Theory of Stochastic Processes
Vol. 21 (37), no. 2, 2016, pp. 4–13

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A NOTE ON WEAK CONVERGENCE OF THE n-POINT MOTIONS OF HARRIS FLOWS

In this note we extend the main results of [2] and [8], which concern the weak convergence of the n-point motions of smooth Harris flows to those of the Arratia flow, to the case when the covariance functions of these Harris flows converge pointwise to a covariance function whose support is of zero Lebesgue measure.

The main aim of this note is to generalize the results of [2] and [8] concerning the weak convergence of the n-point motions of Harris flows.

We begin by recalling the definition of a Harris flow (e. g., see [3, Definition 1.2]).

Definition 1. A random field $\{x(u,t),\ u \in \mathbb{R},\ t \geqslant 0\}$ is called a *Harris flow* with covariance function Γ if it satisfies the following conditions:

(i) for any $u \in \mathbb{R}$ the stochastic process $\{x(u,t),\ t \geqslant 0\}$ is a Brownian motion with respect to the common filtration $(\mathcal{F}_t := \sigma\{x(v,s),\ v \in \mathbb{R},\ 0 \leqslant s \leqslant t\})_{t \geqslant 0}$ such that $x(u,0) = u$;

(ii) for any $u, v \in \mathbb{R}$, if $u \leqslant v$, then $x(u,t) \leqslant x(v,t)$ for all $t \geqslant 0$;

(iii) for any $u, v \in \mathbb{R}$ the joint quadratic variation of the martingales $\{x(u,t),\ t \geqslant 0\}$ and $\{x(v,t),\ t \geqslant 0\}$ is given by

$$\langle x(u,\cdot), x(v,\cdot)\rangle_t = \int_0^t \Gamma(x(u,s) - x(v,s))\, ds, \quad t \geqslant 0.$$

Note that the function Γ is necessarily non-negative definite and, in particular, symmetric. Besides, without loss of generality we always assume that

$$\Gamma(0) = 1,$$

so that the one-point motions of Harris flows we consider are standard Brownian motions.

The existence of random fields satisfying the above conditions (i), (ii) and (iii) under mild assumptions on the covariance function was proved in [4].

A Harris flow with covariance function $\Gamma = 1\!\!1_{\{0\}}$ is called the *Arratia flow* (here $1\!\!1_A(z) \equiv 1\!\!1\{z \in A\}$ stands for the indicator function of the set A). It is one of the first examples of Harris flows and was initially constructed in [1] as the weak limit of a family of coalescing simple random walks. Throughout this paper the Arratia flow will be denoted by $\{x_0(u,t),\ u \in \mathbb{R},\ t \geqslant 0\}$.

It is convenient to construct Harris flows with a smooth covariance function as solutions of stochastic differential or integral equations. To be more precise, let us consider the following stochastic integral equation:

$$x(u,t) = u + \int_0^t \int_{\mathbb{R}} \varphi(x(u,s) - q)\, W(dq, ds), \quad t \geqslant 0, \tag{1}$$

2010 *Mathematics Subject Classification.* 60G60, 60B12, 60H20.

Key words and phrases. Harris flows, Brownian stochastic flows, n-point motions, weak convergence.

where $u \in \mathbb{R}$ plays the role of a parameter, W is a Wiener sheet on $\mathbb{R} \times [0; +\infty)$ and the function $\varphi \in C_0^\infty(\mathbb{R}, [0; +\infty))$ (i. e. infinitely differentiable and with compact support) is symmetric and has a unit L_2-norm.

It is known [2] that under these conditions on the function φ this equation has a unique strong solution for every $u \in \mathbb{R}$ and the random field $\{x(u,t),\ u \in \mathbb{R},\ t \geqslant 0\}$ is a Harris flow with covariance function Γ given by

$$\Gamma(z) := \int_{\mathbb{R}} \varphi(z+q)\varphi(q)\, dq, \quad z \in \mathbb{R}.$$

Now we can formulate the main results of [2] and [8]. Although these results were proved for the case of the finite time interval $[0; 1]$, their proofs remain valid for the more general case of the infinite time interval $[0; +\infty)$ and it is in this form that we formulate them below.

Theorem 2. [2, Theorem 3] *For $\varepsilon > 0$ define*

$$\varphi_\varepsilon(q) := \frac{1}{\sqrt{\varepsilon}} \varphi\left(\frac{q}{\varepsilon}\right), \quad q \in \mathbb{R}, \tag{2}$$

and let $\{x_\varepsilon(u,t),\ u \in \mathbb{R},\ t \geqslant 0\}$ be the Harris flow formed by the solutions of the stochastic integral equation (1) *with φ_ε instead of φ. Then for any $n \in \mathbb{N}$ and for any $u_1, \ldots, u_n \in \mathbb{R}$ the weak convergence*

$$(x_\varepsilon(u_1, \cdot), \ldots, x_\varepsilon(u_n, \cdot)) \xrightarrow{w} (x_0(u_1, \cdot), \ldots, x_0(u_n, \cdot)), \quad \varepsilon \to 0+,$$

takes place in the space $C([0; +\infty), \mathbb{R}^n)$.

Note that in this case for the covariance function Γ_ε of the Harris flow x_ε we have

$$\forall\, z \in \mathbb{R}: \quad \Gamma_\varepsilon(z) \longrightarrow 1\!\!\mathrm{I}_{\{0\}}(z), \quad \varepsilon \to 0+,$$

and also

$$\varphi_\varepsilon^2 \longrightarrow \delta_0, \quad \varepsilon \to 0+, \tag{3}$$

in the sense of generalized functions (here and below δ_a stands for the delta function at point $a \in \mathbb{R}$).

In [8] it was shown that the assertion of this theorem still holds true even if φ_ε^2 converges to a generalized function distinct from δ_0.

Theorem 3. [8, p. 1538] *For $\varepsilon > 0$ define*

$$\varphi_\varepsilon(q) := \frac{\sqrt{\alpha}}{\sqrt{\varepsilon}} \varphi\left(\frac{q - a_1}{\varepsilon}\right) + \frac{\sqrt{\beta}}{\sqrt{\varepsilon}} \varphi\left(\frac{q - a_2}{\varepsilon}\right), \quad q \in \mathbb{R},$$

where $0 < \alpha, \beta < 1$, $\alpha + \beta = 1$, and $a_1 < a_2$, and let $\{x_\varepsilon(u,t),\ u \in \mathbb{R},\ t \geqslant 0\}$ be the Harris flow formed by the solutions of the stochastic integral equation (1) *with φ_ε instead of φ. Then for any $n \in \mathbb{N}$ and for any $u_1, \ldots, u_n \in \mathbb{R}$ the weak convergence*

$$(x_\varepsilon(u_1, \cdot), \ldots, x_\varepsilon(u_n, \cdot)) \xrightarrow{w} (x_0(u_1, \cdot), \ldots, x_0(u_n, \cdot)), \quad \varepsilon \to 0+,$$

takes place in the space $C([0; +\infty), \mathbb{R}^n)$.

Note that in this case for the covariance function Γ_ε of the Harris flow x_ε we have

$$\forall\, z \in \mathbb{R}: \quad \Gamma_\varepsilon(z) \longrightarrow \sqrt{\alpha\beta} \cdot 1\!\!\mathrm{I}_{\{-b\}}(z) + 1\!\!\mathrm{I}_{\{0\}}(z) + \sqrt{\alpha\beta} \cdot 1\!\!\mathrm{I}_{\{+b\}}(z), \quad \varepsilon \to 0+,$$

where $b := a_2 - a_1$, and also

$$\varphi_\varepsilon^2 \longrightarrow \alpha\delta_{a_1} + \beta\delta_{a_2}, \quad \varepsilon \to 0+, \tag{4}$$

in the sense of generalized functions.

Here we show that the proof presented in [8] can be extended to the case when the right-hand side of (4) is replaced by a discrete probability measure on the real line satisfying some mild conditions. To be more precise, let ν be an arbitrary finite singular measure on the real line having at least one atom, i. e. such that

$$\nu^2(\Delta) > 0,$$

where

$$\nu^2 := \nu \otimes \nu$$

and

$$\Delta := \{\vec{q} = (q_1, q_2) \in \mathbb{R}^2 \mid q_1 = q_2\}.$$

Suppose that the function φ considered above is additionally non-decreasing on $(-\infty; 0]$ and is non-increasing on $[0; +\infty)$ and that φ_ε is defined by (2). Let us set

$$\psi_\varepsilon(z) := c_\varepsilon \int_{\mathbb{R}} \varphi_\varepsilon(z - q)\, \nu(dq), \quad z \in \mathbb{R},$$

where the constant $c_\varepsilon > 0$ is chosen to be such that

$$\int_{\mathbb{R}} \psi_\varepsilon^2(z)\, dz = 1.$$

It is clear that

$$c_\varepsilon = \left[\iint_{\mathbb{R}^2} \Phi_\varepsilon(q_1 - q_2)\, \nu^2(dq_1 dq_2) \right]^{-1/2},$$

where

$$\Phi_\varepsilon(z) := \int_{\mathbb{R}} \varphi_\varepsilon(z + q)\varphi_\varepsilon(q)\, dq, \quad z \in \mathbb{R},$$

and also

$$\psi_\varepsilon \in C^\infty(\mathbb{R}).$$

For $\varepsilon > 0$ let $\{x_\varepsilon(u, t),\ u \in \mathbb{R},\ t \geqslant 0\}$ be the Harris flow formed by the solutions of the stochastic integral equation (1) with ψ_ε instead of φ. The covariance functions of these Harris flows are given by

$$\Gamma_\varepsilon(z) := \int_{\mathbb{R}} \psi_\varepsilon(z + q)\psi_\varepsilon(q)\, dq, \quad z \in \mathbb{R}.$$

The main result of this note is the following theorem.

Theorem 4. *For any $n \in \mathbb{N}$ and for any $u_1, \ldots, u_n \in \mathbb{R}$ the weak convergence*

$$(x_\varepsilon(u_1, \cdot), \ldots, x_\varepsilon(u_n, \cdot)) \xrightarrow{w} (x_0(u_1, \cdot), \ldots, x_0(u_n, \cdot)), \quad \varepsilon \to 0+,$$

takes place in the space $C([0; +\infty), \mathbb{R}^n)$.

Following [8] we divide the proof into several lemmas. We repeat the considerations of [8], where necessary, as concisely as possible and omit the proofs which are similar to those of that paper. The main difference lies in the proof of Lemma 10, since the idea used in the proof of its analogue [8, Lemma 6] cannot be applied to our case. Our proof of Lemma 10 is based on additional Lemmas 6 and 9.

Before proceeding to the proof of the main result, however, we prove an analogue of relations (3) and (4). To formulate it, let ν_0 be a discrete probability measure on the real line defined by

$$\nu_0(A) := \frac{\sum\limits_{k:\ a_k\in A} (\nu(\{a_k\}))^2}{\sum\limits_{k} (\nu(\{a_k\}))^2}, \quad A \in \mathcal{B}(\mathbb{R}),$$

where $\{a_k,\ k \geqslant 1\}$ are the atoms of the measure ν and $\mathcal{B}(\mathbb{R})$ is the Borel σ-field of the real line.

Proposition 5. *For every function $f \in C_0^\infty(\mathbb{R})$ we have*

$$\lim_{\varepsilon\to 0+} \int\limits_{\mathbb{R}} f(z)\psi_\varepsilon^2(z)\,dz = \int\limits_{\mathbb{R}} f(z)\nu_0(dz).$$

Proof. By Fubini's theorem we have

$$\lim_{\varepsilon\to 0+} \int\limits_{\mathbb{R}} f(z)\psi_\varepsilon^2(z)\,dz =$$

$$= \lim_{\varepsilon\to 0+} \left[c_\varepsilon^2 \iint\limits_{\mathbb{R}^2} \left(\int\limits_{\mathbb{R}} f(z)\varphi_\varepsilon(z-q_1)\varphi_\varepsilon(z-q_2)\,dz \right) \nu^2(dq_1dq_2) \right].$$

However, by the dominated convergence theorem

$$\lim_{\varepsilon\to 0+} c_\varepsilon^2 = \left[\iint\limits_{\mathbb{R}^2} \left(\lim_{\varepsilon\to 0+} \Phi_\varepsilon(q_1-q_2) \right) \nu^2(dq_1dq_2) \right]^{-1} = \frac{1}{\nu^2(\Delta)}.$$

Moreover, since for any $q_1, q_2 \in \mathbb{R}$ we have

$$\left| \int\limits_{\mathbb{R}} f(z)\varphi_\varepsilon(z-q_1)\varphi_\varepsilon(z-q_2)\,dz \right| \leqslant \|f\|_\infty \cdot \Phi_\varepsilon(q_1-q_2) \leqslant \|f\|_\infty < +\infty,$$

where

$$\|f\|_\infty := \max_{z\in\mathbb{R}} |f(z)|,$$

by the same theorem

$$\lim_{\varepsilon\to 0+} \int\limits_{\mathbb{R}} f(z)\varphi_\varepsilon(z-q_1)\varphi_\varepsilon(z-q_2)\,dz = f(q_1)1\!\!\mathrm{I}\{q_1 = q_2\}.$$

It remains to note that

$$\frac{1}{\nu^2(\Delta)} \iint\limits_{\mathbb{R}^2} f(q_1)1\!\!\mathrm{I}\{q_1=q_2\}\,\nu^2(dq_1dq_2) = \int\limits_{\mathbb{R}} f(z)\,\nu_0(dz). \qquad \square$$

Now let us set

$$\Delta_z := \{\vec{q} = (q_1, q_2) \in \mathbb{R}^2 \mid q_1 - q_2 = z\}, \quad z \in \mathbb{R}.$$

Then it is easy to see that for every $z \in \mathbb{R}$ we have

$$\lim_{\varepsilon\to 0+} \Gamma_\varepsilon(z) = \Gamma_0(z),$$

where the function Γ_0 is given by

$$\Gamma_0(z) := \frac{\nu^2(\Delta_z)}{\nu^2(\Delta_0)}.$$

Moreover, the set

$$D := \{z \in \mathbb{R} \mid \Gamma_0(z) > 0\} \tag{5}$$

is countable, since the family $\{\Delta_z,\ z \in \mathbb{R}\}$ is a partition of $\mathbb{R}^2$ and $\nu^2(\mathbb{R}^2) < +\infty$.

Lemma 6. *The following assertions hold true:*

$$\lim_{|z|\to+\infty} \Gamma_0(z) = 0, \tag{6}$$

$$\forall\, \delta > 0 : \quad \sup_{|z|\geqslant\delta} \Gamma_0(z) < 1. \tag{7}$$

Proof. To prove (6) note that for any $\varepsilon > 0$

$$0 \leqslant \Gamma_0(z) \leqslant \frac{1}{\nu^2(\Delta_0)} \iint\limits_{\mathbb{R}^2} \Phi_\varepsilon(z + q_1 - q_2)\, \nu^2(dq_1 dq_2)$$

and that by the dominated convergence theorem the last expression converges to zero as $|z| \to +\infty$.

Now suppose that (7) is false, i. e. that there exists some $\delta_0 > 0$ such that

$$\sup_{|z|\geqslant\delta_0} \Gamma_0(z) = 1. \tag{8}$$

It means, in particular, that we can find some $z_1 \geqslant \delta_0$ such that

$$\Gamma_0(z_1) > \frac{1}{2}.$$

Since the function Γ_0 is non-negative definite and $\Gamma_0(0) = 1$, we have (e. g., see [6, p. 22])

$$\forall\, x, y \in \mathbb{R} : \quad |\Gamma_0(x) - \Gamma_0(y)| \leqslant 2\sqrt{1 - \Gamma_0(x - y)},$$

and, in particular, for any $z \in \mathbb{R}$

$$|\Gamma_0(z_1 + z) - \Gamma_0(z_1)| \leqslant 2\sqrt{1 - \Gamma_0(z)}. \tag{9}$$

Using (8) and (9) and the symmetry of Γ_0 we can choose $z_2 \geqslant \delta_0$ such that

$$|\Gamma_0(z_1 + z_2) - \Gamma_0(z_1)| < \Gamma_0(z_1) - \frac{1}{2}$$

and so

$$\Gamma_0(z_1 + z_2) > \frac{1}{2}.$$

Proceeding further in this way we obtain a sequence $\{z_n\}_{n=1}^{\infty}$ such that

$$z_n \geqslant \delta_0, \quad n \geqslant 1,$$

$$\Gamma_0(z_1 + \ldots + z_n) > \frac{1}{2},$$

which contradicts (6). □

Now fix arbitrary $n \in \mathbb{N}$ and $u_1, \ldots, u_n \in \mathbb{R}$, $u_1 < \ldots < u_n$, and consider the family

$$\{\vec{x}_\varepsilon = (x_\varepsilon(u_1, \cdot), \ldots, x_\varepsilon(u_n, \cdot)),\ \varepsilon > 0\}$$

of random elements in the space $C([0; +\infty), \mathbb{R}^n)$ endowed with the distance

$$\rho(\vec{f}, \vec{g}) := \sum_{i=1}^{n} \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\max\limits_{0\leqslant t\leqslant k} |f_i(t) - g_i(t)|}{1 + \max\limits_{0\leqslant t\leqslant k} |f_i(t) - g_i(t)|},$$

$$\vec{f} = (f_1, \ldots, f_n) \in C([0; +\infty), \mathbb{R}^n),$$

$$\vec{g} = (g_1, \ldots, g_n) \in C([0; +\infty), \mathbb{R}^n).$$

Since all stochastic processes $\{x_\varepsilon(u_i,t),\ t\geqslant 0\}$, $1\leqslant i\leqslant n$, are Wiener processes, thus having the same distribution in the complete separable metric space $C([0;+\infty),\mathbb{R})$, using Prohorov's theorem one can easily show that the family $\{\vec{x}_\varepsilon,\ \varepsilon>0\}$ is weakly relatively compact. Let $\vec{x}=(x(u_1,\cdot),\dots,x(u_n,\cdot))$ be one of its limit points (as $\varepsilon\to 0+$).

Lemma 7. *The n-dimensional stochastic process $\{\vec{x}(t),\ t\geqslant 0\}$ is a martingale (with respect to its own filtration).*

Proof. The proof of this lemma is identical to that of [8, Lemma 2] and is therefore omitted. $\square$

Lemma 8. *With probability one for any $i,j\in\{1,\dots,n\}$ we have*

$$0\leqslant \langle x(u_i,\cdot),x(u_j,\cdot)\rangle_t-\langle x(u_i,\cdot),x(u_j,\cdot)\rangle_s\leqslant\int_s^t\Gamma_0(x(u_i,r)-x(u_j,r))\,dr,$$
$$0\leqslant s\leqslant t<+\infty.$$

Proof. Fix arbitrary $i,j\in\{1,\dots,n\}$, $i\neq j$, and in the space $C([0;+\infty),\mathbb{R}^{n+1})$ consider the random elements

$$\vec{x}_\varepsilon^{(ij)}=(x_\varepsilon(u_1,\cdot),\dots,x_\varepsilon(u_n,\cdot),\theta_\varepsilon^{(ij)}),\quad \varepsilon>0,$$

where

$$\theta_\varepsilon^{(ij)}(t):=\langle x_\varepsilon(u_i,\cdot),x_\varepsilon(u_j,\cdot)\rangle_t,\quad t\geqslant 0.$$

As in the proof of [8, Lemma 3] one can show that the family $\{\vec{x}_\varepsilon^{(ij)},\ \varepsilon>0\}$ is weakly relatively compact and that, if

$$\vec{x}_{\varepsilon_n}^{(ij)}\xrightarrow{w}\vec{x}^{(ij)},\quad n\to\infty,$$

in the space $C([0;+\infty),\mathbb{R}^{n+1})$ for some sequence $\{\varepsilon_n\}_{n=1}^\infty$ strictly decreasing to zero, with $\vec{x}^{(ij)}:=(x(u_1,\cdot),\dots,x(u_n,\cdot),\theta^{(ij)})$, then

$$\theta^{(ij)}(t)=\langle x(u_i,\cdot),x(u_j,\cdot)\rangle_t,\quad t\geqslant 0.$$

Now, since the set

$$\{\vec{f}=(f_1,\dots,f_{n+1})\in C([0;+\infty),\mathbb{R}^{n+1})\mid 0\leqslant f_{n+1}(t)-f_{n+1}(s)\leqslant\int_s^t h_\delta(f_i(r)-f_j(r))\,dr\},$$

where $0\leqslant s\leqslant t<+\infty$ and

$$h_\delta(z):=\frac{1}{\nu^2(\Delta_0)}\iint_{\mathbb{R}^2}\Phi_\delta(z+q_1-q_2)\,\nu^2(dq_1dq_2),\quad z\in\mathbb{R},$$

with $\delta>0$, is closed and

$$\Gamma_\varepsilon(z)=c_\varepsilon^2\iint_{\mathbb{R}^2}\Phi_\varepsilon(z+q_1-q_2)\,\nu^2(dq_1dq_2)\leqslant h_\delta(z),\quad z\in\mathbb{R},$$

for $\varepsilon<\delta$, we obtain that

$$\mathbf{P}\left\{0\leqslant\theta^{(ij)}(t)-\theta^{(ij)}(s)\leqslant\int_s^t h_\delta(x(u_i,r)-x(u_j,r))\,dr\right\}\geqslant$$
$$\geqslant\varlimsup_{n\to\infty}\mathbf{P}\left\{0\leqslant\theta_{\varepsilon_n}^{(ij)}(t)-\theta_{\varepsilon_n}^{(ij)}(s)\leqslant\int_s^t h_\delta(x_{\varepsilon_n}(u_i,r)-x_{\varepsilon_n}(u_j,r))\,dr\right\}=1.$$

Thus, for every $\delta > 0$ with probability one

$$0 \leqslant \theta^{(ij)}(t) - \theta^{(ij)}(s) \leqslant \int\limits_s^t h_\delta(x(u_i, r) - x(u_j, r))\, dr, \quad 0 \leqslant s \leqslant t < +\infty, \quad s, t \in \mathbb{Q},$$

and so with probability one

$$0 \leqslant \theta^{(ij)}(t) - \theta^{(ij)}(s) \leqslant \int\limits_s^t \Gamma_0(x(u_i, r) - x(u_j, r))\, dr, \quad 0 \leqslant s \leqslant t < +\infty.$$

The lemma is proved. □

Lemma 9. *With probability one for any $i, j \in \{1, \ldots, n\}$ we have*

$$\lim_{t \to +\infty} (x(u_i, t) - x(u_j, t)) = 0.$$

Proof. Let us fix arbitrary $i, j \in \{1, \ldots, n\}$, $i > j$. The proof of the existence of the limit

$$\lim_{t \to +\infty} (x(u_i, t) - x(u_j, t))$$

is similar to the proof of [7, Lemma 1]. Namely, we note (e. g., see [5, Theorem 18.4]) that with probability one the following representation takes place:

$$x(u_i, t) - x(u_j, t) = (u_i - u_j) + \beta(\tau(t)), \quad t \geqslant 0, \tag{10}$$

where $\{\beta(t),\ t \geqslant 0\}$ is a standard Wiener process (maybe defined on an extended probability space) and

$$\tau(t) := \langle x(u_i, \cdot) - x(u_j, \cdot) \rangle_t = 2t - 2 \langle x(u_i, \cdot), x(u_j, \cdot) \rangle_t, \quad t \geqslant 0. \tag{11}$$

Then

$$x(u_i, t) - x(u_j, t) \geqslant 0, \quad t \geqslant 0,$$

implies that

$$\tau(t) \leqslant \overline{\tau}, \quad t \geqslant 0,$$

where

$$\overline{\tau} := \inf\{t \geqslant 0 \mid \beta(t) = -(u_i - u_j)\} < +\infty \quad \text{a. s.}$$

Therefore, there exists the limit

$$\lim_{t \to +\infty} \tau(t) =: \tau(+\infty) \leqslant \overline{\tau}$$

and so, due to the continuity of β,

$$\lim_{t \to +\infty} (x(u_i, t) - x(u_j, t)) = (u_i - u_j) + \beta(\tau(+\infty)).$$

Now suppose that

$$\tau(+\infty) < \overline{\tau},$$

i. e.

$$\lim_{t \to +\infty} (x(u_i, t) - x(u_j, t)) > 0.$$

Then there exists $\delta_0 > 0$ (depending on ω) such that

$$x(u_i, t) - x(u_j, t) > \delta_0, \quad t \geqslant 0.$$

So using Lemma 8 (with $s = 0$) and Lemma 6 we obtain that

$$\begin{aligned}\tau(t) &= 2t - 2\left\langle x(u_i,\cdot), x(u_j,\cdot)\right\rangle_t \geqslant \\ &\geqslant 2t - 2\int_0^t \Gamma_0(x(u_i,s) - x(u_j,s))\,ds = \\ &= 2\int_0^t \left[1 - \Gamma_0(x(u_i,s) - x(u_j,s))\right] ds \geqslant \\ &\geqslant 2(1 - \sup_{|z|\geqslant\delta_0} \Gamma_0(z)) \cdot t \to +\infty, \quad t \to +\infty,\end{aligned}$$

which contradicts the almost sure finiteness of $\overline{\tau}$. □

Lemma 10. *With probability one for any $i, j \in \{1, \dots, n\}$ we have*

$$\lambda(\{t \geqslant 0 \mid x(u_i,t) - x(u_j,t) \in D \setminus \{0\}\}) = 0,$$

where λ is the one-dimensional Lebesgue measure and D is defined in (5).

Proof. Let us fix $i, j \in \{1, \dots, n\}$, $i > j$, and $z \in (0; u_i - u_j)$ and set

$$\sigma_z := \sup\{t \geqslant 0 \mid x(u_i,t) - x(u_j,t) \geqslant z\}.$$

From Lemma 9 it follows that σ_z is finite almost surely. Also let τ_z be the restriction of the (random) mapping $\tau\colon [0; +\infty) \to [0; +\infty)$ defined in (11) to the set $[0; \sigma_z]$ and τ_z^{-1} be its inverse. Then using (10) we get

$$\begin{aligned}&\lambda(\{t \geqslant 0 \mid x(u_i,t) - x(u_j,t) \in D \cap [z; +\infty)\}) = \\ &= \lambda(\{0 \leqslant t \leqslant \sigma_z \mid \beta(\tau(t)) \in D_{ij}(z)\}) = \lambda(\tau_z^{-1}(C_{ij}(z))),\end{aligned}$$

where

$$D_{ij}(z) := D \cap [z - (u_i - u_j); +\infty)$$

and

$$C_{ij}(z) := \{t \geqslant 0 \mid \beta(t) \in D_{ij}(z)\}.$$

Moreover, since the stochastic process $\{x(u_i,t) - x(u_j,t),\ t \geqslant 0\}$ is a non-negative (continuous) martingale, we have

$$r_z := \inf\{x(u_i,t) - x(u_j,t) \mid 0 \leqslant t \leqslant \sigma_z\} > 0,$$

and so by Lemma 6

$$\rho_z := 1 - \sup_{|z'|\geqslant r_z} \Gamma_0(z') > 0.$$

Thus, we obtain that for any $s, t \in [0; \sigma_z]$, $s < t$, we have

$$2 \geqslant \frac{\tau(t) - \tau(s)}{t - s} \geqslant \frac{1}{t-s}\int_s^t \left[1 - \Gamma_0(x(u_i,r) - x(u_j,r))\right] dr \geqslant 2\rho_z.$$

Therefore, for any $s, t \in [0; \tau(\sigma_z)]$, $s < t$,

$$\frac{1}{2} \leqslant \frac{\tau_z^{-1}(t) - \tau_z^{-1}(s)}{t - s} \leqslant \frac{1}{2\rho_z}.$$

This implies that the function τ_z is absolutely continuous and so it maps the sets of zero Lebesgue measure to the sets with the same property. Thus, from

$$\lambda(C_{ij}(z)) = 0$$

it follows that

$$\lambda(\{t \geqslant 0 \mid x(u_i,t) - x(u_j,t) \in D \cap [z;+\infty)\}) = 0.$$

Finally, since $z \in (0; u_i - u_j)$ was arbitrary and $x(u_i,\cdot) - x(u_j,\cdot) \geqslant 0$, we can conclude that

$$\lambda(\{t \geqslant 0 \mid x(u_i,t) - x(u_j,t) \in D \setminus \{0\}\}) =$$
$$= \lambda\left(\bigcup_{k \geqslant 1} \{t \geqslant 0 \mid x(u_i,t) - x(u_j,t) \in D \cap [1/k;+\infty)\}\right) = 0.$$

The assertion of the lemma now follows trivially. □

To finish the proof of Theorem 4 (obviously, it is enough to consider the case when $u_1 < \ldots < u_n$) suppose that $(x(u_1,\cdot),\ldots,x(u_n,\cdot))$ is one of the weak limits (as $\varepsilon \to 0+$) of the family $\{\vec{x}_\varepsilon = (x_\varepsilon(u_1,\cdot),\ldots,x_\varepsilon(u_n,\cdot)),\ \varepsilon > 0\}$. Then for any $i,j \in \{1,\ldots,n\}$, $i > j$, the stochastic process $\{x(u_i,t) - x(u_j,t),\ t \geqslant 0\}$ is a non-negative martingale and so does not leave zero after hitting it. Since both $x(u_i,\cdot)$ and $x(u_j,\cdot)$ are standard Brownian motions, this implies that

$$\langle x(u_i,\cdot), x(u_j,\cdot)\rangle_t \geqslant \int_0^t 1\!\!1\{x(u_i,s) = x(u_j,s)\}\, ds, \quad t \geqslant 0.$$

However, from Lemma 8 (with $s = 0$) and Lemma 10 it follows that

$$\langle x(u_i,\cdot), x(u_j,\cdot)\rangle_t \leqslant \int_0^t 1\!\!1\{x(u_i,s) = x(u_j,s)\}\, ds, \quad t \geqslant 0.$$

Hence

$$\langle x(u_i,\cdot), x(u_j,\cdot)\rangle_t = \int_0^t 1\!\!1\{x(u_i,s) = x(u_j,s)\}\, ds, \quad t \geqslant 0.$$

Thus, we conclude that any weak limit (as $\varepsilon \to 0+$) of the family $\{\vec{x}_\varepsilon,\ \varepsilon > 0\}$ coincides in distribution with the n-point motion of the Arratia flow, which means that this family converges weakly to the latter.

Acknowledgements. The author is grateful to the anonymous referee for the careful reading of the paper and useful comments.

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