X if $P_{r,s} P_{s,t} = P_{r,t}$ for any $r \leq s \leq t$ and $P_{t,t}(x, \cdot) = \delta_x$. A (time) inhomogeneous Markov process x_t in X is a family of processes, one for each pair of starting time $s \in \mathbb{R}_+$ and starting point $x \in X$, associated to a two-parameter transition semigroup $P_{s,t}$ and satisfying the following inhomogeneous Markov property:

$$E[f(x_t) \mid \mathcal{F}_s] = P_{s,t} f(x_s), \quad P\text{-almost surely.} \tag{2}$$

where $\{\mathcal{F}_t\}$ is the natural filtration of process x_t. If the above holds for a larger filtration $\{\mathcal{F}_t\}$, then the inhomogeneous Markov process x_t is said to be associated to $\{\mathcal{F}_t\}$.

A Markov process x_t in X is called invariant under a measurable map g: $X \to X$, or g-invariant for short, if its transition semigroup P_t, as an operator for each t, is g-invariant, that is, if

$$P_t(g(x), B) = P_t(x, g^{-1}(B)) \tag{3}$$

for any $t \in \mathbb{R}_+$, $x \in X$ and $B \in \mathcal{B}(X)$. Note that (3) is equivalent to

$$E_{gx}[f(x_t)] = E_x[f(gx_t)] \tag{4}$$

for any $t \in \mathbb{R}_+$, $x \in X$ and $f \in \mathcal{B}_b(X)$.

Proposition 1. *A Markov process x_t in X is g-invariant if and only if for any $x \in X$, the process gx_t with $x_0 = x$ has the same distribution as the process x_t with $x_0 = gx$.*

Proof. It is clear that the same distribution implies the g-invariance (4). Now assume (4). Let $\{\mathcal{F}_t\}$ be the natural filtration of process x_t. Then for $0 < t_1 < t_2$ and $f_1, f_2 \in \mathcal{B}_b(X)$, by the simple Markov property,

$$\begin{aligned} & E_{gx}[f_1(x_{t_1}) f_2(x_{t_2})] = E_{gx}\{f_1(x_{t_1}) E_{gx}[f_2(x_{t_2}) \mid \mathcal{F}_{t_1}]\} = E_{gx}[f_1(x_{t_1}) P_{t_2 - t_2} f_2(x_{t_1})] \\ = \; & E_x[f_1(gx_{t_1}) P_{t_2 - t_1} f(gx_{t_1})] = E_x[f_1(gx_{t_1}) P_{t_2 - t_1} (f \circ g) f(x_{t_1})] \\ = \; & E_x\{f_1(gx_{t_1}) E_x[f_2(gx_{t_2}) \mid \mathcal{F}_{t_1}]\} = E_x[f_1(gx_{t_1}) f_2(gx_{t_2})]. \end{aligned}$$

Inductively, it can be shown that for $t_1 < t_2 < \cdots < t_n$,

$$E_{gx}[f_1(x_{t_1})f_2(x_{t_2})\cdots f_n(x_{t_n})] = E_x[f_1(gx_{t_1})f_2(gx_{t_2})\cdots f_n(gx_{t_n})].$$

This proves the same distribution. □

The g-invariance for an inhomogeneous Markov process x_t in X is defined in a similar fashion. The process x_t is called invariant under a measurable map g: $X \to X$, or g-invariant, if its transition semigroup $P_{s,t}$ is g-invariant, that is, if

$$\forall t \geq s \geq 0,\ x \in X \text{ and } B \in \mathcal{B}(X), \qquad P_{s,t}(gx, B) = P_{s,t}(x, g^{-1}(B)). \tag{5}$$

The following proposition may be proved as Proposition 1.

Proposition 2. *An inhomogeneous Markov process x_t is g-invariant if and only if for all $s \in \mathbb{R}_+$ and $x \in X$, the process gx_t, $t \geq s$, with $x_s = x$ has the same distribution as process x_t, $t \geq s$, with $x_s = gx$.*

A Markov process x_t in X is called invariant under the action of a group G, or G-invariant, if its transition semigroup P_t is g-invariant for any $g \in G$. Similarly, an inhomogeneous Markov process x_t in X is called G-invariant if its transition semigroup $P_{s,t}$ is G-invariant.

A process is said to have rcll paths, or called a rcll process, if almost all its paths are right continuous with left limits in its state space. In literature, a rcll process is often called càdlàg (French "continue à droite, limite à gauche"). Most processes in this work will be assumed to be rcll. We will see in Remarks 5, 15, 21 and 28 that many of these processes in fact have rcll versions if they are continuous in distribution.

A Markov process x_t in X is called a Feller process if its transition semigroup P_t is Feller, that is, if for any $f \in C_0(X)$, $P_t f \in C_0(X)$ and $P_t f \to f$ pointwise on X as $t \to 0$, noting that this in fact implies that $P_t f \to f$ uniformly on X. A Feller process has a rcll version, so it will be assumed to be rcll, and it has many other useful properties, including the strong Markov property. See for example [21, Chapter 19] for more details.

3. Lévy processes in topological groups

A Markov process x_t in a topological group G invariant under the action of G on itself by left translation is called left invariant. Its transition semigroup P_t is left invariant in the sense that

$$P_t(f \circ l_g)(x) = P_t f(gx) \tag{6}$$

for $t \in \mathbb{R}_+$, $x, g \in G$ and $f \in \mathcal{B}_b(G)$.

Let x_t be a rcll left invariant Markov process in G. Recall e is the identity element of G. Then $P_t f(x) = P_t(f \circ l_x)(e) = E_e[f(xx_t)]$. By the rcll paths, one sees that P_t is a Feller transition semigroup, and hence x_t is a Feller process.

Let x_t be a process in G with an infinite life time, and let $\{\mathcal{F}_t^x\}$ be its natural filtration. It is said to have independent increments if for $s < t$, $x_s^{-1}x_t$ is independent of $\mathcal{F}_s^x$. It is said to have stationary increments if for $s < t$, the distribution of $x_s^{-1}x_t$ depends only on $t - s$, that is, if $x_s^{-1}x_t$ and $x_0^{-1}x_{t-s}$ have the same distribution. A rcll process x_t is called a Lévy process in G if it has independent and stationary increments.

The classical example is a Lévy process x_t in the d-dim Euclidean space $\mathbb{R}^d$ when $\mathbb{R}^d$ is regarded as an additive group. In this case, the increment $x_s^{-1}x_t$ is written as $x_t - x_s$.

For a Lévy process x_t in G, let

$$x_t^e = x_0^{-1}x_t. \tag{7}$$

It is clear that process x_t^e is also a Lévy process in G starting at e and is independent of x_0.

Let x_t be a Lévy process in G. For $t \in \mathbb{R}_+$, $x \in G$ and $f \in \mathcal{B}_b(G)$, put

$$P_t f(x) = E[f(x x_t^e)]. \tag{8}$$

It is easy to see that this defines a transition semigroup P_t on G, which is conservative, that is, $P_t 1 = 1$, and is left invariant.

For $t > s$,

$$E[f(x_t) \mid \mathcal{F}_s] = E[f(x_s x_s^{-1} x_t) \mid \mathcal{F}_s] = E[f(x x_{t-s}^e)] \mid_{x=x_s} = P_{t-s} f(x_s).$$

This shows that the Lévy process x_t is a left invariant Markov process.

Now let x_t be a rcll left invariant Markov process in G with an infinite life time. Then for $t > s$ and $f \in \mathcal{B}_b(G)$,

$$E[f(x_s^{-1} x_t) \mid \mathcal{F}_s] = E[f(x^{-1} x_t) \mid \mathcal{F}_s] \mid_{x=x_s} = P_{t-s}(f \circ l_{x_s^{-1}})(x_s) = P_{t-s} f(e).$$

This shows that x_t has independent and stationary increments, and hence is a Lévy process. We have proved that the class of Lévy processes in G coincides with the class of left invariant rcll Markov processes in G with infinite life times.

Let x_t be a left invariant rcll Markov process in G with a possibly finite life time. Then $P_t 1(x) = P_t(1 \circ l_x)(e) = P_t 1(e)$, which is right continuous in t due to the right continuity of paths. By the semigroup property and the left invariance of P_t, $P_{s+t} 1(e) = \int P_s(e, dx) P_t 1(x) = P_s 1(e) P_t 1(e)$. It follows that $P_t 1(x) = e^{-\lambda t}$ for all $x \in G$ and for some fixed $\lambda \geq 0$. Let $\hat{P}_t = e^{\lambda t} P_t$. It is easy to see that $\hat{P}_t$ is a conservative and left invariant Feller transition semigroup. The Feller process $\hat{x}_t$ associated to $\hat{P}_t$ is a left invariant rcll Markov process with an infinite life time, and hence is a Lévy process. Let τ be an exponential random variable of rate λ ($\tau = \infty$ if $\lambda = 0$) that is independent of process $\hat{x}_t$, and let x_t' be the process $\hat{x}_t$ killed at time τ, that is, $x_t' = \hat{x}_t$ for $t < \tau$ and $x_t' = \Delta$ for $t \geq \tau$, where Δ is the point at infinity (see Appendix A.3). It is easy to show that x_t' is a rcll Markov process with transition semigroup P_t and hence is identical in distribution to process x_t. To summarize, we have proved the following result.

Theorem 3. *A left invariant rcll Markov process in G with an infinite life time is a Lévy process. Conversely, a Lévy process x_t in G is a left invariant rcll Markov process with an infinite life time and its transition semigroup is given by (8).*

In general, a left invariant rcll Markov process x_t in G with a possibly finite life time is a Feller process and is identical in distribution to a Lévy process $\hat{x}_t$ killed at an independent exponential time of rate $\lambda \geq 0$. The transition semigroup P_t of x_t and $\hat{P}_t$ of $\hat{x}_t$ are related as $P_t = e^{-\lambda t} \hat{P}_t$.

Remark 4. Let x_t be a process in G with an infinite life time. The above proof shows that x_t is a left invariant Markov process if and only if it has independent and stationary increments.

Remark 5. Let x_t be a left invariant Markov process in G without assuming an infinite life time, and let P_t be its transition semigroup. Then $P_t f(x) = P_t(f \circ l_x)(e) = \int_G f(xy) \mu_t(dy)$, where $\mu_t = P_t(e, \cdot)$. If $\mu_t \to \delta_e$ (the unit point mass at e) weakly as $t \to 0$, then P_t is Feller and hence x_t has a rcll version. Therefore, if x_t also has an infinite life time, then after a modification on a null set for each $t \geq 0$, x_t becomes a Lévy process in G.

Given a filtration $\{\mathcal{F}_t\}$ on the underlying probability space $(\Omega, \mathcal{F}, P)$. A Lévy process x_t is called associated to $\{\mathcal{F}_t\}$, or an $\{\mathcal{F}_t\}$-Lévy process, if it is adapted to $\{\mathcal{F}_t\}$ and for any $s < t$, $x_s^{-1} x_t$ is independent of $\mathcal{F}_s$. A Lévy process is clearly associated to its natural filtration. By the proof of Theorem 3, it is easy to see that a Lévy process is associated to a filtration if and only if it is associated to the same filtration as a Markov process.

It is easy to see that if x_t is a Lévy process associated to a filtration $\{\mathcal{F}_t\}$ and if $s > 0$ is fixed, then $x'_t = x_s^{-1} x_{s+t}$ is a Lévy process identical in distribution to the process x_t^e and is independent of $\mathcal{F}_s$. The following theorem says that s may be replaced by a stopping time.

Theorem 6. *Let x_t be a Lévy process associated to a filtration $\{\mathcal{F}_t\}$ in G. If τ is an $\{\mathcal{F}_t\}$-stopping time with $P(\tau < \infty) > 0$, then under the conditional probability $P(\cdot \mid \tau < \infty)$, the process $x'_t = x_\tau^{-1} x_{\tau+t}$ is a Lévy process in G that is independent of $\mathcal{F}_\tau$. Moreover, the process x'_t under $P(\cdot \mid \tau < \infty)$ has the same distribution as the process x_t^e under P.*

Proof. First assume τ takes only discrete values. Fix $0 < t_1 < t_2 < \cdots < t_k$, $\phi \in C_b(G^k)$ and $\xi \in (\mathcal{F}_\tau)_+$. Because $\xi 1_{[\tau=t]} \in (\mathcal{F}_t)_+$, where 1_A denotes the indicator of set A, we have

$$\begin{aligned} & E[\phi(x_\tau^{-1} x_{\tau+t_1}, \cdots, x_\tau^{-1} x_{\tau+t_k})\xi \mid \tau < \infty] \\ = & \sum_{t<\infty} E[\phi(x_t^{-1} x_{t+t_1}, \cdots, x_t^{-1} x_{t+t_k})\xi 1_{[\tau=t]}]/P(\tau < \infty) \\ = & \sum_{t<\infty} E[\phi(x_t^{-1} x_{t+t_1}, \cdots, x_t^{-1} x_{t+t_k})]E(\xi 1_{[\tau=t]})/P(\tau < \infty) \\ = & E[\phi(x_0^{-1} x_{t_1}, \cdots, x_0^{-1} x_{t_k})]E(\xi \mid \tau < \infty) \end{aligned} \tag{9}$$

Setting $\xi = 1$ yields $E[\phi(x_\tau^{-1} x_{\tau+t_1}, \cdots, x_\tau^{-1} x_{\tau+t_k}) \mid \tau < \infty] = E[\phi(x_0^{-1} x_{t_1}, \cdots, x_0^{-1} x_{t_k})]$. Therefore, for a general $\xi \in (\mathcal{F}_\tau^0)_+$, the expression in (9) is equal to

$$E[\phi(x_\tau^{-1} x_{\tau+t_1}, \cdots, x_\tau^{-1} x_{\tau+t_k}) \mid \tau < \infty]\, E(\xi \mid \tau < \infty).$$

This proves the desired result for a discrete stopping time τ.

For a general stopping time τ, let $\tau_n = (k+1)2^{-n}$ on the set $[k \cdot 2^{-n} \leq \tau < (k+1)2^{-n}]$ for $k = 0, 1, 2, \ldots$. Then τ_n are discrete stopping times and $\tau_n \downarrow \tau$ as $n \uparrow \infty$. The result for τ follows from the discrete case and the right continuity of x_t. □

The convolution of two measures μ and ν on G is the measure $\mu * \nu$ on G determined by

$$\mu * \nu(f) = \int f(xy)\mu(dx)\nu(dy) \tag{10}$$

for $f \in \mathcal{B}_+(G)$, and it is a probability measure if so are μ and ν. It is easy to check that the convolution is associative, that is, $(\mu * \nu) * \gamma = \mu * (\nu * \gamma)$, and hence, it is meaningful write a convolution product $\mu_1 * \mu_2 * \cdots * \mu_n$ or an n-fold convolution power $\mu^{*n} = \mu * \mu * \cdots * \mu$.

A family of probability measures μ_t on G, $t \geq 0$, is called a convolution semigroup on G if $\mu_{s+t} = \mu_s * \mu_t$. Although convolution semigroup may be defined for more general measures, in this work, a convolution semigroup will always mean a family of probability measures. It is called continuous if $\mu_t \to \mu_0$ weakly as $t \to 0$. By [18, Theorem 1.5.7], if μ_t is continuous, then $\mu_t \to \mu_s$ weakly as $t \to s$ for all $s > 0$. An alternative proof will be given shortly.

Let x_t be a Lévy process in G. It is easy to see that $P_t(e, \cdot)$, the distributions μ_t of x_t^e, $t \geq 0$, form a continuous convolution semigroup of probability measures on G (the continuity follows from the right continuity of paths) with $\mu_0 = \delta_e$, which will be called the convolution semigroup associated to the Lévy process g_t. On the other hand, if μ_t is a continuous convolution semigroup on G with $\mu_0 = \delta_e$, then

$$P_t f(x) = \mu_t(f \circ l_x), \tag{11}$$

for $f \in C_0(G)$, defines a left invariant conservative Feller transition semigroup P_t on G. There is a Feller process x_t associated to P_t, which is left invariant, and hence is a Lévy process with μ_t as the associated convolution semigroup.

Because x_t is a Feller process, it is stochastically continuous, that is, $x_t = x_{t-}$ almost surely for all $t > 0$ (see for example ([21, Proposition 25.20]). It follows that μ_t is in fact continuous in t under the weak convergence, as mentioned before.

To summarize, we have the following result.

Theorem 7. *For a Lévy process x_t in G, the distributions μ_t of $x_t^e = x_0^{-1}x_t$ form a continuous convolution semigroup on G with $\mu_0 = \delta_e$. conversely, if μ_t is such a convolution semigroup on G, then there is a Lévy process x_t in G with μ_t as distribution of x_t^e.*

Using the Markov property, is easy to show that the finite dimensional distributions of a Lévy process x_t in G with associated convolution semigroup μ_t are given by

$$\begin{aligned} E[f(x_{t_1}, x_{t_2} \ldots, x_{t_n})] &= \int f(x_0x_1, x_0x_1x_2, \ldots, x_0x_1x_2\cdots x_n) \\ &\qquad \mu_0(dx_0)\mu_{t_1}(dx_1)\mu_{t_2-t_1}(dx_2)\cdots\mu_{t_n-t_{n-1}}(dx_n) \end{aligned} \tag{12}$$

for $f \in \mathcal{B}_b(X^n)$ and $0 \le t_1 < t_2 < \cdots < t_n$, where μ_0 is the initial distribution.

Remark 8. In the definition of Lévy processes, if the increments $x_s^{-1}x_t$ for $s < t$ are replaced by $x_t x_s^{-1}$, this will lead to a different definition of Lévy processes, which coincide with Markov processes in G that are invariant under right translations. In [25], the Lévy processes defined earlier and defined here are called respectively left and right Lévy processes. Because the group G is in general non-commutative, left and right Lévy processes are two different classes of processes, but they are in natural duality under the map $x_t \mapsto x_t^{-1}$. All the preceding results established for left Lévy processes hold also for right Lévy processes with suitable changes. For example, $x_t^e = x_0^{-1}x_t$ should be changed to $x_t^e = x_t x_0^{-1}$ and (11) holds with l_x replaced by r_x. Note that a continuous convolution semigroup μ_t on G with $\mu_0 = \delta_e$ can be used to generate either a left Lévy process by (11) or a right Lévy process by the counter part of (11) with l_x replaced by r_x.

In this paper, we will exclusively consider left Lévy processes, unless when explicitly stated otherwise, and we will omit the adjective "left" in its name.

4. Lévy process in topological homogeneous spaces

Let X be a topological space and let G be a topological group that acts transitively on X, both are equipped with lcscH topologies. Fix a point o in X. The isotropy subgroup K of G at o, defined by $K = \{g \in G;\ go = o\}$, is a closed subgroup of G.

For any closed subgroup K of G, the space G/K of left cosets gK, $g \in K$, is called a homogeneous space of G. It is equipped with the quotient topology, under which the natural projection

$$\pi:\ G \to G/K, \quad g \mapsto gK,$$

is continuous and open. Under this topology, the natural action of G on G/K, given by $xK \mapsto gxK$ for $g \in G$, is continuous. Moreover, when K is the isotropy subgroup at $o \in X$ as defined above, then under the map: $gK \mapsto go$, G/K is homeomorphic to X, and the G-action on X is just the natural action of G on G/K (see Theorem 3.2 in [16, chapter II]).

We may identify X and G/K together with the associated G-actions in this way. Under this identification, the natural projection π: $G \to G/K$ is just the map: $G \to X$ given by $g \mapsto go$, where $o = eK$.

Note that if G is a topological group with a lcscH topology, and if K is a compact subgroup, then the homogeneous space $X = G/K$, under the quotient topology, is a lcscH space.

In the rest of this section, we will assume K is compact. We will also assume the homogeneous space G/K has a continuous local section in the sense that there are a neighborhood U of o in X and a continuous map ϕ: $U \to G$ such that $\pi \circ \phi = \mathrm{id}_U$ (the identity map on U). Because the isotropy subgroup at a different point go, $g \in G$, is gKg^{-1}, it can be shown that these assumptions are independent of the choice for the point $o \in X$. Under these assumptions, it can be shown that the map: $(x, k) \mapsto \phi(x)k$ is a homeomorphism from $U \times K$ onto a neighborhood of e of G. Some sufficient conditions for the existence of a continuous local section map may be found in [22, 30], and see also [22] for some non-existence examples.

It is well known and also easy to see that a continuous local section, even a smooth local section, exists if G is a Lie group. By Theorems 4.6 and 4.7, and Corollary 4.8 in [23, Chapter I], the group G of isometries on a connected Riemannian manifold X has a Lie group structure and acts smoothly on X, and the isotropy subgroup K of G at any $o \in X$ is compact. In this case, if G acts transitively on X, then $X = G/K$ satisfies all the assumptions stated here.

A measurable map S: $X \to G$ is called a section map on X if $\pi \circ S = \mathrm{id}_X$. By the existence of a continuous local section, one can always construct a section map on X. In general, it may not be continuous on X, but for any $x \in X$, there is a section map that is continuous on a neighborhood of x (even smooth there when G is a Lie group).

A measure on a lcscH space is called a Radon measure if it has a finite charge on any compact set. Note that the usual definition of Radon measures on Hausdorff spaces includes a regularity condition, which is automatically satisfied on a lcscH space. It is well known that on a topological group G equipped with a lcscH topology, there is a nontrivial left (resp. right) invariant Radon measure μ, called a left (resp. right) Haar measure on G, which is unique up to a multiplicative constant. In general, left Haar measures do not agree with right Haar measures, but when they do, the group G is called unimodular. In this case, we will simply say Haar measures. It is easy to show (by applying right translations and the inverse map to a left Haar measure) that a compact group K is unimodular and there is a unique left invariant probability measure on K, which is also invariant under right translations and the inverse map on K. This measure is called the normalized Haar measure on the compact group K, and is denoted as ρ_K or simply dk in computation.

The convolution between two measures μ and ν on X is the measure $\mu * \nu$ defined by

$$\mu * \nu(f) = \int_{X\times X} \int_K f(S(x)ky) dk \mu(dx) \nu(dy), \tag{13}$$

for any $f \in \mathcal{B}_+(X)$. This definition does not depend on the choice for the section map S (because if S' is another section map, then $S'(x) = S(x)k_x$ for some $k_x \in K$), and reduces to the convolution on G when $K = \{e\}$. Because for any $g \in G$ and $x \in X$, $S(gx) = gS(x)k$ for some $k = k(g, x) \in K$, it is easy to show that the convolution on X, as on G, is associative, that is, $(\mu * \nu) * \gamma = \mu * (\nu * \gamma)$, so the n-fold convolution $\mu_1 * \mu_2 * \cdots * \mu_n$ is well defined.

Recall a measure μ on X is called K-invariant if $k\mu = \mu$ for any $k \in K$. If ν is a K-invariant measure on X, then $\mu * \nu$ can be written a little more concisely as

$$\mu * \nu(f) = \int_{X\times X} f(S(x)y) \mu(dx) \nu(dy) \tag{14}$$

for $f \in \mathcal{B}_+(X)$, which does not depend on the choice for the section map S. Moreover, if μ is also K-invariant, then so is $\mu * \nu$.

A convolution semigroups on X and its continuity are defined in the same way as on G given in §3. Thus, a family of probability measures μ_t on X, $t \in \mathbb{R}_+$, is called a convolution semigroup on X if $\mu_{s+t} = \mu_s * \mu_t$, and it is called continuous if $\mu_t \to \mu_0$ weakly as $t \to 0$. Recall that if μ_t is a continuous convolution semigroup on G, then

$\mu_t \to \mu_s$ weakly as $t \to s$ for any $s > 0$. The same holds for a continuous convolution semigroup μ_t on X, as an easy consequence of Proposition 12 later.

The following result provides a basic relation between invariant measures on G and on $X = G/K$. Recall a measure μ on G is called K-left or K-right invariant if $l_k\mu = \mu$ or $r_k\mu = \mu$ for $k \in K$, and K-bi-invariant if it is both K-left and K-right invariant.

Proposition 9. *The map*

$$\mu \ \mapsto \ \nu = \pi\mu$$

is a bijection from the set of K-right invariant measures μ on G onto the set of measures ν on X. It is also a bijection from the set of K-bi-invariant measures μ on G onto the set of K-invariant measures ν on X. Moreover, if ν is a measure on X, then the unique K-right invariant measure μ on G satisfying $\nu = \pi\mu$ is given by

$$\forall f \in \mathcal{B}_+(G), \quad \mu(f) = \int_X \int_K f(S(x)k)dk\, \nu(dx), \tag{15}$$

where S is any section map on X. Furthermore, the map $\mu \to \pi\mu$ preserves the convolution in the sense that for any measures μ_1 and μ_2 on G,

$$\pi(\mu_1 * \mu_2) = (\pi\mu_1) * (\pi\mu_2), \tag{16}$$

provided one of the following three conditions holds: μ_1 is K-right invariant, or μ_2 is K-left invariant, or μ_2 is K-conjugate invariant.

Proof. For $g \in G$, $S \circ \pi(g) = gk$ for some $k \in K$. If μ is a K-right invariant measure on G with $\pi\mu = \nu$, then for $f \in \mathcal{B}_+(G)$,

$$\mu(f) = \int_K \mu(f \circ r_k)dk = \int_K \mu(f \circ r_k \circ S \circ \pi)dk = \int_K \nu(f \circ r_k \circ S)dk.$$

This shows that μ satisfies (15). Conversely, using the K-right invariance of dk, it is easy to show that μ given by (15) is K-right invariant with $\pi\mu = \nu$. It is also clear that μ is K-bi-invariant if and only if ν is K-invariant. For two measures μ_1 and μ_2 on G, satisfying one of the three conditions stated above, let $\nu_1 = \pi\mu_1$ and $\nu_2 = \pi\mu_2$. Then for $f \in \mathcal{B}_+(X)$,

$$\begin{aligned}
&\pi(\mu_1 * \mu_2)(f) = \int f(\pi(g_1g_2))\mu_1(dg_1)\mu_2(dg_2) \\
= &\int f(\pi(g_1kg_2))\mu_1(dg_1)\rho_K(dk)\mu_2(dg_2) \\
= &\int f(g_1k\pi(g_2))\mu_1(dg_1)\rho_K(dk)\mu_2(dg_2) = \int f(gky)\mu_1(dg)\rho_K(dk)\nu_2(dy) \\
= &\int f(S(\pi(g))k'ky)\mu_1(dg)\rho_K(dk)\nu_2(dy) \quad \text{(for some } k' \in K) \\
= &\int f(S(x)ky)\nu_1(dx)\rho_K(dk)\, \nu_2(dy) = \nu_1 * \nu_2(f).
\end{aligned}$$

□

Proposition 10. *Let ρ_G be a left Haar measure on G.*
(a) ρ_G is K-right invariant.
(b) $\rho_X = \pi\rho_G$ is a G-invariant measure on $X = G/K$, and any G-invariant measure on X is $c\rho_X$ for some constant $c \geq 0$. In particular, if G is compact and if ρ_G is the normalized Haar measure on G, then ρ_X is the unique G-invariant probability measure on X.

Proof. For $k \in K$, $r_k\rho_G$ is left invariant and so $r_k\rho_G = \lambda(k)\rho_G$ for some $\lambda(k) > 0$. For $f \in C_c(G)$, $\rho_G(f \circ r_k) = \lambda(k)\rho_G(f)$. This shows $k \mapsto \lambda(k)$ is continuous. Then $k \mapsto \lambda(k^{-1})$ is a continuous group homomorphism from K into the multiplicative group $(0, \infty)$. Its range as a compact subgroup is necessarily $\{1\}$, and hence $r_k\rho_G = \rho_G$. This proves (a). It is clear that $\rho_X = \pi\rho_G$ is G-invariant. For any G-invariant measure ν on X, the unique K-right invariant measure μ on G with $\pi\mu = \nu$, given by (15), is clearly left invariant, and so $\mu = c\rho_G$ for some constant $c \geq 0$. This implies $\nu = c\rho_X$ and proves (b). □

In the literature (for example, in [14]), the convolution of measures in $X = G/K$ has been defined by identifying the measures on X with the K-right invariant measures in G and then use the convolution on G. By Proposition 9, this definition is consistent with our definition given in (13).

Convolution of functions on $X = G/K$ have appeared in literature under various contexts, but in the present setting, they all take the following form (see for example [41]): For $f_1, f_2 \in \mathcal{B}_+(X)$,

$$f_1 * f_2(gK) = \int_G f_1(hK)f_2(h^{-1}gK)\rho(dh), \quad g \in G, \tag{17}$$

where ρ denotes a left Haar measure on G. This definition is compatible with our definition of convolution of measures on X by the following proposition.

Proposition 11. *If μ_1 and μ_2 are measures on $X = G/K$ with densities f_1 and f_2 with respect to $\pi\rho$, then $\mu_1 * \mu_2$ has density $f_1 * f_2$.*

Proof. By Proposition 10 (a), ρ is K-right invarint. For $f \in \mathcal{B}_+(X)$, writing dg and dh for $\rho(dg)$ and $\rho(dh)$ respectively,

$$\mu_1 * \mu_2(f) = \int f(S(x)ky)dk\mu(dx)\nu(dy) = \int f(S(hK)kgK)dkf_1(hK)f_2(gK)dhdg$$
$$= \int f(hgK)f_1(hK)f_2(gK)dhdg = \int_G f(gK)[\int_G f_1(hK)f_2(h^{-1}gK)dh]dg$$
$$= \int f(gK)(f_1 * f_2)(gK)dg.$$

□

A probability measure μ on G satisfying $\mu * \mu = \mu$ is called an idempotent. By [18, Theorem 1.2.10], if μ is an idempotent, then $\mu = \rho_H$ for some compact subgroup H of G. For a convolution semigroup μ_t on G, $\mu_0 * \mu_0 = \mu_0$, so $\mu_0 = \rho_H$ for some compact subgroup H of G. Then μ_t is H-bi-invariant because $\mu_t = \mu_0 * \mu_t = \mu_t * \mu_0$.

Proposition 12. *(a) If a convolution semigroup μ_t on G is K-right invariant, that is, if each μ_t is K-right invariant, then it is K-bi-invariant.*
(b) If ν_t is a convolution semigroup on $X = G/K$, then each ν_t is K-invariant, and $\nu_0 = \pi\rho_H$ for some compact subgroup H of G containing K.
(c) The map

$$\mu_t \;\mapsto\; \nu_t = \pi\mu_t$$

is a bijection from the set of K-bi-invariant convolution semigroups μ_t on G onto the set of convolution semigroups ν_t on $X = G/K$. Moreover, μ_t is continuous if and only if so is ν_t.

Proof. By the preceding discussion, $\mu_0 = \rho_H$ and μ_t is H-bi-invariant for some compact subgroup H of G. The K-right invariance of μ_0 implies $K \subset H$. This proves (a). Let ν_t be a convolution semigroup on X, and let μ_t be the unique K-right invariant probability measure on G with $\pi\mu_t = \nu_t$. By Proposition 9 and (a), μ_t is a K-bi-invariant convolution semigroup on G, and hence $\nu_t = \pi\mu_t$ is K-invariant. Moreover, $\mu_0 = \rho_H$ for some compact subgroup H of G. Because μ_0 is K-bi-invariant, $K \subset H$. This proves (b). Now (c) follows from (b) and Proposition 9. Note that to derive the continuity of μ_t from that of ν_t, (15) is used, where $\int_K f(S(x)k)\rho_K(dk)$ is continuous in x if f is bounded continuous on G. This is because the integral does not depend on S, and S may be chosen to be continuous near any x (that is, in a neighborhood of x). □

Because a left invariant rcll Markov process in G with an infinite life time is a Lévy process in G, a G-invariant rcll Markov process x_t in $X = G/K$ with an infinite life time will be called a Lévy process in X. Let P_t be its transition semigroup. By the G-invariance of P_t, it is clear that $\mu_t = P_t(o,\cdot)$, $t \in \mathbb{R}_+$, are K-invariant measures. We now show that they form a continuous convolution semigroup on G with $\mu_0 = \delta_o$, which will be called the convolution semigroup associated to the Lévy process x_t. For $f \in \mathcal{B}_b(X)$ and $s, t \in \mathbb{R}_+$,

$$\begin{aligned} \mu_{s+t}(f) = P_{s+t}f(o) = P_sP_tf(o) = \int_X \mu_s(dx)P_tf(x) = \int_X \mu_s(dx)P_tf(S(x)o) \\ = \int_X \mu_s(dx)P_t(f\circ S(x))(o) \quad \text{(by the G-invariance of P_t)} \\ = \int_{X\times X} \mu_s(dx)\mu_t(dy)f(S(x)y) = \mu_s * \mu_t(f). \end{aligned}$$

This shows that $\mu_t = P_t(o,\cdot)$, $t \in \mathbb{R}_+$, form a convolution semigroup on X, which is continuous by the right continuity of x_t.

On the other hand, given a continuous convolution semigroup μ_t on X with $\mu_0 = \delta_o$, let

$$P_tf(x) = \int f(S(x)y)\mu_t(dy) \tag{18}$$

for $t \in \mathbb{R}_+$, $x \in X$ and $f \in \mathcal{B}_+(X)$. Because μ_t is K-invariant, this expression does not depend on the choice for the section map S, and defines a conservative G-invariant Feller transition semigroup P_t on X (recall that S may be chosen to be continuous near any point in X). The associated Feller process x_t is a G-invariant rcll Markov process with an infinite life time, and hence is a Lévy process in X with μ_t as the associated convolution semigroup.

For a G-invariant rcll Markov process x_t in X with a possibly finite life time, it can be shown as on G that its transition semigroup P_t satisfies $P_t1(x) = e^{-\lambda t}$ for some $\lambda \geq 0$, and hence x_t is equal in distribution to a Lévy process killed at an independent exponential time of rate λ. To summarize, we have the following result.

Theorem 13. *Let x_t be a Lévy process in $X = G/K$, that is, a G-invariant rcll Markov process with an infinite life time, and let P_t be its transition semigroup. Then $\mu_t = P_t(o,\cdot)$ is a continuous convolution semigroup on X with $\mu_0 = \delta_o$. conversely, if μ_t is such a convolution semigroup on X, then there is a Lévy process x_t in X with $x_0 = o$ such that μ_t is the distribution of x_t.*

In general, a G-invariant rcll Markov process x_t in X with a possibly finite life time is identical in distribution to a Lévy process $\hat{x}_t$ in X killed at an independent exponential time of rate $\lambda \geq 0$. The transition semigroup P_t of x_t and $\hat{P}_t$ of $\hat{x}_t$ are related as $P_t = e^{-\lambda t}\hat{P}_t$.

Although there is no natural product structure on the homogeneous space $X = G/K$, the integrals like

$$\begin{aligned} \int f(xy)\mu(dy) &= \int f(S(x)y)\mu(dy), \\ \int f(xy, xyz)\mu(dy)\nu(dz) &= \int f(S(x)y, S'(S(x)y)z)\mu(dy)\nu(dz) \\ &= \int f(S(x)y,\, S(x)S'(y)z)\mu(dx)\nu(dy), \end{aligned} \tag{19}$$

are well defined for K-invariant measures μ and ν on X, that is, they do not depend on the choice for section maps S and S', noting that $S'(S(x)y) = S(x)S'(y)k_{x,y}$ for some $k_{x,y} \in K$. In this notation, the formula (12) for the finite dimensional distributions of a Lévy process in G holds also for a Lévy process x_t in $X = G/K$. This follows from the Markov property of x_t and (18).

The following result says that a Lévy process in X may also be characterized by independent and stationary increments, just like a Lévy process in G.

Theorem 14. *Let x_t be a rcll process in $X = G/K$ with an infinite life time and let $\{\mathcal{F}_t^x\}$ be its natural filtration. If x_t is a Lévy process with associated convolution semigroup μ_t, then for any section map S and $s < t$, $x_s^{-1}x_t = S(x_s)^{-1}x_t$ is independent of $\mathcal{F}_s^x$ and has distribution μ_{t-s}. Consequently, this distribution is K-invariant, and depends only on $t - s$ and not on the choice of S. Conversely, if for some section map S and any $s < t$, $x_s^{-1}x_t = S(x_s)^{-1}x_t$ is independent of $\mathcal{F}_s^x$, and its distribution is K-invariant and depends only on $t - s$, then x_t is a Lévy process in X.*

Proof. Let x_t be a Lévy process in X with associated convolution semigroup μ_t. Then from its finite dimensional distributions given in (12), it can be shown that $x_s^{-1}x_t = S(x_s)^{-1}x_t$ is independent of $\mathcal{F}_s^x$ and its distribution is μ_{t-s}. For simplicity, we will prove that for $r < s < t$, $x_s^{-1}x_t$ is independent of x_r and has distribution μ_{t-s}. Fix an arbitrary section map S, we will write $x_1x_2\cdots x_{k-1}x_k$ for $S(x_1)S(x_2)\cdots S(x_{k-1})x_k$, and x^{-1} for $S(x)^{-1}$. By (12), for $f, g \in \mathcal{B}_+(X)$,

$$\begin{aligned} & E[f(x_r)g(x_s^{-1}x_t)] \\ = & \int \mu_0(dx_0)\mu_r(dx_1)\mu_{s-r}(dx_2)\mu_{t-s}(dx_3)f(x_0x_1)g((x_0x_1x_2)^{-1}x_0x_1x_2x_3)) \\ = & \int \mu_0(dx_0)\mu_r(dx_1)\mu_{s-r}(dx_2)\mu_{t-s}(dx_3)f(x_0x_1)g(x_3) \\ = & E[f(x_r)]E[g(x_{t-s})]. \end{aligned}$$

This shows that $S(x_s)^{-1}x_t$ is independent of x_r and has the distribution μ_{t-s}. There is no difficulty in this proof to replace x_r by $(x_{r_1}, x_{r_2}, \ldots, x_{r_k})$ for $r_1 < r_2 < \cdots < r_k \le s$, except the expressions will be much longer. Conversely, assume for some section map S and any $s < t$, $x_s^{-1}x_t = S(x_s)^{-1}x_t$ is independent of $\mathcal{F}_s^x$, and its distribution, denoted as $\mu_{s,t}$, is K-invariant and depends only on $t - s$. Then for $f \in \mathcal{B}_+(X)$,

$$\begin{aligned} E[f(x_t) \mid \mathcal{F}_s^x] &= E[f(S(x_s)S(x_s)^{-1}x_t) \mid \mathcal{F}_s^x] = \int_X f(S(x_s)y)\mu_{s,t}(dy) \\ &= \int_X f(S(x_s)y)\mu_{0,t-s}(dy). \end{aligned}$$

Because $\mu_{s,t}$ is K-invariant, by (14), $\mu_{r,s} * \mu_{s,t} = \mu_{r,t}$ for $r < s < t$. Because $\mu_{s,t}$ depends only on $t - s$, it then is easy to show that $\mu_t = \mu_{0,t}$ is a continuous convolution semigroup on G and x_t is a G-invariant Markov process in X with associated convolution semigroup μ_t, and hence x_t is a Lévy process in X. □

We note that by Theorem 14, for a Lévy process x_t in X and a section map S on X,

$$x_t^o = S(x_0)^{-1} x_t \tag{20}$$

is a Lévy process in X with $x_0^o = o$ and is independent of x_0, and its distribution does not depend on the choice for S. Moreover, the process x_t is equal in distribution to $S(x_0)x_t^o$.

Remark 15. Let x_t be a G-invariant Markov process in $X = G/K$ with transition semigroup P_t and let $\mu_t = P_t(o, \cdot)$. Then $P_t f(x) = P_t(f \circ S(x))(e) = \int_G f(S(x)y)\mu_t(dy)$. If $\mu_t \to \delta_o$ weakly as $t \to 0$, then P_t is Feller and hence x_t has a rcll version. In this case, if x_t has an infinite life time, then this version of x_t is a Lévy process, and μ_t is a continuous convolution semigroup on X.

The following three results deal with some simple relations between processes in G and in $X = G/K$.

Proposition 16. *Let g_t be a Markov process in G. Assume its transition semigroup P_t is K-right invariant, that is, $(P_t f) \circ r_k = P_t(f \circ r_k)$ for $f \in \mathcal{B}_b(G)$ and $k \in K$. Then $x_t = \pi(g_t) = g_t o$ is a Markov process in $X = G/K$ with transition semigroup*

$$Q_t f(x) = P_t(f \circ \pi)(S(x)), \qquad x \in X \quad \text{and} \quad f \in \mathcal{B}_b(X), \tag{21}$$

which does not depend on the choice for the section map S. Moreover, if g_t is a Feller process in G, then so is x_t in X.

Proof. The K-right invariance of P_t shows that Q_t defined by (21) does not depend on the choice for S. Note that (21) may be written as $(Q_t f) \circ \pi = P_t(f \circ \pi)$. Then

$$\begin{aligned} Q_{s+t} f(x) &= P_{s+t}(f \circ \pi)(S(x)) = P_s[P_t(f \circ \pi)](S(x)) = P_s[(Q_t f) \circ \pi](S(x)) \\ &= (Q_s Q_t f) \circ \pi(S(x)) = Q_s Q_t f(x). \end{aligned}$$

This shows that Q_t is a transition semigroup. Since S may be chosen to be continuous near any fixed point x in X, it is easy to see that if P_t is a Feller transition semigroup on G, then so is Q_t on X. The Markov property of x_t follows from $E[f(x_{s+t}) \mid \mathcal{F}_s^g] = E[f \circ \pi(g_{s+t}) \mid \mathcal{F}_s^g] = P_t(f \circ \pi)(g_s) = (Q_t f) \circ \pi(g_s) = Q_t f(x_s)$. □

Proposition 17. *Let g_t be a right Lévy process in G as defined in Remark 8, that is, a rcll Markov process with a right invariant transition semigroup P_t and an infinite life time. Let $\mu_t = P_t(e, \cdot)$ be the associated convolution semigroup. Then for any $z \in X = G/K$, $x_t = g_t z$ is a Feller process in X with transition semigroup Q_t given by (21). Moreover,*

$$Q_t f(x) = \int_G f(gx)\mu_t(dg), \qquad x \in X \quad \text{and} \quad f \in \mathcal{B}_b(X). \tag{22}$$

Proof. Because $g_t = g_t^e g_0$ and $g_t z = g_t^e g_0 z$, replacing z by $g_0 z$, we may assume $g_0 = e$. If the reference point o in X is replaced by z, then the natural projection π: $G \to X$ given by $g \mapsto go$ should be replaced by π_z: $G \to X$ given by $g \mapsto gz$. If S is a section map on X with respect to o, then $S_z(\cdot) = S(\cdot)S(z)^{-1}$ is a section with respect to z because $S_z(x)z = S(x)S(z)^{-1}z = S(x)o = x$ for any $x \in X$. Because P_t is right invariant, by Proposition 16, $x_t = g_t z$ is a Feller process in X with transition semgroup

$$\begin{aligned} &Q_t f(x) = P_t(f \circ \pi_z)(S(x)S(z)^{-1}) = P_t(f \circ \pi_z \circ r_{S(x)S(z)^{-1}})(e) \\ = &\int_G P_t(e, dg) f(gS(x)S(z)^{-1}z) = \int_G \mu_t(dg) f(gS(x)o) = \int_G \mu_t(dg) f(gx). \end{aligned}$$

This proves (22). From above, $Q_t f(x) = \int_G \mu_t(dg) f(gS(x)o) = \int_G P_t(e, dg) f(gS(x)o) = \int_G P_t(S(x), dg) f(go) = P_t(f \circ \pi)(S(x))$. This is (21). □

Recall that for $g \in G$, c_g: $G \to G$ is the conjugation map $x \mapsto gxg^{-1}$. A Lévy process g_t in G is called K-conjugate invariant if its transition semigroup is K-conjugate invariant, that is, if $P_t(f \circ c_k) = (P_t f) \circ c_k$ for $f \in \mathcal{B}_b(G)$ and $k \in k$. By the left invariance of P_t, this is equivalent to the K-right invariance of P_t. In terms of the process, this is the same as saying that for any $t \in \mathbb{R}_+$ and $k \in K$, $kg_t^e k^{-1} = g_t^e$ in distribution.

Theorem 18. *Let g_t be an K-conjugate invariant Lévy process in G. Then $x_t = g_t o$ is a Lévy process in $X = G/K$ with transition semigroup Q_t given by (21).*

Note: It can be shown that any Lévy process in $X = G/K$ may be obtained from an K-conjugate invariant Lévy process in G as in Theorem 18 when G is a Lie group, see [25, Theorem 2.2].

Proof of Theorem 18. By Proposition 16, $x_t = g_t o$ is a rcll Markov process in X with transition semigroup Q_t and infinite life time. It remains to show that Q_t is G-invariant. This follows from the left invariance and K-right invariance of P_t, because for $f \in \mathcal{B}_b(X)$ and $g \in G$, $Q_t(f \circ g)(x) = P_t(f \circ g \circ \pi)(S(x)) = P_t(f \circ \pi \circ l_g)(S(x)) = P_t(f \circ \pi)(gS(x)) = P_t(f \circ \pi)(S(gx)k)$ (for some $k \in K$) $= P_t(f \circ \pi \circ r_k)(S(gx)) = P_t(f \circ \pi)(S(gx)) = Q_t f(gx)$. □

5. Inhomogeneous Lévy processes

As before, let G be a topological group equipped with a lcscH topology. An inhomogeneous Markov process x_t in G with transition semigroup $P_{s,t}$ is called left invariant if it is invariant under the action of left translations, that is, if $P_{s,t}(f \circ l_g) = (P_{s,t} f) \circ l_g$ for $g \in G$ and $f \in \mathcal{B}_b(G)$.

Assume x_t has an infinite life time. By the simple Markov property and the left invariance, for $s < t$ and $f \in \mathcal{B}_b(G)$,

$$E[f(x_s^{-1} x_t) \mid \mathcal{F}_s^x] = P_{s,t}(f \circ l_{x_s^{-1}})(x_s) = P_{s,t} f(e).$$

This shows that $x_s^{-1} x_t$ is independent of $\mathcal{F}_s^x$ and has the distribution $P_{s,t}(e, \cdot)$. Thus, like a left invariant homogeneous Markov process considered in §3, x_t has independent increments. However, these increments are in general not stationary.

A rcll process x_t in G, with an infinite life time and independent increments, will be called an inhomogeneous Lévy process. In contrast, the Lévy processes defined in §3 may be called homogeneous. Note that the class of inhomogeneous Lévy processes includes homogeneous ones as special cases. As for homogeneous Lévy processes, it is easy to show that an inhomogeneous Lévy process x_t is a left invariant inhomogeneous Markov process. To summarize, we have the following result.

Theorem 19. *The class of left invariant inhomogeneous Markov processes in G, that have rcll paths and infinite life times, coincides with the class of inhomogeneous Lévy processes in G. For such a process x_t with transition semigroup $P_{s,t}$, $P_{s,t}(e, \cdot)$ is the distribution of the increment $x_s^{-1} x_t$, $0 \le s \le t$.*

A family of probability measures $\mu_{s,t}$ on G, $0 \le s \le t$, is called a two-parameter convolution semigroup on G if $\mu_{r,t} = \mu_{r,s} * \mu_{s,t}$ for $r \le s \le t$. It is called right continuous if $\mu_{s,t} \to \mu_{u,v}$ weakly as $s \downarrow u$ and $t \downarrow v$. It is called continuous if $\mu_{s,t}$ is continuous in (s, t) under the weak convergence. Note that if μ_t is a continuous convolution semigroup, then $\mu_{s,t} = \mu_{t-s}$ is a continuous two-parameter convolution semigroup. For simplicity, a two-parameter convolution semigroup may also be called a convolution semigroup.

It is easy to see that if x_t is an inhomogeneous Lévy process in G, then by the rcll paths, the distributions $\mu_{s,t}$ of its increments $x_s^{-1} x_t$, $s \le t$, form a right continuous convolution semigroup with $\mu_{t,t} = \delta_e$ for all $t \ge 0$, which is called the convolution semigroup associated to process x_t. Moreover, $\mu_{s,t}$ is continuous if and only if x_t is stochastically continuous.

Now let $\mu_{s,t}$ be a continuous (two-parameter) convolution semigroup on G with $\mu_{t,t} = \delta_e$. For $u, t \in \mathbb{R}_+$, $x \in G$ and $f \in C_0(\mathbb{R}_+ \times G)$, let

$$R_t f(u,x) = \int_G f(u+t, xy)\mu_{u,u+t}(dy). \tag{23}$$

It is easy to show that this defines a conservative Feller transition semigroup R_t on the product space $\mathbb{R}_+ \times G$. Let z_t be the associated Feller process in $\mathbb{R}_+ \times G$. Then z_t has an infinite life time, and given $z_0 = (u,x)$, $z_t = (u+t, y_t)$ is such that for $f \in C_c(G)$, $E_{(u,x)}[f(y_t)] = \int_G f(xy)\mu_{u,u+t}(dy)$. Let $x_t = y_{t-u}$ for $t \geq u$. Then x_t is a left invariant inhomogeneous Markov process with an infinite life time, and its transition semigroup $P_{s,t}$ is given by $P_{s,t}(e,\cdot) = \mu_{s,t}$, and hence x_t is an inhomogeneous Lévy process associated to the convolution semigroup $\mu_{s,t}$. As a component of a Feller process, x_t is stochastically continuous.

To summarize, we have established the following result.

Theorem 20. *Let $\mu_{s,t}$, $0 \leq s \leq t$, be a two-parameter convolution semigroup on G with $\mu_{t,t} = \delta_e$. Then $\mu_{s,t}$ is associated to a stochastically continuous inhomogeneous Lévy process in G if and only if $\mu_{s,t}$ is continuous.*

By the Markov property, it can be shown that the finite dimensional distributions of an inhomogeneous Lévy process x_t in G with initial distribution μ_0 and associated convolution semigroup $\mu_{s,t}$ are given by

$$\begin{aligned} E[f(x_{t_1}, x_{t_2}, \ldots, x_{t_n})] &= \int f(x_0x_1, x_0x_1x_2, \ldots, x_0x_1x_2\cdots x_n) \\ &\quad \mu_0(dx_0)\mu_{0,t_1}(dx_1)\mu_{t_1,t_2}(dx_2)\cdots\mu_{t_{n-1},t_n}(dx_n) \end{aligned} \tag{24}$$

for $0 \leq t_1 < t_2 < \cdots < t_n$ and $f \in \mathcal{B}_b(G^n)$.

Remark 21. Let x_t be a process in G with an infinite life time. By the proof of Theorem 19, x_t is a left invariant inhomogeneous Markov process if and only if it has independent increments. In this case, if for any $s \geq 0$, $x_s^{-1}x_t \to e$ in distribution as $t \to s$, then the distributions $\mu_{s,t}$ of $x_s^{-1}x_t$, $s \leq t$, form a continuous convolution semigroup with $\mu_{t,t} = \delta_e$ for $t \geq 0$. It follows that the transition semigroup R_t of the process $z_t = (u+t, x_{u+t})$, given in (23), is Feller, and hence x_t has a rcll version that is an inhomogeneous Lévy process in G.

More generally, let x_t be a left invariant inhomogeneous Markov process in G without assuming an infinite life time, and let $P_{s,t}$ be its transition semigroup. Then $P_{s,t}f(x) = P_{s,t}(f \circ l_x)(e) = \int_G f(xy)\mu_{s,t}(dy)$, where $\mu_{s,t} = P_{s,t}(e,\cdot)$. Assume $\mu_{s,t}$ is continuous in (s,t) under the weak convergence. Although $\mu_{s,t}$ is a sub-probability, R_t defined by (23) is still a Feller transition semigroup on $\mathbb{R}_+ \times G$, and hence x_t has a rcll version.

Recall that a Radon measure has a finite charge on any compact set. A measure is called diffuse if it does not charge points. The reader is referred to [21, Chapter 12] for the standard definition of a Poisson random measure.

Theorem 22. *Let x_t be a left invariant inhomogeneous Markov process in G with a possibly finite life time and transition semigroup $P_{s,t}$. Assume x_t has rcll paths and is stochastically continuous. Then there are a diffuse Radon measure λ on $\mathbb{R}_+$ and a stochastically continuous inhomogeneous Lévy process $\hat{x}_t$ in G with transition semigroup $\hat{P}_{s,t}$ such that*

$$P_{s,t} = e^{-\lambda((s,t])}\hat{P}_{s,t}. \tag{25}$$

This means that process x_t is equal in distribution to $\hat{x}_t$ killed at the random time point of an independent Poisson random measure ξ on $\mathbb{R}_+$ with intensity measure λ. More precisely, if for any starting time s, let τ be the first point of ξ on (s, ∞), and let $x'_t = \hat{x}_t$ for $s \leq t < \tau$ and $x'_t = \Delta$ for $t \geq \tau$, then x'_t is an inhomogeneous Markov process with transition semigroup $P_{s,t}$.

Proof. To prove (25), note that by the left invariance and the semigroup property of transition semigroup, $P_{s,t}1 = P_{s,t}1(e)$ and for $r < s < t$, $P_{r,t}1(e) = \int P_{r,s}(e, dx)P_{s,t}1(x) = P_{r,s}1(e)P_{s,t}1(e)$. By the right continuity of $P_{s,t}1(e)$ in t, it follows that there is a Radon measure λ on $\mathbb{R}_+$ such that $P_{s,t}1(e) = e^{-\lambda((s,\, t])}$. Let $\hat{P}_{s,t} = e^{\lambda((s,\, t])}P_{s,t}$. Then $P_{s,t}$ is a conservative two-parameter transition semigroup and $\hat{\mu}_{s,t} = \hat{P}_{s,t}(e, \cdot)$ is a two-parameter convolution semigroup of probability measures on G. By the stochastic continuity of x_t, λ is a diffuse measure and hence $\hat{\mu}_{s,t}$ is continuous. By Theorem 20, there is a stochastically continuous inhomogeneous Lévy process $\hat{x}_t$ in G associated to $\hat{\mu}_{s,t}$. This proves (25). □

As for a homogeneous Lévy process, we define an inhomogeneous Lévy process x_t to be associated to a filtration $\{\mathcal{F}_t\}$ if x_t is adapted to $\{\mathcal{F}_t\}$ and for $s < t$, $x_s^{-1}x_t$ is independent of $\mathcal{F}_s$. This is equivalent to saying that x_t is associated to the filtration $\{\mathcal{F}_t\}$ as an inhomogeneous Markov process.

Let x_t be an inhomogeneous Lévy process x_t in G with associated convolution semigroup $\mu_{s,t}$. If it is associated to a filtration $\{\mathcal{F}_t\}$, then for fixed $r \in \mathbb{R}_+$, the process $x'_t = x_r^{-1}x_{r+t}$, $t \in \mathbb{R}_+$, is an inhomogeneous Lévy process with associated convolution semigroup $\nu_{s,t} = \mu_{r+s,r+t}$, and is independent of $\mathcal{F}_r$. In a certain sense, this holds when r is replaced by a stopping time. The following result is an inhomogeneous analog of Theorem 6.

Theorem 23. *Let x_t be an inhomogeneous Lévy process in G with associated convolution semigroup $\mu_{s,t}$. Assume it is associated to a filtration $\{\mathcal{F}_t\}$. If τ is an $\{\mathcal{F}_t\}$-stopping time, then for $t_1 < t_2 < \cdots < t_n$ and $f \in \mathcal{B}_b(G^n)$,*

$$\begin{aligned} & E[f(x_\tau^{-1}x_{\tau+t_1}, x_\tau^{-1}x_{\tau+t_2}, \ldots, x_\tau^{-1}x_{\tau+t_n})1_{[\tau<\infty]} \mid \mathcal{F}_\tau] \\ = & \int f(x_1, x_1x_2, \ldots, x_1x_2\cdots x_n) \\ & \qquad\qquad \mu_{\tau,\tau+t_1}(dx_1)\mu_{\tau+t_1,\tau+t_2}(dx_2)\cdots\mu_{\tau+t_{n-1},\tau+t_n}(dx_n)1_{[\tau<\infty]}. \end{aligned}$$

This implies that under the conditional distribution given τ, $x'_t = x_\tau^{-1}x_{\tau+t}$ is an inhomogeneous Lévy process in G with associated convolution semigroup $\nu_{s,t} = \mu_{\tau+s,\tau+t}$, and is independent of $\mathcal{F}_\tau$. More precisely, this means that for any bounded $\mathcal{F}_\tau$-measurable H,

$$\begin{aligned} & E[Hf(x_\tau^{-1}x_{\tau+t_1}, x_\tau^{-1}x_{\tau+t_2}, \ldots, x_\tau^{-1}x_{\tau+t_n})1_{[\tau<\infty]} \mid \sigma(\tau)] \\ = & E[H \mid \sigma(\tau)] \int f(x_1, \ldots, x_1x_2\cdots x_n)\mu_{\tau,\tau+t_1}(dx_1)\cdots\mu_{\tau+t_{n-1},\tau+t_n}(dx_n)1_{[\tau<\infty]}. \end{aligned}$$

Proof. We may assume $f \in C_b(G^n)$. As in the proof of Theorem 6, first assume τ has discrete values. Then

$$\begin{aligned} & E[f(x_\tau^{-1}x_{\tau+t_1}, x_\tau^{-1}x_{\tau+t_2}, \ldots, x_\tau^{-1}x_{\tau+t_n})1_{[\tau<\infty]} \mid \mathcal{F}_\tau] \\ = & \sum_{t<\infty} E[f(x_t^{-1}x_{t+t_1}, x_t^{-1}x_{t+t_2}, \ldots, x_t^{-1}x_{t+t_n}) \mid \mathcal{F}_t]1_{[\tau=t]} \\ = & \sum_{t<\infty} \int f(x_1, x_1x_2, \ldots, x_1x_2\cdots x_n) \\ & \qquad \mu_{t,t+t_1}(dx_1)\mu_{t+t_1,t+t_2}(dx_2)\cdots\mu_{t+t_{n-1},t+t_n}(dx_n)1_{[\tau=t]} \\ = & \int f(x_1, x_1x_2, \ldots, x_1x_2\cdots x_n) \\ & \qquad \mu_{\tau,\tau+t_1}(dx_1)\mu_{\tau+t_1,\tau+t_2}(dx_2)\cdots\mu_{\tau+t_{n-1},\tau+t_n}(dx_n)1_{[\tau<\infty]}. \end{aligned}$$

For a general stopping time τ, choose discrete stopping times $\tau_m \downarrow \tau$. The result follows after taking the limit of the above expression with $\tau = \tau_m$ as $m \to \infty$ and using the right continuity of $\mu_{s,t}$. □

The basic theory of inhomogeneous Lévy processes in a homogeneous space $X = G/K$ may be developed parallel to the case of Lévy processes in G/K discussed in §4 with suitable modifications, assuming K is compact and G/K has a continuous local section. Thus, a G-invariant inhomogeneous Markov process x_t in X, that has rcll paths and an infinite life time, will be called an inhomogeneous Lévy process in X. Let $P_{s,t}$ be its transition semigroup and let $\mu_{s,t} = P_{s,t}(o, \cdot)$. Then $\mu_{s,t}$ are K-invariant, and by the Markov property and the G-invariance of $P_{s,t}$, it can be shown that the finite dimensional distributions of x_t are given in (24) if the integral like $\int f(xy, xyz)\mu(dy)\nu(dz)$ is understood in the sense of (19) with a choice of a section map S (but not dependent on the choice).

By (24), it can be shown that an inhomogeneous Lévy process in X can be characterized by independent increments just like an inhomogeneous Lévy process in G. This is similar to the homogeneous case, see the proof of Theorem 14. The result is summarized below.

Theorem 24. *Let x_t be a rcll process in $X = G/K$ with an infinite life time and let $\{\mathcal{F}_t^x\}$ be its natural filtration. If x_t is an inhomogeneous Lévy process with associated transition semigroup $P_{s,t}$, then for any section map S and $s < t$, $x_s^{-1}x_t = S(x_s)^{-1}x_t$ is independent of $\mathcal{F}_s^x$, and its distribution is $\mu_{s,t} = P_{s,t}(o, \cdot)$, so is K-invariant and does not depend on the choice for S. Conversely, if for some section map S and any $s < t$, $x_s^{-1}x_t = S(x_s)^{-1}x_t$ is independent of $\mathcal{F}_s^x$ and its distribution is K-invariant, then x_t is an inhomogeneous Lévy process in X.*

A (two-parameter) convolution semigroup $\mu_{s,t}$ on $X = G/K$, $0 \le s \le t$, and its continuity and right continuity, are defined just as on G. For a convolution semigroup $\mu_{s,t}$ on G, because for any t, $\mu_{t,t} * \mu_{t,t} = \mu_{t,t}$, it follows that $\mu_{t,t} = \rho_H$ for some compact subgroup H of G (which may depend on t). Because $\mu_{s,t} = \mu_{s,s}*\mu_{s,t} = \mu_{s,t}*\mu_{t,t}$, it follows that if each $\mu_{s,t}$ is K-right invariant, then each $\mu_{s,t}$ is K-bi-invariant. The following simple result may be derived from Proposition 9 in the same way as Proposition 12.

Proposition 25. *(a) If $\mu_{s,t}$ is a convolution semigroup on G such that each measure $\mu_{s,t}$ is K-right invariant, then each $\mu_{s,t}$ is K-bi-invariant.*
(b) If $\nu_{s,t}$ is a convolution semigroup on $X = G/K$, then each $\nu_{s,t}$ is K-invariant.
(c) The map

$$\mu_{s,t} \ \mapsto\ \nu_{s,t} = \pi\mu_{s,t}$$

is a bijection from the set of K-bi-invariant convolution semigroups $\mu_{s,t}$ on G onto the set of convolution semigroups $\nu_{s,t}$ on $X = G/K$. Moreover, $\mu_{s,t}$ is continuous (resp. right continuous) if and only if so is $\nu_{s,t}$.

For an inhomogeneous Lévy process x_t in X, the distribution $\mu_{s,t}$ of its increments $x_s^{-1}x_t = S(x_s)^{-1}x_t$, $s \leq t$, form a right continuous convolution semigroup on X with $\mu_{t,t} = \delta_o$ for all $t \geq 0$ (not dependent on the choice for the section map S), which is called the convolution semigroup associated to process x_t. The right continuity requires a short proof given below.

Because a lcscH space is paracompact, there is a partition of unity $\{\psi_j\}$ on X. This is a collection of functions $\psi_j \in C_c(X)$ with $0 \leq \psi_j \leq 1$ such that $\sum_j \psi_j = 1$ is a locally finite sum, that is, any point of X has a neighborhood on which only finitely many terms ψ_j are nonzero. We may also assume that for each j, there is a section map S_j on X that is continuous on the support of ψ_j. Then by (24), for $s < t$, $f \in C_b(X)$ and a section map S on X,

$$\begin{aligned} \mu_{s,t}(f) &= E[f(S(z_s)^{-1}z_t)] = \sum_j E[\psi_j(z_s)f(S(z_s)^{-1}z_t)] \\ &= \sum_j E[\psi_j(z_s)f(S_j(z_s)^{-1}z_t)] = E[\sum_j \psi_j(z_s)f(S_j(z_s)^{-1}z_t)] \end{aligned} \tag{26}$$

This shows that $\mu_{s,t}(f)$ is right continuous in s and t.

The following two results may be proved as for Theorems 20 and 22, respectively.

Theorem 26. *Let $\mu_{s,t}$ be a convolution semigroup on $X = G/K$ with $\mu_{t,t} = \delta_o$. Then $\mu_{s,t}$ is associated to a stochastically continuous inhomogeneous Lévy process in X if and only if $\mu_{s,t}$ is continuous.*

Theorem 27. *A stochastically continuous G-invariant inhomogeneous Markov process x_t in $X = G/K$, with rcll paths and a possibly finite life time, is identical in distribution to a stochastically continuous inhomogeneous Lévy process $\hat{x}_t$ killed at the random time point of an independent Poisson random measure on $\mathbb{R}_+$ with a Radon intensity measure λ. The transition semigroups $P_{s,t}$ of x_t and $\hat{P}_{s,t}$ of $\hat{x}_t$ are related by (25).*

Remark 28. Let x_t be a G-invariant inhomogeneous Markov process in $X = G/K$. It can be shown as in Remark 21 that if $\mu_{s,t} = P_{s,t}(o, \cdot)$ is continuous in (s,t) under the weak convergence, then x_t has a rcll version. If x_t also has an infinite life time, then this verion is an inhomogeneous Lévy process in X.

As for a Lévy process, an inhomogeneous Lévy process g_t in G is called K-conjugate invariant if its transition semigroup $P_{s,t}$ is K-conjugate invariant, that is, if $P_{s,t}(f \circ c_k) = (P_{s,t}f) \circ c_k$ for $f \in \mathcal{B}_b(G)$ and $k \in K$. This is equivalent to saying that for any $t > s \geq 0$ and $k \in K$, $k(g_s^{-1}g_t)k^{-1} \stackrel{d}{=} g_s^{-1}g_t$ (equal in distribution). For $g_0 = e$, this is also equivalent to saying that for any $k \in K$, the two processes g_t and kg_tk^{-1} are equal in distribution. The following result can be easily proved as for Theorem 18, and its converse holds on a homogeneous space of a Lie group, see [27, Theorem 33].

Theorem 29. *Let g_t be a K-conjugate invariant inhomogeneous Lévy process in G. Then $x_t = g_to$ is an inhomogeneous Lévy process in $X = G/K$. Moreover, the transition semigroups $P_{s,t}$ of g_t and $Q_{s,t}$ of x_t are related by $(Q_{s,t}f) \circ \pi = P_{s,t}(f \circ \pi)$ for $f \in \mathcal{B}_b(X)$.*

6. Markov processes under a non-transitive action

Let X be a topological space under the continuous action of a topological group G, both are assumed to be lcscH, and let x_t be a G-invariant rcll Markov process in X with transition semigroup P_t. Suppose the G-action on X is non-transitive. Then X is a collection of disjoint G-orbits. In this case, under some suitable regularity condition, we may obtain a decomposition of the process into two components, one transversal to the G-orbits and the other along an orbit, with the former preserving the Markov property and the latter preserving the G-invariance.

We first derive a simple result under a general setting. Let X/G be the space of G-orbits and let J: $X \to (X/G)$ be the projection map: $x \mapsto Gx$. Equip X/G with the quotient topology induced by J so that J is continuous and open. Note that if G is compact, then X/G is lcscH.

Theorem 30. *$J(x_t)$ is a rcll Markov process in X/G with transition semigroup Q_t given by*

$$Q_t f(y) = P_t(f \circ J)(x), \quad y = J(x) \in X/G \text{ and } f \in \mathcal{B}_b(X/G) \tag{27}$$

(Q_t does not depend on the choice of x in Gy due to the G-invariance of P_t). Moreover, when G is compact, if x_t is a Feller process, then so is $J(x_t)$.

Proof. For $f \in \mathcal{B}_b(X/G)$ and $y \in (X/G)$,

$$E[f \circ J(x_{t+s}) \mid \mathcal{F}_t^x] = P_s(f \circ J)(x_t) = Q_s f(J(x_t)).$$

This proves that $J(x_t)$ is a Markov process in X/G with transition semigroup Q_t. When G is compact, if $f \in C_0(X/G)$, then $f \circ J \in C_0(X)$ and $(Q_t f) \circ J = P_t(f \circ J) \in C_0(X)$, which implies $Q_t f \in C_0(X/G)$. The Feller property of $J(x_t)$ follows from that of x_t. □

Let Y be a topological subspace of X that is transversal to the action of G in the sense that it intersects each G-orbit at exactly one point, that is,

$$\forall y \in Y, \quad (Gy) \cap Y = \{y\} \quad \text{and} \quad X = \cup_{y \in Y} Gy. \tag{28}$$

Then Y will be called a transversal subspace of X (under the G-action). Let J_1: $X \to Y$ be the projection map $J_1(x) = y$ for $x \in Gy$. Note that Y is naturally identified with the orbit space X/G via the map $y \mapsto Gy$ and J_1 identified with J, but the subspace topology on Y may not be the same as the quotient topology on X/G induced by J. However, when G is compact and Y is closed in X, then the two topologies on Y agree and J_1 is continuous.

In the rest of this section, we will assume the two topologies agree even when G is not compact. Then J_1 is continuous. The following result is an immediate consequence of Theorem 30.

Theorem 31. *$y_t = J_1(x_t)$ is a rcll Markov process in the transversal subspace Y with transition semigroup Q_t given by*

$$Q_t f(y) = P_t(f \circ J_1)(y), \quad y \in Y \text{ and } f \in \mathcal{B}_b(Y). \tag{29}$$

Moreover, if G is compact and if x_t is a Feller process in X, then y_t is a Feller process in Y.

Now assume the isotropy subgroup of G at every point $y \in Y$ is the same compact subgroup K of G. This assumption is often satisfied if the transversal subspace Y is properly chosen. For example, consider $\mathbb{R}^d$ without the origin under the action of the group $O(d)$ of orthogonal transformations on $\mathbb{R}^d$. The $O(d)$-orbits are spheres centered at the origin, and any curve from the origin to infinity is transversal if the curve intersects each of these spheres only once. When the curve is straight, that is, if it is a half line, all its points will have the same isotropy subgroup of $O(d)$.

Let $Z = G/K$. For $y \in Y$ and $z \in Z$, the product $zy = gy$, where $z = gK$, is well defined. For $y \in Y$, zy traces out the G-orbit through y as z varies over Z, and hence Z may be regarded as the standard G-orbit. Note that map F: $(Y \times Z) \to X$ given by $(y, z) \mapsto zy$ is bijective. Because the restricted action map $G \times Y \to X$, $(g, y) \mapsto gy$, is continuous and the projection map $G \times Y \to Y \times Z$, $(g, y) \mapsto (y, gK)$, is open, it follows that F is continuous. If G is compact, then it is easy to show that F has a continuous inverse and hence F: $Y \times Z \to X$ is a homeomorphism.

We will now take this to be part of our assumption. By slightly changing the notation, our general setup may be stated as follows. Let $X = Y \times Z$ be a topological product with $Z = G/K$, where G is a topological group and K is a compact subgroup. Assume both Y and G are lcscH. Let G act on X by its natural action on G/K, that is, $g(y, z) = (y, gz)$ for $g \in G$, $y \in Y$ and $z \in Z$. Then $Y \times \{o\}$ is transversal to the G-action and K is the common isotropy subgroup of G at every point of $Y \times \{o\}$. Let

$$J_1 : X \to Y, \quad \text{and} \quad J_2 : X \to Z \tag{30}$$

be the natural projections $(y, z) \mapsto y$ and $(y, z) \mapsto z$, respectively.

The processes $y_t = J_1(x_t)$ and $z_t = J_2(x_t)$ are called respectively the radial and angular parts of process x_t. By Theorem 31, the radial part y_t is a Markov process in Y. As the angular process z_t lives in the standard G-orbit Z which is invariant under the G-action, it is natural to expect that it should inherit the G-invariance of x_t in some sense. Before discussing the properties of the angular process z_t, we present some examples of spaces under group actions for which the topological assumptions made here are satisfied.

Example 1: Let $X = \mathbb{R}^{m+n} = Y \times Z$ with $Y = \mathbb{R}^m$ and $Z = \mathbb{R}^n$. In this case, $G = \mathbb{R}^n$ (additive group), and K consists of only the origin in $\mathbb{R}^n$.

Example 2: Consider the action of the orthogonal group $G = O(d)$ on $X = \mathbb{R}^d$ $(d \geq 2)$. Any half line Y from the origin is a transversal subspace. Except the origin, all the points in Y have the same isotropy subgroup K of $O(d)$, which may be identified with the orthogonal group $O(d-1)$ on a $(d-1)$-dimensional linear subspace of $\mathbb{R}^d$. The $O(d)$-orbits are spheres in $\mathbb{R}^d$ centered at origin.

Let X' be the $\mathbb{R}^d$ without the origin. Then $X' = Y' \times Z$, where $Y' = (0, \infty)$, and Z is the unit sphere $S^{d-1} = O(d)/O(d-1)$, via the map $(r, z) \mapsto rz$ from $Y' \times Z$ to X'.

Example 3: Consider the space X of $n \times n$ real symmetric matrices $(n \geq 2)$ under the action of $G = O(n)$ by conjugation: $(g, x) \mapsto gxg^{-1}$ for $g \in O(n)$ and $x \in X$. Because symmetric matrices in X with the same set of eigenvalues are $O(n)$-conjugate to each other, the set Y of all $n \times n$ diagonal matrices with non-ascending diagonal elements is a transversal subspace.

Let Y' be the subset of Y consisting of diagonal matrices with strictly descending diagonals. Then all $y \in Y'$ have the same isotropy subgroup K of $G = O(n)$, which is the finite subgroup of $O(n)$ consisting of diagonal matrices with ± 1 along diagonal. Note that an element of Y' not in Y has a larger isotropy subgroup. Let X' be the subset of X consisting of symmetric matrices with distinct eigenvalues. Then X' may be identified with $Y' \times (G/K)$ via the map $(y, gK) \mapsto gyg^{-1}$ from $Y' \times (G/K)$ to X'.

Example 4: For readers who know symmetric spaces, we mention one more example. Let $X = G/K$ be a symmetric space of noncompact type, with a fixed Weyl chamber $\mathfrak{a}_+$ and the centralizer M of $\mathfrak{a}_+$ in K, see [16]. Let $A_+ \subset G$ be the image of $\mathfrak{a}_+$ under the Lie group exponential map from the Lie algebra of G to G. The set X' of regular points in X is diffeomorphic to $A_+ \times (K/M)$ under the map: $A_+ \times (K/M)$ given by $(a, kM) \mapsto kaK$, so X' under the K-action satisfies our assumption with $Y = A_+$ and $Z = K/M$. A special case is the space X of $n \times n$ real positive definite symmetric matrices of unit determinant, and X' is the space of those matrices with distinct eigenvalues. This is a subset of the space in Example 3.

It may occur as in Examples 2, 3 and 4 that not all the points of the transversal subspace Y share the same isotropy subgroup of G, but by restricting to a slightly smaller open subset Y' of Y and the G-invariant open subset X' of X that is the union of G-orbits through Y', our assumptions are satisfied. In this case, if x_t is a G-invariant rcll Markov process in X, then its restriction to X', defined below, is a G-invariant rcll Markov process in X'.

The restriction of the process x_t to X' is the process $\hat{x}_t$ in X' defined by

$$\hat{x}_t = x_t \quad \text{for} \quad t < \zeta \qquad \text{and} \qquad \hat{x}_t = \Delta \quad \text{for} \quad t \geq \zeta, \tag{31}$$

where $\zeta = \inf\{t \geq 0;\, x_t \notin X' \text{ or } x_{t-} \notin X'\}$ ($\inf \emptyset = \infty$ by convention) and Δ is the point at infinity (see Appendix A.3). Then $\hat{x}_t$ is a rcll process in X'. Moreover, ζ is a stopping time under the natural filtration $\{\mathcal{F}_t^x\}$ of process x_t because $[\zeta = 0] = [x_0 \notin X']$ and for any $t > 0$,

$$[t < \zeta] \;=\; \cup_{n>1}\{[x_t \in X_n'] \cap [x_r \in X_n' \text{ for any } r \in \mathbb{Q} \text{ with } 0 < r < t]\} \;\in\; \mathcal{F}_t^x,$$

where X_n' are open subsets of X with closures $\bar{X}_n'$ contained in X' and $X_n' \uparrow X'$ as $n \uparrow \infty$, and $\mathbb{Q}$ is the set of rational numbers.

Proposition 32. *If x_t is a G-invariant rcll Markov process in X, then $\hat{x}_t$ is a G-invariant rcll Markov process in X' with transition semigroup $\hat{P}_t$ given by*

$$\hat{P}_t f(x) = E_x[f(x_t); t < \zeta] \qquad \text{for} \quad f \in \mathcal{B}_+(X') \quad \text{and} \quad x \in X'. \tag{32}$$

Proof. Let θ_t be the time shift (see Appendix A.3). For $s < t$, $1_{[t<\zeta]} = 1_{[s<\zeta]}(1_{[t-s<\zeta]} \circ \theta_s)$ and

$$\begin{aligned} & E_x[f(\hat{x}_t) \mid \mathcal{F}_s^x] = E_x[f(x_t)1_{[t<\zeta]} \mid \mathcal{F}_s^x] = E_x\{[f(x_{t-s})1_{[t-s<\zeta]}] \circ \theta_s \mid \mathcal{F}_s^x]1_{[s<\zeta]} \\ = \;\; & E_{x_s}[f(x_{t-s})1_{[t-s<\zeta]}]1_{[s<\zeta]} = \hat{P}_{t-s}f(\hat{x}_s)1_{[s<\zeta]} = \hat{P}_{t-s}f(\hat{x}_s), \end{aligned}$$

under the convention that any function on X or X' vanishes at Δ. This shows that $\hat{x}_t$ is a Markov process in X' with transition semigroup $\hat{P}_t$. By Proposition 1, the G-invariance of process x_t implies the G-invariance of $\hat{P}_t$. ☐

Remark 33. We may also define the restriction of process x_t in X' by stopping x_t at the first time τ when it exits X', where $\tau = \inf\{t > 0;\, x_t \notin X'\}$. Thus, let x_t' be defined by (31) with $\hat{x}_t$ and ζ replaced by x_t' and τ. However, τ in general is not a stopping time under $\mathcal{F}_t^x$ and x_t' may not be rcll in X'. By the standard stochastic analysis, τ is a stopping times under the filtration $\{\mathcal{F}_t\}$ that is the completion of $\{\mathcal{F}_{t+}^x\}$, and if the original process x_t is quasi-left-continuous, that is, if $x_\sigma = \lim_n x_{\sigma_n}$ almost surely on $[\sigma < \infty]$ for any $\{\mathcal{F}_t\}$-stopping times $\sigma_n \uparrow \sigma$, then the two processes x_t' and $\hat{x}_t$ are identical almost surely. To show this, let X_n' be the open subsets of X' with $X_n' \uparrow X'$ as defined earlier, and let τ_n be the first times when x_t exits X_n'. It is easy to see that $\tau_n \uparrow \zeta$, and so by the quasi-left-continuity, $x_\zeta = \lim_n x_{\tau_n}$ almost surely. Because X_n' is open, $x_{\tau_n} \notin X_n'$, and hence $x_\zeta \notin X'$ almost surely. This shows $\zeta = \tau$ almost surely. It is well known that a Feller process is quasi-left-continuous (see [21, Proposition 25.20]).

7. Angular part

As discussed in the previous section §6, when discussing an invariant Markov process under a non-transitive action, by suitably restricting the process, it may be possible to work under the following setup, which will be assumed throughout this section. Let $X = Y \times Z$ be a topological product with $Z = G/K$, where G is a topological group with identity element e and K is a compact subgroup. Assume both Y and G are lcscH. Let G act on X by its natural action on G/K. Then with $o = eK$, $Y \times \{o\}$ is transversal to the G-action and K is the common isotropy subgroup of G at every point of $Y \times \{o\}$.

We will also assume G/K has a continuous local section (as defined in §4). As in §6, let J_1: $X \to Y$ and J_2: $X \to Z$ be the natural projections.

We mention an important case when this setup holds. When X is a manifold under the (smooth) action of a Lie group G, and Y is a submanifold of X transversal to the G-action, if the tangent space of X at any $y \in Y$ is the direct sum of the tangent space of Y at y and that of the orbit through y, that is, if

$$\forall y \in Y, \quad T_y X = T_y(Gy) \oplus T_y Y \quad \text{(a direct sum).} \tag{33}$$

then by [17, II.Lemma 3.3], for any $y \in Y$, there is a neighborhood U of y in Y, and a neighborhood V of $o = eK$ in $Z = G/K$, where e is the identity element of G as before, such that the map $(u, v) \mapsto vu$ is a diffeomorphism from $U \times V$ onto a neighborhood of y in X. From this, it is easy to show that F: $Y \times Z \to X$ is a diffeomorphism. The existence of a continuous local section holds automatically in the Lie group case.

Let x_t be a G-invariant Markov process in X. By Theorem 31, the radial process $y_t = J_1(x_t)$ is a Markov process in Y. The purpose of this section is to study the angular process $z_t = J_2(x_t)$ under the conditional distribution given the radial process y_t. We now begin with some preparation.

Let (S, ρ) be a complete and separable metric space, and let $D(S)$ be the space of rcll paths in S, that is, the space of rcll maps: $\mathbb{R}_+ \to S$. Under the Skorohod metric, $D(S)$ is a complete and separable metric space (see [10, chapter 3]). By Proposition 5.3(c) in [10, chapter 3], the convergence $x_n \to x$ in $D(S)$ means that up to a time change that is asymptotically an identity, $x_n(t)$ converge to $x(t)$ uniformly for bounded t. Precisely, this means that for any $T > 0$, there are bijections λ_n: $\mathbb{R}_+ \to \mathbb{R}_+$ (necessarily continuous with $\lambda_n(0) = 0$) such that $|\lambda_n(t) - t| \to 0$ and $\rho(x_n(t), x(\lambda_n(t))) \to 0$ as $n \to \infty$ uniformly for $t \in [0, T]$. By Proposition 7.1 in [10, chapter 3], the Borel σ-algebra on $D(S)$ is generated by the coordinate maps: $D(S) \to S$ given by $x \mapsto x(t)$ for $t \in \mathbb{R}_+$. It is easy to show that the topology on $D(S)$ induced by the Skorohod metric is determined completely by the topology of (S, ρ), and does not depend on the choice of ρ.

Let $X_\Delta = X \cup \{\Delta\}$ be the one-point compactification of X. Because X is a lcscH space, X_Δ is metrizable (see Theorem 12.12 in [5, chapter I]), and as being compact, X_Δ may be equipped with a complete and separable metric. Let $D'(X)$ be the space of rcll paths in X with possibly finite life times. These are rcll paths in X_Δ such that each path $x(\cdot)$ is associated to a life time $\zeta \in [0, \infty]$ with $x(t) \in X$ for $t < \zeta$ and $x(t) = \Delta$ for $t \geq \zeta$. By Proposition 34 below, $D'(X)$ is a Borel subset of $D(X_\Delta)$, and hence $D'(X)$ is a Borel space (see Appendix A.7 for the definition of Borel spaces).

Proposition 34. *$D'(X)$ is a Borel subset of $D(X_\Delta)$, and hence $D'(X)$ is a Borel space.*

Proof. Let X_n be open subsets of X such that $\bar{X}_n \subset X_{n+1}$ and $X_n \uparrow X$. Then $D'(X) = \cap_{m>0} A(m)$, where $A(m) = \cup_{n=1}^{\infty} \cup_{r \in \mathbb{Q}, r \geq 0} A(m, n, r)$ and $A(m, n, r) = \{x \in D(X_\Delta);\ x(s) = \Delta$ for $s \in \mathbb{Q} \cap [r, \infty)$ and $x(s) \in X_n$ for $s \in \mathbb{Q} \cap [0, r - 1/m]\}$. □

For $b > 0$, let $D'_b(X)$ be the subset of $D'(X)$ consisting of all paths with a common constant life time b. Then $D'_b(X)$ is a closed in $D'(X)$ under the Skorohod metric, and $D'_\infty(X) = D(X)$.

Recall J_1: $X \to Y$ and J_2: $X \to Z$ are the natural projections $Y \times Z \to Y$ and $Y \times Z \to Z$. We will also use J_1 and J_2 to denote the maps

$$J_1 : D'(X) \to D'(Y), \quad x(\cdot) \mapsto y(\cdot), \quad \text{and} \quad J_2 : D'(X) \to D'(Z), \quad x(\cdot) \mapsto z(\cdot),$$

respectively, given by the decomposition $x(\cdot) = (y(\cdot).z(\cdot))$.

Recall x_t is a G-invariant rcll Markov process in X. It may be regarded as the coordinate process on the canonical path space $D'(X)$, that is, $x_t(\omega) = \omega(t)$ for $\omega \in \Omega$. The radial part $y_t = J_1(x_t)$ and the angular part $z_t = J_2(x_t)$ are thus the coordinate processes on $D'(Y)$ and $D'(Z)$, respectively. For $t > s \geq 0$, let $\mathcal{F}^Y_{s,t} = \sigma\{y_u;\ u \in [s, t]\}$, the σ-algebra on $D'(Y)$ generated by the radial process on the time interval $[s,\, t]$, and set $\mathcal{F}_t^Y = \mathcal{F}_{0,t}^Y$ and $\mathcal{F}_\infty^Y = \sigma\{\cup_{t>0}\mathcal{F}_t^\infty\}$.

The distribution P_x of the process x_t with $x_0 = x$ is a probability kernel from X to Ω. By the standard probability theory, if the underlying probability space $(\Omega, \mathcal{F}, P)$ is a Borel space, then for any σ-algebra $\mathcal{G} \subset \mathcal{F}$, there is a regular conditional distribution $Q(\omega, B)$ of P given $\mathcal{G}$, which is a probability kernel from $(\Omega, \mathcal{G})$ to $(\Omega, \mathcal{F})$, such that for $B \in \mathcal{F}$, $P(B \mid \mathcal{G}) = Q(\cdot, B)$ P-almost surely. This result can be extended to our probability kernel P_x (by suitably adapting the standard proof as in [21, Theorem 6.3]). Let $P'_x(x(\cdot), \cdot)$ be of regular conditional distribution of P_x given the radial process $y_t = J_1 x_t$, that is, given $J_1^{-1}(\mathcal{F}_\infty^Y)$. This is a probability kernel from $(X \times \Omega, \mathcal{B}(X) \times J_1^{-1}(\mathcal{F}_\infty^Y))$ to $(\Omega, \mathcal{F})$. Because it is $\mathcal{B}(X) \times J_1^{-1}(\mathcal{F}_\infty^Y)$-measurable, for any $y(\cdot) \in D'(Y)$, $P'_x(x(\cdot), \cdot)$ is constant on the set $J_1^{-1}(\{y(\cdot)\})$. Let $P_x^Y(y(\cdot), \cdot) = P'_x((y(\cdot), o(\cdot)), \cdot)$, where $o(\cdot) \in D'(Z)$ is the constant path $o(t) = o$ for all $t \geq 0$.

For $x \in X$ with decomposition $x = (y, z)$, let

$$Q_y = J_1 P_x = P_x(J_1^{-1}(\cdot)). \tag{34}$$

This is the distribution of the radial process $y_t = J_1 x_t$ with $y_0 = y$, and it does not depend on x in the orbit Gy. For $z \in Z$, $y(\cdot) \in D'(Y)$ and $x = (y(0), z)$, let

$$R_z^{y(\cdot)}(\cdot) = P_x^Y(y(\cdot), J_2^{-1}(\cdot)). \tag{35}$$

This is a probability kernel from $Z \times D'(Y)$ to $D'(Z)$ such that for any $y \in Y$, $z \in Z$ and measurable $F \subset D'(Z)$, with $x = (y, z)$,

$$R_z^{y(\cdot)}(F) = P_x[J_2^{-1}(F) \mid J_1^{-1}(\mathcal{F}_\infty^Y)] \quad \text{for } Q_y\text{-almost all } y(\cdot). \tag{36}$$

Thus, $R_z^{y(\cdot)}$ is the conditional distribution of the angular process $z_t = J_2(x_t)$ given the radial path $y(\cdot)$ under P_x with $x = (y(0), z)$.

The life time ζ of process x_t is equal to that of the radial process y_t, and hence is $J_1^{-1}(\mathcal{F}_\infty^Y)$-measurable. Thus, given $J_1^{-1}(\mathcal{F}_\infty^Y)$ or under $R_z^{y(\cdot)}$, ζ is a constant. For $z \in Z$ and Q_y-almost all $y(\cdot) \in D'(Y)$, the measure $R_z^{y(\cdot)}$ is supported $D'_\zeta(X)$.

For $t \in \mathbb{R}_+$, the time shift θ_t^Y on $D'(Y)$ is defined as usual by $\theta_t^Y y(\cdot) = y(\cdot + t)$. Then $y_s \circ \theta_t^Y = y_{s+t}$ for the coordinate process y_s on $D'(Y)$.

Recall Lévy processes in Lie groups and homogeneous spaces, including inhomogeneous ones, are defined to have infinite life times, but these definitions may be easily modified to include processes defined on a finite time interval $[0,\, T]$ or $[0,\, T)$ for some constant $T > 0$. The following result says that almost surely, given the radial process y_t, the conditioned angular process z_t is an inhomogeneous Lévy process in $Z = G/K$ for $0 \leq t < \zeta$.

Theorem 35. *For $y \in Y$ and $z \in Z$, and Q_y-almost all $y(\cdot)$ in $D'(Y)$, the coordinate process z_t on $D'_\zeta(Z)$ is an inhomogeneous Lévy process under $R_z^{y(\cdot)}$. The associated convolution semigroup $\mu_{s,t}$, setting $\mu_{s,t} = 0$ for $t \geq \zeta$, is $\mathcal{F}_{s,t}^Y$-measurable for any $s \leq t$ (that is, $\mu_{s,t}(B)$ is $\mathcal{F}_{s,t}^Y$-measurable for $B \in \mathcal{B}(Z)$) and has the time shift property*

$$\mu_{s,t} = \mu_{0,t-s} \circ \theta_s^Y. \tag{37}$$

Moreover, the transition semigroup P_t of the Markov process x_t is given by

$$\forall f \in \mathcal{B}_+(X) \text{ and } x = (y, z) \in X, \quad P_t f(x) = Q_y[\int \mu_{0,t}(dz_1) f(y_t, z z_1) 1_{[\zeta > t]}]. \tag{38}$$

Proof. Recall Q_t is the transition semigroup of the radial process y_t. By the existence of a regular conditional distribution, there is a probability kernel $R_t(y, y_1, \cdot)$ from $Y^2 = Y \times Y$ to Z such that for $y \in Y$, $P_t((y, o), dy_1 \times dz_1) = Q_t(y, dy_1) R_t(y, y_1, dz_1)$. Then let $R_t(y, y_1, \cdot) = J_2 R'_t(y, (y_1, o), \cdot)$, noting $R'_t(y, (y_1, z), \cdot)$ is constant in $z \in Z$ and supported by $J_1^{-1}(\{y_1\})$.

The G-invariance of P_t implies that $P_t((y,o),\cdot)$ is K-invariant, and hence the measure $R_t(y,y_1,\cdot)$ is K-invariant for $Q_t(y,\cdot)$-almost all y_1. Modifying R_t on the exceptional set $\{(y,y_1);\ R_t(y,y_1,\cdot)$ is not K-invariant$\}$, its y-section has zero $Q_t(y,\cdot)$-measure for all $y\in Y$, we may assume $R_t(y,y_1,\cdot)$ is K-invariant for all $y,y_1\in Y$. Therefore, for $z\in Z$, it is meaningful to write $R_t(y,y_1,z^{-1}dz_1)=R_t(y,y_1,S(z)^{-1}dz_1)$ and $\int f(zz_1)R_t(y,y_1,dz_1)=\int f(S(z)z_1)R_t(y,y_1,dz_1)$ because they are independent of choice of section map S. We then have, for $x=(y,z)\in X$,

$$\forall y\in Y \text{ and } z\in Z,\quad P_t(x,\,dz_1\times dy_1)=Q_t(y,dy_1)R_t(y,y_1,z^{-1}dz_1). \tag{39}$$

For $0<s_1<s_2<\cdots<s_k<\infty$, $y\in Y$, $z\in Z$, $h\in C_b(Y^k)$ and $f\in C_b(Z^k)$,

$$\begin{aligned}
&E_x[h(y_{s_1},\ldots,y_{s_k})f(z_{s_1},\ldots,z_{s_k})1_{[\zeta>s_k]}]\\
=\ &\int\int P_{s_1}(x,\,dz_1\times dy_1)\\
&\quad P_{s_2-s_1}((y_1,z_1),\,dz_2\times dy_2)\cdots P_{s_k-s_{k-1}}((y_{k-1},z_{k-1}),\,dz_k\times dy_k)\\
&\quad h(y_1,y_2,\ldots,y_k)f(z_1,z_2,\ldots,z_k)\\
=\ &\int Q_{s_1}(y,dy_1)Q_{s_2-s_1}(y_1,dy_2)\cdots Q_{s_k-s_{k-1}}(y_{k-1},dy_k)h(y_1,y_2,\ldots,y_k)\\
&\int R_{s_1}(y,y_1,dz_1)R_{s_2-s_1}(y_1,y_2,dz_2)\cdots R_{s_k-s_{k-1}}(y_{k-1},y_k,dz_k)\\
&\quad f(zz_1,zz_1z_2,\ldots,zz_1\cdots z_k)\\
=\ &E_y[h(y_{s_1},y_{s_2},\ldots,y_{s_k})\int R_{s_1}(y,y_{s_1},dz_1)R_{s_2-s_1}(y_{s_1},y_{s_2},dz_2)\cdots\\
&\qquad R_{s_k-s_{k-1}}(y_{s_{k-1}},y_{s_k},dz_k)f(zz_1,zz_1z_2,\ldots,zz_1\cdots z_k)1_{[\zeta>s_k]}].
\end{aligned}$$

This implies that P_x-almost surely on $[\zeta>s_k]$,

$$\begin{aligned}
E_x[f(z_{s_1},\ldots,z_{s_k})\mid y_{s_1},\ldots,y_{s_k}]=&\int R_{s_1}(y,y_{s_1},dz_1)R_{s_2-s_1}(y_{s_1},y_{s_2},dz_2)\cdots\\
&R_{s_k-s_{k-1}}(y_{s_{k-1}},y_{s_k},dz_k)f(zz_1,zz_1z_2,\ldots,zz_1\cdots z_k)].
\end{aligned} \tag{40}$$

For an integer $m\geq 1$, let Γ_m be the set of dyadic numbers $i/2^m$ for $i=0,1,2,\ldots$ and let $\Gamma=\cup_{m=1}^{\infty}\Gamma_m$. For $s<t\leq T$ in Γ_m, let $0=s_0<s_1<s_2<\cdots<s_k=T$ be a partition of $[0,\,T]$ spaced by $1/2^m$ with $s=s_i$ and $t=s_j$, and let

$$\mu^m_{s,t}=R_{s_{i+1}-s_i}(y_{s_i},y_{s_{i+1}},\cdot)*R_{s_{i+2}-s_{i+1}}(y_{s_{i+1}},y_{s_{i+2}},\cdot)*\cdots*R_{s_j-s_{j-1}}(y_{s_{j-1}},y_{s_j},\cdot). \tag{41}$$

By (40), K-invariance of $P_t((y,o),y_1,\cdot)$ and the measurability of $\mu^m_{s,t}$ in $y_{s_i},\ldots,y_{s_j}$,

$$\mu^m_{s,t}(f)=E_x[f(z_s^{-1}z_t)\mid y_{s_1},\ldots,y_{s_k}]=E_x[f(z_s^{-1}z_t)\mid y_{s_i},\ldots,y_{s_j}] \tag{42}$$

P_x-almost surely on $[\zeta>T]$ for $f\in C_b(Z)$, which is independent of the choice for the section map S to represent $z_s^{-1}z_t=S(z_s)^{-1}z_t$.

By the right continuity of y_t, as $m\to\infty$, $\sigma\{y_{s_1},\ldots,y_{s_k}\}\uparrow\mathcal{F}^Y_T$ and $\sigma\{y_{s_i},\ldots,y_{s_j}\}\uparrow\mathcal{F}^Y_{s,t}$, it follows that as $m\to\infty$, $\mu^m_{s,t}(f)\to E_x[f(z_s^{-1}z_t)\mid\mathcal{F}^Y_T]=E_x[f(z_s^{-1}z_t)\mid\mathcal{F}^Y_{s,t}]$ P_x-almost surely on $[\zeta>T]$. The exceptional set may be chosen simultaneously for countably many $f\in C_b(Z)$ and hence for all $f\in C_b(Z)$. It follows that P_x-almost surely on $[\zeta>T]$, there is an K-invariant probability measure $\mu_{s,t}$ such that $\mu^m_{s,t}\to\mu_{s,t}$ weakly and for $f\in C_b(Z)$,

$$\mu_{s,t}(f)=E_x[f(z_s^{-1}z_t)\mid\mathcal{F}^Y_T]=E_x[f(z_s^{-1}z_t)\mid\mathcal{F}^Y_{s,t}]\quad P_x\text{-almost surely on } [\zeta>T]. \tag{43}$$

Note that $\mu^m_{s,t}$ is independent of choice $T\in\Gamma_m$ with $T\geq t$, and hence $\mu_{s,t}$ is defined for any $s,t\in\Gamma$ with $s<t<\zeta$. Set $\mu_{t,t}=\delta_o$ and $\mu_{s,t}=0$ for $t\geq\zeta$. By (41), $\mu_{s,t}$ is an $\mathcal{F}^Y_{s,t}$-measurable random measure independent of starting point x and has the time shift property (37) for $s,t\in\Gamma$. Moreover, $\mu_{r,s}*\mu_{s,t}=\mu_{r,t}$ for $r\leq s\leq t$ in Γ. Because Γ is countable, the exceptional set of zero P_x-measure for the weak convergence $\mu^m_{s,t} \to \mu_{s,t}$ may be chosen simultaneously for all $s \le t$ and T in Γ.

For $0 \le t_1 < t_2 < \cdots < t_n \le T$ in Γ, it can be shown from (40) and by choosing a partition $s_1 < s_2 < \cdots < s_k$ of $[0, T]$ from Γ containing all t_i, spaced by $1/2^m$, that almost surely on $[\zeta > T]$, for $f \in C_b(Z^n)$,

$$E_x[f(z_{t_1}, \dots, z_{t_n}) \mid \mathcal{F}^Y_T] = \lim_{m\to\infty} E_x[f(z_{t_1}, \dots, z_{t_n}) \mid y_{s_1}, \dots, y_{s_k}]$$
$$= \lim_{m\to\infty} \int f(zz_1, zz_1z_2, \dots, zz_1\cdots z_n)\mu^m_{0,t_1}(dz_1)\mu^m_{t_1,t_2}(dz_2)\cdots\mu^m_{t_{n-1},t_n}(dz_n).$$

This implies that for $0 \le t_1 < \cdots < t_n \le T$ in Γ, $y \in Y$ and $z \in Z$, and Q_y-almost all $y(\cdot) \in [\zeta > T]$,

$$R^{y(\cdot)}_z[f(z_{t_1}, \dots, z_{t_n})] = E_x[f(z_{t_1}, \dots, z_{t_n}) \mid \mathcal{F}^Y_T]$$
$$= \int f(zz_1, zz_1z_2, \dots, zz_1\cdots z_n)\mu_{0,t_1}(dz_1)\mu_{t_2,t_1}(dz_2)\cdots\mu_{t_{n-1},t_n}(dz_n). \tag{44}$$

Then the finite dimensional distribution of the conditioned angular process z_t under $R^{y(\cdot)}_z$, when restricted to time points in $\Gamma \cap [0, T]$, has the form consistent with an inhomogeneous Lévy process in Z. To prove that the conditioned process z_t is an inhomogeneous Lévy processes in Z, restricted to time interval $[0, T]$, it remains to extend $\mu_{s,t}$ in (43) to all real $s \le t$ in $[0, T]$ and prove (44) for real times $t_1 < t_2 < \cdots < t_n$ in $[0, T]$.

By a computation similar to the one leading to (26), it can be show that for $s < t \le T$ in Γ and $f \in C_b(Z)$,

$$\mu_{s,t}(f) = E_x[\sum_j \psi_j(z_s)f(S_j(z_s)^{-1}z_t) \mid \mathcal{F}^Y_T] \quad P_x\text{-almost surely on } [\zeta > T],$$

where $\{\psi_j\}$ is a partition of unity on Z and for each j, S_j is a section map on $Z = G/K$ that is continuous on the support of ψ_j. The above expression for $\mu_{s,t}(f)$ extends to real times $s < t$, and it is clearly right continuous in s and t. Moreover, considering $P_x(\cdot \mid \mathcal{F}^Y_T)$ as a regular conditional distribution of P_x given $\mathcal{F}^Y_T$, no additional exceptional set is produced. Because for real $s < t$, $\mu_{s,t}$ is the weak limit of $\mu_{p,q}$ for $p, q \in \Gamma$ as $p \downarrow s$ and $q \downarrow t$, taking limit in (44) shows that it holds on real time points. It is also clear that $\mu_{s,t}$ is $\mathcal{F}^Y_{s,t}$-measurable and has the time shift property (37).

It remains to prove (38) which follows from

$$P_tf(x) = E_x[f((y_t, z_t))1_{[\zeta>t]}] = E_x\{R^{y(\cdot)}_z[f((y', z_t))]_{y'=y_t}1_{[\zeta>t]}\}$$
$$= Q_y[\int \mu_{0,t}(dz_1)f((y_t, zz_1))1_{[\zeta>t]}],$$

where the last equality is due to (44). □

The following result provides a converse to Theorem 35.

Theorem 36. *Let y_t be a rcll Markov process in Y with life time ζ and for any $y \in Y$, let Q_y be its distribution on $D'(Y)$ with $Q_y(y_0 = y) = 1$. Assume for any $y \in Y$, $z \in Z = G/K$ and Q_y-almost all $y(\cdot)$, there is a probability measure $R^{y(\cdot)}_z$ on $D'(Z)$ such that under $R^{y(\cdot)}_z$, the coordinate process z_t on $D'(Z)$ is an inhomogeneous Lévy process z_t in Z for $t < \zeta$, with $z_0 = z$, and the associated convolution semigroup $\mu_{s,t}$ (setting $\mu_{s,t} = 0$ for $t \ge \zeta$) is $\mathcal{F}^Y_t$-measurable and has the time shift property (37) as in Theorem 35. Then $x_t = (y_t, z_t)$ is a G-invariant rcll Markov process in X with transition semigroup P_t given by (38).*

Proof. For $z \in Z$ and $y(\cdot) \in D'(Y)$, let $R_z^{y(\cdot)}$ be the distribution on $D'(Z)$ of the inhomogeneous Lévy process z_t in Z for $t < \zeta$ with $z_0 = z$, associated to convolution semigroup $\mu_{s,t}$. Then for $0 \le t_1 < t_2 < \cdots < t_n < \zeta$ and $f \in C_b(Z^n)$,

$$R_z^{y(\cdot)}[f(z_{t_1}, z_{t_2}, \ldots, z_{t_n})] = \int f(zz_1, zz_1z_2, \ldots, zz_1z_2\cdots z_n) \mu_{0,t_1}(dz_1)\mu_{t_1,t_2}(dz_2)\cdots\mu_{t_{n-1},t_n}(dz_n).$$

Because $\mu_{s,t}$ is $\mathcal{F}_t^Y$ measurable, $R_z^{y(\cdot)}[f(z_{t_1}, z_{t_2}, \ldots, z_{t_n})]$ is measurable on $Z \times D'(Y)$. Then a simple monotone class argument shows that $R_z^{y(\cdot)}$ is a kernel from $Z \times D'(Y)$ to $D'(Z)$.

For $x \in X$ with $x = (y, z)$, let P_x be the measure on $D'(Y) \times D'(Z)$ defined by

$$\forall F \in \mathcal{B}_b(D'(Y) \times D'(Z)), \qquad P_x(F) = Q_y\{R_z^{y(\cdot)}[F(y(\cdot), \cdot)]\}.$$

Then P_x is a probability measure supported by the closed subset H of $D'(Y) \times D'(Z)$ consisting of $(y(\cdot), z(\cdot))$ with the same life time for $y(\cdot)$ and $z(\cdot)$. The map: $x(\cdot) \mapsto (J_1x(\cdot), J_2x(\cdot))$ is continuous and bijective from $D'(X)$ onto H, and its inverse is also continuous under the Skorohod metric. We will identify $D'(X)$ with H via this map, and then P_x may be regarded as a probability measure on $D'(X)$.

For $0 \le s_1 < s_2 < \cdots < s_n = s < t$, $h \in C_b(X^n)$ and $f \in C_b(X)$,

$$\begin{aligned}
& E_x[h(x_{s_1}, x_{s_2}, \ldots, x_{s_n})f(x_t)] \\
= \;& Q_y\{R_z^{y(\cdot)}[h(z_{s_1}y_1, z_{s_2}y_2, \ldots, z_{s_n}y_n)f(z_ty')]_{y_1=y(s_1),y_2=y(s_2),\ldots,y_n=y(s_n),y'=y(t)}\} \\
= \;& Q_y\{\int h(zz_1y_{s_1}, zz_1z_2y_{s_2}, \ldots, zz_1z_2\cdots z_ny_{s_n})f(zz_1z_2\cdots z_nz'y_t) \\
& \quad \mu_{0,s_1}(dz_1)\mu_{s_1,s_2}(dz_2)\cdots\mu_{s_{n-1},s_n}(dz_n)\mu_{s,t}(dz')\} \\
= \;& Q_y\{\int h(zz_1y_{s_1}, zz_1z_2y_{s_2}, \ldots, zz_1z_2\cdots z_ny_{s_n})\mu_{0,s_1}(dz_1)\mu_{s_1,s_2}(dz_2)\cdots\mu_{s_{n-1},s_n}(dz_n) \\
& \quad Q_y[(\int \mu_{0,t-s}\circ(dz')f(zz_1z_2\cdots z_nz'y_{t-s}))\circ\theta_s^Y \mid \mathcal{F}_s^Y]\} \\
= \;& Q_y\{\int h(zz_1y_{s_1}, zz_1z_2y_{s_2}, \ldots, zz_1z_2\cdots z_ny_{s_n})\mu_{0,s_1}(dz_1)\mu_{s_1.s_2}(dz_2)\cdots\mu_{s_{n-1},s_n}(dz_n) \\
& \quad Q_{y_s}[\int \mu_{0,t-s}(dz')f(zz_1z_2\cdots z_nz'y_{t-s})]\} \\
= \;& Q_y\{R_z^{y(\cdot)}\{h(z_{s_1}y_1, z_{s_2}y_2, \ldots, z_{s_n}y_n) \\
& \quad Q_{y_n}[\int \mu_{0,t-s}(dz')f(\tilde{z}z'y_{t-s})]_{\tilde{z}=z_s}\}_{y_i=y(s_i),1\le i\le n}\} \\
= \;& P_x\{h(x_{s_1}, x_{s_2}, \ldots, x_{s_n})P_{t-s}f(x_s)\}.
\end{aligned}$$

This shows that under P_x, x_t is a Markov process in X with transition semigroup P_t given by (38). It follows directly from (38) that P_t is G-invariant. □

References

1. D. Applebaum, *Lévy processes and stochastic calculus*, second ed., Cambridge Univ. Press, 2009.
2. D. Applebaum, *Probability on compact Lie groups*, Springer, 2014.
3. M. Babullot, *Comportement asymptotique due mouvement Brownien sur une variété homogène à courbure négative ou nulle*, Ann. Inst. H. Poincaré (prob et Stat) **27** (1991), 61–90.
4. J. Bertoin, *Lévy processes*, Cambrige Univ. Press, 1996.
5. G. E. Bredon, *Topology and geometry*, Springer, 1997.
6. E. Born, *An explicit Lévy-Hinčin formula for convolution semigroups on locally compact groups*, J. Theor. Prob. **2** (1989), 325–342.

7. S. G. Dani and M. McCrudden, *Embeddability of infinitely divisible distributions on linear Lie groups*, Invent. Math. **110** (1992), 237–261.
8. S. G. Dani and M. McCrudden, *Convolution roots and embedding of probability measures on Lie groups*, Adv. Math. **209** (2007), 198–211.
9. E. B. Dynkin, *Nonnegative eigenfunctions of the Laplace-Betrami operators and Brownian motion in certain symmetric spaces*, Dokl. Akad. Nauk SSSR **141** (1961), 1433–1426.
10. S. N. Ethier and T. G. Kurtz, *Markov processes, characterization and convergence*, John Wiley, 1986.
11. P. Feinsilver, *Processes with independent increments on a Lie group*, Trans. Amer. Math. Soc. **242** (1978), 73–121.
12. H. Furstenberg and H. Kesten, *Products of random matrices*, Ann. Math. Statist. **31** (1960), 457–469.
13. H. Furstenberg, *Noncommuting random products*, Trans. Am. Math. Soc. **108** (1963), 377–428.
14. R. Gangolli, *Isotropic infinitely divisible measures on symmetric spaces*, Acta Math. **111** (1964), 213–246.
15. Y. Guivarc'h and A. Raugi, *Frontière de Furstenberg, propriétés de contraction et convergence*, Z. Wahr. Gebiete **68** (1985), 187–242.
16. S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Academic Press, 1978.
17. S. Helgason, *Groups and geometric analysis*, Academic Press, 1984.
18. H. Heyer, *Probability measures on locally compact groups*, Springer-Verlag, 1977.
19. H. Heyer and G. Pap, *Martingale characterizations of increment processes in a local compact group*, in Infinite Dimensional Analysis, Quantum Probability and Related Topics **6** (2003), 563–595.
20. G. A. Hunt, *Semigroups of measures on Lie groups*, Trans. Am. Math. Soc. **81** (1956), 264–293.
21. O. Kallenberg, *Foundations of modern probability, second ed.*, Springer-Verlag, 2002.
22. T. Karube, *On the local cross sections in locally compact groups*, J. Math. Soc. Japan **10** (1958), 343–347.
23. S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, vols I and II, Interscience Publishers, 1963 and 1969.
24. M. Liao, *Lévy processes in semi-simple Lie groups and stability of stochastic flows*, Trans. Am. Math. Soc. **350** (1998), 501-522.
25. M. Liao, *Lévy processes in Lie groups*, Cambridge Univ. Press, 2004.
26. M. Liao, *A decomposition of Markov processes via group actions*, J. Theoret. Probab. **22** (2009), 164–185.
27. M. Liao, *Inhomogeneous Lévy Processes in Lie Groups and Homogeneous Spaces*, J. Theoret. Probab. **27** (2014), 315–357.
28. M. Liao, *Convolution of probability measures on Lie groups and homogeneous spaces*, Potential Analysis **43** (2015), 707–715.
29. M. P. Malliavin and P. Malliavin, *Factorizations et lois limites de la diffusion horizontale audessus d'un espace Riemannien symmetrique*, Lecture Notes Math. **404** (1974), 164–271.
30. K. Nagami, *Cross sections in locally compact groups*, J. Math. Soc. Japan **15** (1963), 301–303.
31. G. Pap, *General solution of the functional central limit problems on a Lie group*, in Infin. Dim. Anal. Quantum Probab. Rel. Top. **7** (2004), 43–87.
32. J. R. Norris, L.C.G. Rogers and D. Williams, *Brownian motion of ellipsoids*, Trans. Am. Math. Soc. **294** (1986), 757–765.
33. A. Orihara, *On random ellipsoids*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **17** (1970), 73–85.
34. A. Raugi, *Fonctions harmoniques et théorèmes limites pour les marches gléatoires sur les groupes*, Bull. Soc. Math. France, mémoire **54** (1997).
35. K. Sato, *Lévy processes and infinitely devisible distributions*, translated from 1990 Japanese original and revised by the author, Cambridge Studies Adv. Math. **68**, Cambridge Univ. Press, 1999.
36. E. Siebert, *Fourier analysis and limit theorems for convolution semigroups on a locally compact groups*, Adv. Math. **39** (1981), 111–154.
37. D. W. Stroock and S.R.S. Varadhan, *Limit theorems for random walks on Lie groups*, Sankhyā: Indian J. of Stat. Ser. A **35** (1973), 277–294.
38. J. C. Taylor, *The Iwasawa decomposition and the limiting behavior of Brownian motion on a symmetric space of non-compact type*, in Geometry of random motion, ed. by R. Durrett and M. A. Pinsky, Contemp. Math. **73** (1988), Am. Math. Soc., 303–332.
39. J. C. Taylor, *Brownian motion on a symmetric space of non-compact type: asymptotic behavior in polar coordinates*, Can. J. Math. **43** (1991), 1065–1085.
40. V. N. Tutubalin, *Limit theorems for a product of random matrices*, Theory Probab. Appl. **10** (1965), 25–27.

41. Valery V. Volchkov and Vitaly V. Volchkov, *Harmonic Analysis of mean periodic functions on symmetric spaces and the Heisenberg group*, Springer-Verlag, 2009.
42. A. D. Virtser, *Central limit theorem for semi-simple Lie groups*, Theory Probab. Appl. **15** (1970), 667–687.

Department of Mathematics and Statistics, Auburn University, Auburn, AL 36849, USA.
E-mail address: liaomin@auburn.edu