Theory of Stochastic Processes
Vol. 21 (37), no. 1, 2016, pp. 12–16

ALEXANDER IKSANOV AND SERGUEI POLOTSKIY

TAIL BEHAVIOR OF SUPREMA OF PERTURBED RANDOM WALKS

We prove a lattice version of Goldie's result on tail behavior of suprema of perturbed random walks.

1. Introduction

Let $(\xi_k, \eta_k)_{k\in\mathbb{N}}$ be a sequence of i.i.d. two-dimensional random vectors with generic copy (ξ, η). No condition is imposed on the dependence structure between ξ and η. Let $(S_n)_{n\in\mathbb{N}_0}$ be the zero-delayed ordinary random walk with increments ξ_n for $n \in \mathbb{N}$, i.e., $S_0 = 0$ and $S_n = \xi_1 + \ldots + \xi_n$, $n \in \mathbb{N}$. Then define its perturbed variant $(T_n)_{n\in\mathbb{N}}$, that we call *perturbed random walk (PRW)*, by

$$T_n := S_{n-1} + \eta_n, \quad n \in \mathbb{N}. \tag{1}$$

Recently it has become a rather popular object of research. A non-exhaustive list of recent publications includes [1, 9, 10, 13, 14] as well as the papers on tail behavior cited below. The so defined perturbed random walks are ubiquitous in applied probability. These are closely related to perpetuities, the Bernoulli sieve, the GEM distribution and the Poisson-Dirichlet distribution; processes with regenerative increments and many other models (see [2] for more details).

Set $T_* := \sup_{n\geq 1} T_n$ and note that T_* is a.s. finite if, and only if, $\lim_{n\to\infty} T_n = -\infty$ a.s. According to Theorem 2.1 in [2] the latter is equivalent to

$$\lim_{n\to\infty} S_n = -\infty \text{ a.s.} \quad \text{and} \quad \int_{(0,\infty)} \frac{x}{\int_0^x \mathbb{P}\{-\xi > y\}\mathrm{d}y} \mathrm{d}\mathbb{P}\{\eta \leq x\} < \infty.$$

Often, a simpler sufficient condition suffices: $\mathbb{E}\xi \in (-\infty, 0)$ and $\mathbb{E}\eta^+ < \infty$.

We are interested in tail behavior of T_* in one situation that remained untouched in the previous works on the same topic [4, 5, 7, 8, 11, 12, 15]. Recall that a distribution is called *nonlattice* if it is not concentrated on any lattice $\delta\mathbb{Z}$, $\delta > 0$. A distribution is called *δ-lattice* if it is concentrated on the lattice $\delta\mathbb{Z}$ and not concentrated on any lattice $\delta_1\mathbb{Z}$ for $\delta_1 > \delta$.

Theorem 1.1 given next which treats the nonlattice case is contained in Theorem 5.2 of [5]. It is stated here for the ease of comparison with Theorem 1.2 which treats the lattice case and forms the main contribution of the present paper.

Theorem 1.1. *Suppose that there exists positive a such that*

$$\mathbb{E}e^{a\xi} = 1, \quad \mathbb{E}e^{a\xi}\xi^+ < \infty \text{ and } \mathbb{E}e^{a\eta} < \infty. \tag{2}$$

If the distribution of e^ξ is nonlattice, then

$$\lim_{x\to\infty} e^{ax}\mathbb{P}\{T_* > x\} = C,$$

where $C := \mathbb{E}\big(e^{a\eta_1} - e^{a(\xi_1+T'_*)}\big)\,\mathbb{1}_{\{\xi_1+T'_*\leq\eta_1\}} \in (0,\infty)$ and $T'_* := \sup_{n\geq 2}(T_n - \xi_1)$.

2000 *Mathematics Subject Classification.* Primary 60G50; Secondary 60K05.

Key words and phrases. perturbed random walk; supremum; tail behavior.

Theorem 1.2. *Suppose that* (2) *holds. If the distribution of* e^{ξ} *is* δ*-lattice, then, for each* $x \in \mathbb{R}$,

$$\lim_{k\to\infty} e^{(\delta k+x)a}\mathbb{P}\{T_* > \delta k + x\} = C(x)$$

for some positive δ*-periodic function* $C(x)$.

2. Proof of Theorem 1.2

We need a version of the key renewal theorem for the lattice distributions *concentrated on the whole line*. Even though the result is widely used in the literature, we are not aware of any reference which would give a proof.

Proposition 2.1. *Assume that* ξ *has a* δ*-lattice distribution concentrated on* $\mathbb{R}$ *and* $\mu = \mathbb{E}\xi \in (0,\infty)$. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a function that satisfies* $\sum_{j\in\mathbb{Z}} |f(x+\delta j)| < \infty$ *for some* $x \in \mathbb{R}$. *Then*

$$\lim_{n\to\infty} \mathbb{E}\sum_{k\geq 0} f(x+\delta n - S_k) = \mu^{-1}\delta \sum_{j\in\mathbb{Z}} f(x+\delta j).$$

Proof. By considering f^+ and f^- separately, without loss of generality f may be assumed nonnegative.

Suppose first that $\xi \geq 0$ a.s. Set $u(\delta n) := \sum_{k\geq 0} \mathbb{P}\{S_k = \delta n\}$. By the classical Blackwell theorem $\lim_{n\to\infty} u(\delta n) = \mu^{-1}\delta$. Hence, for any $\varepsilon \in (0, \mu^{-1}\delta)$ there exists a $j_0 \in \mathbb{N}$ such that

$$\mu^{-1}\delta - \varepsilon \leq u(\delta j) \leq \mu^{-1}\delta + \varepsilon$$

whenever $j \geq j_0 + 1$. Using this we obtain

$$\begin{aligned}
&\mathbb{E}\sum_{k\geq 0} f(x+\delta n - S_k)\\
&= \sum_{j=0}^{j_0} f\big(x+\delta(n-j)\big)u(\delta j) + \sum_{j\geq j_0+1} f\big(x+\delta(n-j)\big)u(\delta j)\\
&\leq \sum_{j=0}^{j_0} f\big(x+\delta(n-j)\big)u(\delta j) + (\mu^{-1}\delta+\varepsilon)\sum_{j=-\infty}^{n-j_0-1} f\big(x+\delta j\big). \qquad (3)
\end{aligned}$$

The assumption $\sum_{j\in\mathbb{Z}} f(x+\delta j) < \infty$ ensures $\lim_{n\to\infty} f(x+\delta n) = 0$, whence

$$\limsup_{n\to\infty} \mathbb{E}\sum_{k\geq 0} f(x+\delta n - S_k) \leq \mu^{-1}\delta \sum_{j\in\mathbb{Z}} f(x+\delta j)$$

on letting in (3) first $n\to\infty$ and then ε to zero. The converse inequality for the lower limit follows analogously.

The general case when ξ takes values of both signs will now be handled by reducing it to the case $\xi > 0$ a.s. via a stopping time argument. We use the representation

$$\mathbb{E}\sum_{k\geq 0} f(x+\delta n - S_k) = \mathbb{E}\sum_{j\geq 0}\sum_{i=\tau_j}^{\tau_{j+1}-1} f(x+\delta n - S_i) = \mathbb{E}\sum_{k\geq 0} f^*(x+\delta n - S_{\tau_k})$$

where $(\tau_k)_{k\in\mathbb{N}_0}$ are successive strictly increasing ladder epochs for (S_n) given by $\tau_0 = 0$ and $\tau_n = \inf\{k > \tau_{n-1} : S_k > S_{\tau_{n-1}}\}$ for $n \in \mathbb{N}$, and $f^*(x) := \mathbb{E}\sum_{j=0}^{\tau-1} f(x - S_j)$, $x \in \mathbb{R}$ (we write τ for τ_1). The sequence $(S_{\tau_k})_{k\in\mathbb{N}_0}$ is an ordinary random walk with positive jumps having the same distribution as S_τ. Observe that $\mathbb{E}S_\tau = \mu\mathbb{E}\tau$ by Wald's identity, and that the distribution of S_τ is δ-lattice. Since

$$\begin{aligned}\sum_{j\in\mathbb{Z}} f^*(x+\delta j) &= \sum_{j\in\mathbb{Z}}\mathbb{E}\sum_{k=0}^{\tau-1}\sum_{i\leq 0} f\big(x+\delta(j-i)\big)\,\mathbb{1}_{\{S_k=\delta i\}}\\ &= \mathbb{E}\sum_{k=0}^{\tau-1}\sum_{i\leq 0}\mathbb{1}_{\{S_k=\delta i\}}\sum_{j\in\mathbb{Z}} f(x+\delta(j-i))\\ &= \mathbb{E}\tau\sum_{j\in\mathbb{Z}} f(x+\delta j)<\infty,\end{aligned}$$

an application of the already proved result in the case $\xi\geq 0$ a.s. yields

$$\lim_{n\to\infty}\mathbb{E}\sum_{k\geq 0} f(x+\delta n - S_k) = \frac{\delta}{\mathbb{E}S_\tau}\sum_{j\in\mathbb{Z}} f^*(x+\delta j) = \frac{\delta}{\mu}\sum_{j\in\mathbb{Z}} f(x+\delta j).$$

The proof of Proposition 2.1 is complete. □

The referee kindly reminded us that another proof of Proposition 2.1 could be based on the Blackwell theorem for the whole line.

Proof of Theorem 1.2. Assume that the distribution of e^ξ is δ-lattice. We shall use the random variables which appear in the following representation

$$T_* = \max(\eta_1, \xi_1+\sup(\eta_2, \xi_2+\eta_3, \xi_2+\xi_3+\eta_4,\ldots)) = \max(\eta_1, \xi_1+T_*')\quad \text{a.s.}$$

where $T_*' = \sup_{n\geq 2}(T_n-\xi_1)$ is independent of (ξ_1,η_1) and has the same distribution as T_*.

Set, for $x\in\mathbb{R}$,

$$P(x) := e^{ax}\mathbb{P}\{T_*>x\}$$

and

$$Q(x) := e^{ax}\bigg(\mathbb{P}\{T_*>x\}-\mathbb{P}\{\xi_1+T_*'>x\}\bigg).$$

Since

$$e^{ax}\mathbb{P}\{\xi_1+T_*'>x\} = \int_{\mathbb{R}} P(x-t)\mathrm{d}\mathbb{P}\{\xi'\leq t\},\quad x\in\mathbb{R},$$

where ξ' is a random variable with distribution $\mathbb{P}\{\xi'\in \mathrm{d}x\} = e^{ax}\mathbb{P}\{\xi\in\mathrm{d}x\}$, we conclude that P is a (locally bounded) solution to the renewal equation

$$P(x) = \int_{\mathbb{R}} P(x-t)\mathrm{d}\mathbb{P}\{\xi'\leq t\} + Q(x),\quad x\in\mathbb{R}. \tag{4}$$

It is well-known that

$$P(x) = \mathbb{E}\sum_{j\in\mathbb{Z}} Q(x-S_j'),\quad x\in\mathbb{R},$$

where $(S_k')_{k\in\mathbb{N}_0}$ is a zero-delayed ordinary random walk with jumps having the distribution of ξ'. Observe that $\mathbb{E}e^{b\xi}\xi^-<\infty$ for all $b>0$. In particular, $\mathbb{E}e^{a\xi}\xi^-<\infty$ which in combination with the second condition in (2) ensures $\mathbb{E}e^{a\xi}\xi\in\mathbb{R}$. The convexity of $m(x) := \mathbb{E}e^{x\xi}$ on $[0,a]$ together with $m(0)=m(a)=1$ implies that m is increasing at the left neighborhood of a whence the left derivative $m'(a)$ is positive. Since

$\mathbb{E}\xi' = \mathbb{E}e^{a\xi}\xi = m'(a)$, we have proved that $\mathbb{E}\xi' \in (0,\infty)$. Further,

$$\begin{aligned}
0 &\le e^{-ax}\sum_{j\in\mathbb{Z}} Q(x+\delta j) \\
&= \sum_{j\in\mathbb{Z}} e^{a\delta j}\big(\mathbb{P}\{\max(\eta_1, \xi_1+T'_*) > x+\delta j\} - \mathbb{P}\{\xi_1+T'_* > x+\delta j\}\big) \\
&= \sum_{j\in\mathbb{Z}} e^{a\delta j}\big(\mathbb{P}\{\eta_1 > x+\delta j, \xi_1+T'_* < \eta_1\} - \mathbb{P}\{\xi_1+T'_* > x+\delta j, \xi_1+T'_* < \eta_1\}\big) \\
&\le \sum_{j\in\mathbb{Z}} e^{a\delta j}\mathbb{P}\{\eta_1 > x+\delta j\}.
\end{aligned}$$

The assumption $\mathbb{E}e^{a\eta} < \infty$ guarantees that the last series converges for each $x \in \mathbb{R}$. Thus, we have checked that the series $\sum_{j\in\mathbb{Z}} Q(x+\delta j)$ converges for each $x \in \mathbb{R}$.

By the key renewal theorem for the lattice case (Proposition 2.1)

$$\lim_{n\to\infty} P(x+\delta n) = \frac{\delta}{\mathbb{E}e^{a\xi}\xi}\sum_{j\in\mathbb{Z}} Q(x+\delta j) =: C(x). \tag{5}$$

It remains to show that $C(x) > 0$. To this end, pick $y \in \mathbb{R}$ such that $p := \mathbb{P}\{\eta > y\} > 0$. For any fixed $x > 0$, there exists $i \in \mathbb{Z}$ such that $x - y \in [\delta i, \delta(i+1))$. With the help of Lemma[1] 2.2 in [3] we obtain, for large enough n,

$$\begin{aligned}
P(x+\delta n) &= e^{a(x+\delta n)}\mathbb{P}\{T_* > x+\delta n\} \\
&\ge pe^{ay}e^{a(x-y+\delta n)}\mathbb{P}\{\sup_{k\ge 0} S_k > x-y+\delta n\} \\
&\ge pe^{a(y-1)}e^{a\delta(i+n+1)}\mathbb{P}\{\sup_{k\ge 0} S_k > \delta(n+i+1)\}.
\end{aligned}$$

Therefore, it suffices to prove that

$$\liminf_{n\to\infty} e^{a\delta n}\mathbb{P}\{\sup_{k\ge 0} S_k > \delta n\} > 0. \tag{6}$$

For $x \ge 0$, set $\tau(x) := \inf\{k \in \mathbb{N} : S_k > x\}$, with the usual convention that $\inf \oslash = \infty$, and $\tau := \tau(0)$. Define a new probability measure[2] $\mathbb{P}_a$ by

$$\mathbb{E}_a h(S_0, \ldots, S_k) = \mathbb{E}e^{aS_k} h(S_0, \ldots, S_k), \quad k \in \mathbb{N} \tag{7}$$

for each Borel function $h : \mathbb{R}^{k+1} \to [0,\infty)$, where $\mathbb{E}_a$ is the corresponding expectation. Since the $\mathbb{P}$-distribution of ξ' is the same as the $\mathbb{P}_a$-distribution of S_1, we have $\mathbb{E}_a S_1 = \mathbb{E}\xi' \in (0,\infty)$. Therefore, $(S_n)_{n\in\mathbb{N}_0}$, under $\mathbb{P}_a$, is an ordinary random walk with the positive drift whence $\mathbb{E}\tau(x) < \infty$ for each $x \ge 0$ and thereupon $\mathbb{E}_a S_\tau = \mathbb{E}_a S_1 \mathbb{E}_a \tau \in (0,\infty)$. Further, for each $x > 0$,

$$\begin{aligned}
e^{ax}\mathbb{P}\{\sup_{k\ge 0} S_k > x\} = e^{ax}\mathbb{P}\{\tau(x) < \infty\} &= e^{ax}\mathbb{E}_a e^{-aS_{\tau(x)}}\,\mathbb{1}_{\{\tau(x)<\infty\}} \\
&= \mathbb{E}_a e^{-a(S_{\tau(x)}-x)}
\end{aligned}$$

having utilized (7) for the second equality. Since S_1, under $\mathbb{P}_a$, has a δ-lattice distribution, an application of Theorem 10.3(ii) on p. 104 in [6] yields $S_{\tau(\delta n)} - \delta n$ converges in $\mathbb{P}_a$-distribution as $n \to \infty$ to a random variable Y with $\mathbb{P}_a\{Y = \delta k\} = \frac{\delta}{\mathbb{E}_a S_\tau}\mathbb{P}_a\{S_\tau \ge \delta k\}$, $k \in \mathbb{N}_0$. This immediately implies that

$$\lim_{n\to\infty} e^{a\delta n}\mathbb{P}\{\sup_{k\ge 0} S_k > \delta n\} = \lim_{n\to\infty}\mathbb{E}_a e^{-a(S_{\tau(\delta n)}-\delta n)} = \mathbb{E}_a e^{-aY} > 0,$$

[1] This result states that $\mathbb{P}\{T_* > x\} \ge \mathbb{P}\{\eta > y\}\mathbb{P}\{\sup_{n\ge 0} S_n > x-y\}$ for all $x, y \in \mathbb{R}$.

[2] This is indeed a probability measure because, in view of the first condition in (2), $(e^{aS_n})_{n\in\mathbb{N}_0}$ is a nonnegative martingale with respect to the natural filtration.

a result that is stronger than (6). The proof of Theorem 1.2 is complete. □

The authors thank the referee for several useful comments.

References

1. G. Alsmeyer, A. Iksanov and A. Marynych, *Functional limit theorems for the number of occupied boxes in the Bernoulli sieve*, Stoch. Proc. Appl. (2017), to appear, preprint available at http://arxiv.org/abs/1601.04274.
2. G. Alsmeyer, A. Iksanov and M. Meiners, *Power and exponential moments of the number of visits and related quantities for perturbed random walks*, J. Theoret. Probab. **28** (2015), 1–40.
3. G. Alsmeyer, A. Iksanov and U. Rösler, *On distributional properties of perpetuities*, J. Theoret. Probab. **22** (2009), 666–682.
4. V. F. Araman and P. W. Glynn, *Tail asymptotics for the maximum of perturbed random walk*, Ann. Appl. Probab. **16** (2006), 1411–1431.
5. C. M. Goldie, *Implicit renewal theory and tails of solutions of random equations*, Ann. Appl. Probab. **1** (1991), 126–166.
6. A. Gut, *Stopped random walks. Limit theorems and applications.* 2nd Edition, Springer, 2009.
7. X. Hao, Q. Tang and L. Wei, *On the maximum exceedance of a sequence of random variables over a renewal threshold*, J. Appl. Probab. **46** (2009), 559–570.
8. A. Iksanov, *On the supremum of perturbed random walk*, Bulletin of Kiev University **1** (2007), 161–164 (in Ukrainian).
9. A. Iksanov and A. Pilipenko, *On the maximum of a perturbed random walk*, Stat. Probab. Letters. **92** (2014), 168–172.
10. M. M. Meerschaert and S. A. Stoev, *Extremal limit theorems for observations separated by random power law waiting times*, J. Stat. Planning and Inference **139** (2009), 2175–2188.
11. Z. Palmowski and B. Zwart, *Tail asymptotics of the supremum of a regenerative process*, J. Appl. Probab. **44** (2007), 349–365.
12. Z. Palmowski and B. Zwart, *On perturbed random walks*, J. Appl. Probab. **47** (2010), 1203–1204.
13. E. I. Pancheva and P. K. Jordanova, *Functional transfer theorems for maxima of iid random variables*, Comptes Rendus de l'Académie Bulgare des Sciences **57** (2004), 9–14.
14. E. Pancheva, I. K. Mitov and K. V. Mitov, *Limit theorems for extremal processes generated by a point process with correlated time and space components*, Stat. Probab. Letters. **79** (2009), 390–395.
15. C. Y. Robert, *Asymptotic probabilities of an exceedance over renewal thresholds with an application to risk theory*, J. Appl. Probab. **42** (2005), 153–162.

Faculty of Cybernetics, Taras Shevchenko National University of Kyiv, Kyiv-01601, Ukraine
E-mail address: iksan@univ.kiev.ua

Faculty of Cybernetics, Taras Shevchenko National University of Kyiv, Kyiv-01601, Ukraine
E-mail address: sergiy.polotskiy@gmail.com