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ON THE EVOLUTION OF GIBBS STATES OF THE LATTICE GRADIENT STOCHASTIC DYNAMICS OF INTERACTING OSCILLATORS

Grand canonical correlation functions of stochastic(Brownian) lattice linear oscillators interacting via a pair short-range potential are found in the thermodynamic limits at low activities and on a finite time interval. It is proved that their sequence is a weak solution of the BBGKY-type gradient diffision hierarchy. The initial correlation functions are Gibbsian, which corresponds to many-body positive finite-range and short-range non-positive pair interaction potentials. The utilized technique is based on an application of the Feynman–Kac formula for solutions of the Smoluchowski equation and a representation of the time-dependent correlation functions in terms of correlation functions of a Gibbs lattice oscillator path system with many-body interaction potentials.

1. Introduction and main result

The lattice gradient stochastic oscillator dynamics with pair interaction is described by the infinite-component evolution equation

$$\dot{q}_x(t) = -\partial u^0(q_x) - \sum_{y \neq x} \partial_x u^0_{x,y}(q_x, q_y) + \beta^{-\frac{1}{2}} \dot{w}_x(t), \qquad x \in \mathbb{Z}^d,$$

where $q_x \in \mathbb{R}$, $u^0_{x,y}(q_x, q_y)$ is the pair interaction potentials, u^0 is an external potential, $\dot{w}_x(t)$ are independent processes of white noise, $\partial_x = \frac{\partial}{\partial q_x}$, the summation is performed over $\mathbb{Z}^d \backslash x = x^c$. We assume that $u^0(q)$ is an even bounded from below polynomial of the $2n$-th degree and the potential $u^0_{x,y}$ is polynomial and short-range. The existence of solutions for the lattice stochastic system for the simplest pair interaction was established in [9],[5], and their special properties were described in [3], [4], [2], [1].

Physical states of the stochastic dynamics are described by probability measures on the infinite Cartesian product $\mathbb{R}^Y, Y = \mathbb{Z}^d$ or, equivalently, by correlation functions of the canonical or grand canonical ensemble. The associated gradient diffusion BBGKY-type hierarchy for the correlation functions is given by

$$\frac{\partial}{\partial t}\rho(q_X; t) = \sum_{x \in X} \partial_x \{\beta^{-1}\partial_x \rho(q_X; t) + \rho(q_X; t)\partial_x U^0(q_X) +$$

$$+ \sum_{y \in X^c} \int (\partial_x u^0_{x,y})(q_x, q_y)\rho(q_{X \cup y}; t) dq_y\}, \tag{1.1}$$

where

$$U^0(q_X) = \sum_{x \in X} u^0(q_x) + \sum_{x \neq y \in X} u^0_{x,y}(q_x, q_y),$$

$X^c = \mathbb{Z}^d \backslash X$, $|X| < \infty$, the integration is performed over $\mathbb{R}^{|Y|}$, $|Y|$ is the number of sites in Y, and β is the inverse temperature. It is derived from the finite-volume gradient

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diffusion hierarchy, coinciding with (1.1), in which the summation in the integral term is performed over $\Lambda\backslash X$ instead of X^c, in the thermodynamic limit $\Lambda \to \mathbb{Z}^d$. Formal arguments show that this finite-volume hierarchy is satisfied by the finite-volume grand canonical correlation functions

$$\rho^{\Lambda}(q_X;t) = \Xi_{\Lambda}^{-1}\chi_{\Lambda}(X)\sum_{Y\subseteq\Lambda\backslash X} z^{|X|+|Y|}\int \rho^0(q_{X\cup Y};t)dq_Y,$$

(1.2)

$$\Xi_{\Lambda} = \sum_{Y\subseteq\Lambda} z^{|Y|}\int \rho^0(q_Y;t)dq_Y,$$

where χ_{Λ} is the characteristic function of a hypercube Λ, z is the activity, and the distribution (integrable, positive) function $\rho^0(q_X;t)$ satisfies the Smoluchowski equation which coincides with (1.1), in which the integral term is omitted. The stationary solution of the Smoluchowski equation coincides with $\exp\{-\beta U^0(q_X)\}$. A rigorous derivation of the finite-volume diffusion hierarchy demands $\rho^0(q_X;t)$ to be sufficiently smooth and have a sufficient decrease at infinity in the oscillator variables. We establish that the sequence ρ^{Λ} of the correlation functions is a weak solution of the lattice finite-volume diffusion hierarchy.

We say that that the sequence ρ of the correlation functions is a weak solution of the lattice diffusion hierarchy (1.1) if, for a twice differentiable function $f(q_X)$ which is bounded together with its derivatives, the following equality is true:

$$\frac{d}{dt}\int f(q_X)\rho(q_X;t)dq_X = \sum_{x\in X}\int \{\rho(q_X;t)[\beta^{-1}\partial_x^2 f(q_X) - (\partial U^0)(q_X)\partial_x f(q_X)]-$$

(1.2′)

$$-(\partial_x f)(q_X)\sum_{y\in X^c}\int (\partial_x u^0_{x,y})(q_x,q_y)\rho(q_{X\cup y};t)dq_y\}dq_X.$$

In this paper, we will find the grand canonical correlation functions in the thermodynamic limit for the initial Gibbsian correlation functions generated by many-body potentials and prove that their sequence is a weak solution of the gradient diffusion hierarchy on a finite time interval and at a small enough activity. The initial functions, ρ^0, are given by

$$\rho^0(q_X) = e^{-\beta(2^{-1}U^0(q_X)+U_1(q_X))}, \qquad U_1(q_X) = \sum_{x\in X} u_1(q_x) + \sum_{|Y|\ge 2, Y\subseteq X} u_{1;Y}(q_Y),$$

where the summation in Y is performed over subsets of X, u_1 is a bounded from below even polynomial of the $2n_1$-th degree, the potentials $u_{1;Y}$ are polynomial for all Y, finite-range, and positive for $|Y|>2$, such that $|u_{1;Y}(q_Y)| \le J_Y^1\sum_{y\in Y}(q_y^{2l_{|Y|}}+1)$, $|\partial_y u_{1;Y}(q_Y)| \le J_Y^1\sum_{y\in Y}(|q_y|^{2l_{|Y|}-1}+1)$, where $l_{|Y|} = 2(n-1)$ if $|Y|>2$ and $l_{|Y|} = 2l_1 < 2n_1$ if $|Y|=2$ (this implies $u_{1;Y}(q_Y)=0$ for $|Y|>2n_1-1$), $||J^1||_1 = \max_x \sum_{x\in Y} J_Y^1 < \infty$, where the summation in Y is performed over subsets of $\mathbb{Z}^d$ containing x.

We will require also that $u^0_{x,y}(q_x,q_y) = J^0_{x-y}u^0(q_x,q_y)$, $J^0_x = J^0_{-x}$,

$$||\sqrt{J^0}||_1 = \sum_x \sqrt{J^0_x} < \infty,$$

where the summation is performed over $\mathbb{Z}^d$ and

$$u^0(q,q') = \sum_{s=1}^{l_0}\sum_{l+k=2s}\phi^0_{l,k}(q^l q'^k + q'^l q^k), \qquad 2l_0 \le 2(n-1),$$

where $\phi^0_{l,k} = \phi^0_{k,l}$ are real numbers. Our main result is formulated as the following theorem.

Theorem 1.1. *There exists a weak solution $\rho(q_X;t)$ of the diffusion hierarchy (1.1) which is the thermodynamic limit of $\rho^\Lambda(q_X;t)$ and given as an expansion in powers of z convergent in a disc $|z| \le g_*^{-1}$ such that*

$$|\rho(q_X;t\beta) - \rho^\Lambda(q_X;t\beta)| \le \epsilon_*(\lambda)\exp\{-\beta\sum_{x\in X}\bar{u}(q_x)\}g^{-|X|}, \quad X\in\Lambda(\lambda), \tag{1.3}$$

$$\rho(q_X;t\beta) \le N_*\exp\{-\beta\sum_{x\in X}\bar{u}(q_x)\}g^{-|X|}, \quad \bar{u}(q) = \frac{1}{2}u^0(q) - \gamma q^{2(n-1)} - \bar{\gamma}, \tag{1.4}$$

where $\epsilon_*(\lambda), g_*, g, N_*, \bar{\gamma}$ are positive locally bounded functions of (t,β) on $\mathbb{R}^+ \times R^+\backslash 0$, independent of X, q_X, growing at infinity in t, γ is a positive constant, $g_* > g$, $\Lambda(\lambda)$ is the set of lattice sites located in Λ, whose distance from the boundary of Λ is greater than λ, and $\epsilon_*(\lambda)$ is a continuous function tending to zero at infinity in λ.

We prove this theorem by reducing the problem of the thermodynamic limit for $\rho^\Lambda(q_X;t)$ to the problem of the thermodynamic limit for the complex Gibbsian path correlation functions $\rho^\Lambda(\omega_X)$ depending on the Wiener paths $\omega_X = (w_x, w^*_x, x\in X)$

$$\rho^\Lambda(\omega_X) = \Xi^{-1}_\Lambda\chi_\Lambda(X)\sum_{Y\subseteq\Lambda\backslash X} z^{|X|+|Y|}\int\exp\{-\beta U(\omega_{X\cup Y})\}P(d\omega_Y), \tag{1.5}$$

where $P(d\omega_Y) = \prod_{y\in Y} P(d\omega_y)$, $P(d\omega) = e^{-\beta u(w)}dqP_q(dw)P_0(dw^*) = P'(dw)P_0(dw^*)$, the integration is performed over $\mathbb{R}^{|Y|}\times\Omega_0^{2|Y|}$, Ω_0 is the probability space of Wiener paths, $P_q(dw)$ is the Wiener measure concentrated on paths starting from q, the complex U, u depend on the Wiener paths on the interval [0,t], the grand partition function Ξ_Λ coincides with the numerator for $X=\emptyset$ and

$$U(\omega_X) = \sum_{|Y|\ge 2, Y\subseteq X} u_Y(\omega_Y), \tag{1.6}$$

where $u^1_Y(\omega_Y) = Re u_Y(\omega_Y) = u_{1,Y}(w_Y(t))$, $u_{*(Y)}(\omega_Y) = Im u_Y(\omega_Y) = 0$ for $|Y| > 2$. The correlation functions $\rho^\Lambda(q_X;t\beta)$ are expressed in terms of the complex path correlation functions as follows:

$$\rho^\Lambda(q_X;t\beta) = \int\rho^\Lambda(\omega_X)e^{-\beta\sum_{x\in X}u(w_x)}P_0(dw^*_X)P_{q_X}(dw_X), \tag{1.7}$$

where the integration is performed over $\Omega_0^{2|X|}$, $P_{q_X}(dw_X) = \prod_{x\in X} P_{q_x}(dw_x)$. It is worth to add also that the real-valued part U^1 of U depends only on w_X, and the imaginary part U_* of U is generated by the pair potential $u_{*(x,y)}$. The presence of β on the left-hand side of (1.7) makes the expressions of the path potentials u, u_Y more simple. Note that the introduction of the measure $dqP_q(dw)$ is simplified by the fact that the Wiener measure is translation invariant: $\int P_q(dw)f(w) = \int P_0(dw)f(w+q)$.

The fact that the many-body potentials u^1_Y are finite-range and positive enables us to solve the symmetrized (with respect to the superstability condition) KS (Kirkwood–Salzburg) resolvent-type equation which is satisfied by the sequence of the path correlation functions $\rho^\Lambda(\omega_X)$ in the thermodynamic limit $\rho(\omega_X)$. We solve the symmetrized KS equation with the help of the resolvent expansion of the symmetrized KS operator showing that it is bounded in the Banach space $\mathbb{E}_{\xi,f}$ ($\mathbb{E}_\xi = \mathbb{E}_{\xi,0}$) of sequences of measurable functions with the norm

$$||F||_{\xi,f} = \max_X \xi^{-|X|} ess \sup_{w_X} \exp\{-\sum_{x\in X} f(\omega_x)\}|F_X(\omega_X)|.$$

The choice of f will depend exclusively on the pair potentials $u^0_{x,y}, u_{1;x,y}$ and guarantees that $N_0 = \int e^{f(\omega)}P(d\omega) < \infty$. As a result, the thermodynamic limit $\rho(q_X;t)$ of $\rho^\Lambda(q_X;t)$ will be expressed in terms of $\rho(\omega_X)$ by

$$\rho(q_X;t\beta) = \int \rho(\omega_X)P_0(dw^*_X)P_{q_X}(dw_X). \tag{1.8}$$

The analog of Theorem 1.1 for the canonical correlation functions of the oscillator stochastic lattice dynamics with Gibbsian initial correlation functions, generated by a pair potential, was formulated in [13], where a polynomial expansion was utilized. The polynomial expansion was proven to converge in the thermodynamic limit for the Gibbsian canonical correlation functions with a pair potential at high-temperatures in [7]. The idea to express correlation functions of the stochastic dynamics in terms of correlation functions of a Gibbs path system with a ternary interaction potential was applied for particle systems in [14]. The similar idea was utilized in [15], where the author showed that the lattice stochastic oscillator dynamics admits a long-range order for an initial canonical Gibbs state.

Our paper is organized as follows: in Section 2, we derive (1.7); in Section 3, we write down the finite-volume, infinite-volume, and symmetrized KS equations for the complex path correlation functions and derive also the recursion relation for the KS kernels; in Section 4, we find the norm of the KS and symmetrized KS operators; and in Section 5 by Theorem 5.1, we establish the existence of solutions of the infinite-volume symmetrized KS equation, i.e. we find the thermodynamic limit of the path correlation functions and prove Theorem 1.1 by using (1.8).

2. Heat equation and Feynman–Kac formula

A derivation of (1.7) is fulfilled in three steps. The first step is the transformation of the Smoluchowski equation into the heat equation. In the second one, we solve the latter with the help of the Feynman–Kac (FK) formula. Indeed, after the substitution

$$\rho^0(q_X;t) = e^{-\frac{\beta}{2}U^0(q_X)}\psi(q_X;t), \tag{2.1}$$

the following heat equation for ψ is obtained:

$$\frac{\partial}{\partial t}\psi(q_X;t) = \beta^{-1}\sum_{x\in X}\partial_x^2\psi(q_X;t) - U_2(q_X)\psi(q_X;t), \tag{2.2}$$

$$U_2(q_X) = \frac{1}{2}\sum_{x\in X}[-\partial_x^2 U^0(q_X) + \frac{\beta}{2}(\partial_x U^0(q_X))^2].$$

We solve the L^2-Cauchy problem for the heat equation with the help of the well-known [12] FK formula

$$\psi(q_X;\beta t) = \int P_{q_X}(dw_X)e^{-\beta\int\limits_0^t U_2(w_X(\tau))d\tau}\psi_0(w_X(t)), \tag{2.3}$$

where $\psi_0 \in L^2(\mathbb{R}^{|X|})$ is the initial data. The derivation of the FK formula is based on the application of the Trotter product formula [12, T.X.51]. A proof of the Trotter product formula demands the Hamiltonian to be essentially self-adjoint on the intersection of the domains of the Laplacian and its unbounded perturbation U_2. This is guaranteed by example X.9.3 from [12] for a bounded from below polynomial U_2 (by adding a finite positive constant, this polynomial can be transformed into a positive one).

The solution $\rho^0(q_X;t)$ of the Cauchy problem for the oscillator Smoluchowski equation with the initial data $\rho^0(q_X) = e^{-\beta(2^{-1}U^0(q_X)+U_1(q_X))}$ is obtained from (2.1) and (2.3) putting $\psi_0(q_X) = e^{-\beta U_1(q_X)}$ in it. That is,

$$\rho^0(q_X;\beta t) = \int P_{q_X}(dw_X)e^{-\beta\tilde{U}(w_X)}, \tag{2.4}$$

where

$$\tilde{U}(w_X) = 2^{-1}U^0(q_X) + U_1(\omega_X(t)) + \int_0^t U_2(w_X(\tau))d\tau, \qquad w_X(0) = q_X.$$

The following representation for the second term in the expression for U_2 holds:

$$\sum_{x\in X}(\partial_x U^0(q_X))^2 = \sum_{x\in X}(\partial_x u^0(q_x))^2 + \sum_{x\neq y\in X} u'_{x,y}(q_x,q_y) + U'_2(q_X). \tag{2.5}$$

Here,

$$u'_{x,y}(q_x,q_y) = J^0_{x-y}u_2(q_x,q_y), \qquad U'_2(q_X) = \sum_{x\in X}(\sum_{y\neq x, y\in X}\partial_x u^0_{x,y}(q_x,q_y))^2,$$

$$u_2(q_x,q_y) = (\partial_x u^0(q_x,q_y))\partial_x u^0(q_x) + (\partial_y u^0(q_x,q_y))\partial_y u^0(q_y)).$$

The similar representation for the first term in the expression for U_2 looks as

$$\sum_{x\in X}\partial_x^2 U^0(q_X) = \sum_{x\in X}\partial^2 u^0(q_x) + \sum_{x\neq y\in X} u^{(2)}_{x,y}(q_x,q_y), \tag{2.6}$$

where

$$u^{(2)}_{x,y}(q_x,q_y) = \frac{1}{2}[\partial_x^2 u^0_{x,y}(q_x,q_y) + \partial_y^2 u^0_{x,y}(q_x,q_y)] = \frac{1}{2}J^0_{x-y}[\partial_x^2 u^0(q_x,q_y) + \partial_y^2 u^0(q_x,q_y)]$$

On the third step of the derivation of (1.7), we construct the imaginary part U_* of U from U'_2, by utilizing the formula

$$\exp\{-\int_0^t f^2(\tau)d\tau\} = \int \exp\{-i\int_0^t f(\tau)dw^*(\tau)\}P_0(dw^*),$$

where the right-hand side contains the stochastic integral [8] determined as the integral of f with the generalized Gaussian process of white noise. This stochastic integral is determined as a strong limit of the sequence of Riemannian sums of cylinder functions in the space of quadratically integrable functions. As a result,

$$\exp\{-\frac{\beta^2}{4}\int_0^t U'_2(w_X(\tau))d\tau\} =$$

$$= \int \exp\{-i\frac{\beta}{2}\sum_{x\in X}\int_0^t \sum_{x\neq y\in X}\partial_x u^0_{x,y}(w_x(\tau),w_y(\tau))dw^*_x(\tau)\}P_0(dw^*_X).$$

or

$$\exp\{-\frac{\beta^2}{4}\int_0^t U'_2(w_X(\tau))d\tau\} = \int \exp\{-i\beta U_*(\omega_X)\}P_0(dw^*_X). \tag{2.7}$$

where

$$U_*(\omega_X)=\sum_{x\neq y\in X}u_{*(x,y)}(\omega_x,\omega_y),\qquad u_{*(x,y)}(\omega_x,\omega_y)=J^0_{x-y}u_*(\omega_x,\omega_y),$$

and

$$u_*(\omega_x,\omega_y)=\frac{1}{2}(f_*(w_x,w_y|w_x^*)+f_*(w_y,w_x|w_y^*)),\tag{2.8}$$

$$f_*(w_x,w_y|w_x^*)=\int_0^t dw_x^*(\tau)\partial_x u^0(w_x(\tau),w_y(\tau)).$$

Our stochastic integrals depend on the Wiener paths without stars, and the sequence of the Riemannian sums depends also on them. Hence, the limiting function of the sums will be measurable in all the Wiener paths since the Wiener measure is concentrated on continuous paths, and the strong limit of a convergent sequence in a space of quadratically integrable functions contains a subsequence converging almost everywhere to a measurable function. Let

$$U^1(w_X)=\tilde U(w_X)-\frac{\beta}{4}\int_0^t U_2'(w_X(\tau))d\tau-\sum_{x\in X}u(w_x),$$

where $u(\omega_x)=\frac{1}{2}u^0(q_x)+u_1(w_x(t))+u_2(w_x)$,

$$u_2(w_x)=\int_0^t\tilde u(w_x(\tau))d\tau,\qquad \tilde u(q_x)=-\frac{1}{2}\partial^2u^0(q_x)+\frac{\beta}{4}(\partial u^0(q_x))^2.$$

From (2.4-8), it follows that the functions u,U,u_Y appeared in (1.6-7) are given by the last two equalities and

$$U(\omega_X)=U^1(w_X)+iU_*(\omega_X),\qquad u_Y(\omega_Y)=u^1_Y(w_Y)+iu_{*(Y)}(\omega_Y),\tag{2.9}$$

where

$$u^1_{x,y}(w_x,\omega_y)=\frac{1}{2}u^0_{x,y}(q_x,q_y)+u_{1;x,y}(w_x(t),w_y(t))+u_{2;x,y}(w_x,w_y),$$

and

$$u_{2;x,y}(w_x,w_y)=\int_0^t\tilde u_{x,y}(w_x(\tau),w_y(\tau))d\tau,$$

$$\tilde u_{x,y}(q_x,q_y)=-\frac{1}{2}u^{(2)}_{x,y}(q_x,q_y)+\frac{\beta}{4}u'_{x,y}(q_x,q_y).$$

The expression for the pair potential $u^0(q,q')$ allows one to derive the bounds

$$|\partial^s u^0(q,q')|\le\frac{1}{2}[v_{0,s}(q)+v_{0,s}(q')],\quad |\partial'\partial u^0(q,q')|\le\frac{1}{2}[v_{0,2}(q)+v_{0,2}(q')],\tag{2.10}$$

where $v_{0,s}(q)=a_s(q^{2l_0-s}+1),\quad s=0,1,2,\ v_{0,0}=v_0$. We assume that $2J^1_{x,y}=J^1_{x-y}=J^1_{-x+y}$, that is,

$$|u_{1;x,y}(q,q')|\le\frac{1}{2}J^1_{x-y}[v_1(q)+v_1(q')],\qquad v_1(q)=q^{2l_1}+1.\tag{2.11}$$

Since $u^0(q)$ is a bounded from below polynomial of the $2n$-th degree, we have $|\partial u^0(q)|\le a_1^0|q|^{2n-1}$, $|\partial^2u^0(q)|\le a_2^0(|q|^{2(n-1)}+1)$, and relation (2.10) yields

$$|u_2(q_x,q_y)|\le[|\partial_x u^0(q_x)|+|\partial_y u^0(q_y)|][|\partial_x u^0(q_x,q_y)|+|\partial_y u^0(q_y,q_x)|]\le$$

$$\leq a_1a_1^0(|q_x|^{2n-1}+|q_y|^{2n-1})(|q_x|^{2l_0-1}+|q_y|^{2l_0-1}+2)\leq$$

$$\leq a_1a_1^0[(|q_x|+|q_y|)^{2(n+l_0-1)}+2(|q_x|+|q_y|)^{2n-1}]\leq$$

$$\leq a_1a_1^0 2^{2(l_0+n)}(|q_x|^{2(n+l_0-1)}+|q_y|^{2(n+l_0-1)}+2).$$

Here, we used the inequalities $(a+b)^n\geq a^n+b^n, (a+b)^n\leq 2^n(a^n+b^n), a^n\leq a^m+1, a,b\geq 0, n<m$. This leads to

$$|\tilde{u}_{x,y}(q_x,q_y)|\leq \frac{J^0_{x-y}}{2}[\tilde{v}(q_x)+\tilde{v}(q_y)],\qquad \tilde{v}(q)=$$

$$= (\beta a_1a_1^0 2^{2(l_0+n-1)}+a_2)(q^{2(n+l_0-1)}+1). \tag{2.11'}$$

The last inequality and (2.11) result in the following inequality which will be used in the next section for symmetrizing the KS equation:

$$|u^1_{x,y}(\omega_x,\omega_y)|\leq \frac{J_{x-y}}{2}[v^1(w_x)+v^1(w_y)]. \tag{2.12}$$

Here,

$$v^1(w)=2^{-1}v_0(q)+v_1(w(t))+v_2(w),\quad J_x=\max(J^0_x,J^1_x),\quad v_2(w)=\int\limits_0^t\tilde{v}(w(\tau))d\tau.$$

3. KS equation

A derivation of the KS equation for $\rho^\Lambda(\omega_X)$ is based on the application of the equality

$$F(\omega_X)=\sum_{S\subseteq X}\sum_{S'\subseteq S}(-1)^{|S\backslash S'|}F(\omega_{S'}) \tag{3.1}$$

which follows from the simple equality

$$n=|X|,\quad \sum_{S\in X}(-1)^{|S|}=\sum_{l=0}^{n}(-1)^lC^l_n=0,\qquad C^l_n=\frac{n!}{l!(n-l)!}.$$

Indeed, let us consider the coefficient before $F(\omega_{X\backslash x})$ on the right-hand side of the previous equality for arbitrary x. It corresponds either to the case $S=X$ or $S=X\backslash x$ and $S'=X\backslash x$. The signs before F are different for these options, and this coefficient is equal to zero. Further, one has to take $S=X, S=X\backslash x_1, S=X\backslash x_2, S=X\backslash x_1\cup x_2$, $S'=X\backslash x_1\cup x_2$ and check that the coefficient before $F(\omega_{X\backslash x_1\cup x_2})$, i.e. the last equality for $n=2$, is equal to zero. In the same fashion, one has to calculate the coefficients before $F(\omega_{X\backslash x_1\cup x_2...\cup x_n})$, corresponding to the choice $S'=X\backslash x_1\cup x_2...\cup x_n$, and check that it coincides with the above sum with the binomial coefficients .

Let $x\in X$ and $X\cap Y=\emptyset$. Then

$$e^{-\beta U(\omega_X,\omega_Y)}=e^{-\beta W(\omega_x|\omega_{X\backslash x},\omega_Y)}e^{-\beta U(\omega_{X\backslash x},\omega_Y)}, \tag{3.1'}$$

where

$$W(\omega_x|\omega_Y)=U(\omega_x,\omega_Y)-U(\omega_Y),\qquad x\cap Y=\emptyset$$

From $(3.1-1')$, it follows that

$$e^{-\beta W(\omega_x|\omega_{X\backslash x},\omega_Y)}=\sum_{S\subseteq Y}K(\omega_x|\omega_{X\backslash x};\omega_S), \tag{3.2}$$

where

$$K(\omega_x|\omega_{X\backslash x};\omega_Y)=\sum_{S\subseteq Y}(-1)^{|Y\backslash S|}e^{-\beta W(\omega_x|\omega_{X\backslash x},\omega_S)}. \tag{3.3}$$

Then, substituting (3.1′), (3.2-3) into the expression for finite-volume grand canonical correlation functions, one obtains

$$\rho^\Lambda(\omega_X) = \Xi_\Lambda^{-1}\chi_\Lambda(X)\sum_{Y\subseteq\Lambda\backslash X} z^{|Y\cup X|}\sum_{S\subseteq Y}\int P(d\omega_Y)K(\omega_x|\omega_{X\backslash x};\omega_S)e^{-\beta U(\omega_{X\cup Y\backslash x})}$$
$$= z\sum_{Z\subseteq\Lambda\backslash X}\int P(d\omega_Z)K(\omega_x|\omega_{X\backslash x};\omega_Z)\Xi_\Lambda^{-1}\chi_\Lambda(X\cup Z)$$
$$\times\sum_{Y\subseteq\Lambda\backslash(Z\cup X)} z^{|Y\cup X\cup Z|-1}\int P(d\omega_Y)e^{-\beta U(\omega_{Z\cup X\backslash x},\omega_Y)}.$$

The equality

$$\rho^\Lambda(\omega_{X\backslash x}) = \Xi_\Lambda^{-1}\chi_\Lambda(X\backslash x)\sum_{Y\subseteq(\Lambda\backslash X)\cup x} z^{|Y\cup X|-1}\int P(d\omega_Y)e^{-\beta U(\omega_{X\backslash x},\omega_Y)}.$$

leads to

$$\Xi_\Lambda^{-1}\chi_\Lambda(X\cup Z)\sum_{Y\subseteq\Lambda\backslash(Z\cup X)} z^{|Y\cup X\cup Z|-1}\int P(d\omega_Y)e^{-\beta U(\omega_{Z\cup X\backslash x},\omega_Y)} =$$
$$= \chi_\Lambda(x)(\rho^\Lambda(\omega_{X\backslash x\cup Z}) - \int P(d\omega_x)\rho^\Lambda(\omega_{X\cup Z})).$$

It is clear that the terms with $x\in Y$ in the sum, representing the first summand in the round brackets, are cancelled by the second term in the brackets. This completes the derivation of the KS equation, if one takes also into account that $\rho^\Lambda(\omega_\emptyset)=1$. It is given for $x\in X, |X|>1$ by

(3.3′)

$$\rho^\Lambda(\omega_X) = z\chi_\Lambda(x)\sum_{Z\subseteq\Lambda\backslash X}\int K(\omega_x|\omega_{X\backslash x};\omega_Z)[\rho^\Lambda(\omega_{X\backslash x\cup Z}) - \int P(d\omega_x)\rho^\Lambda(\omega_{X\cup Z})]P(d\omega_Z)$$

and, for $X=x$, by

$$\rho^\Lambda(\omega_x) = z\chi_\Lambda(x)\{1 - \int\rho^\Lambda(\omega_x)P(d\omega_x) + \sum_{|Z|\geq 1, Z\subseteq\Lambda\backslash x}\int K(\omega_x|\omega_Z)[\rho^\Lambda(\omega_Z)-$$
$$-\int P(d\omega_x)\rho^\Lambda(\omega_{Z\cup x})]P(d\omega_Z)\}.$$

Here, one has to take the following equality into account:

$$\Xi_\Lambda^{-1}\chi_\Lambda(x)\sum_{Y\subseteq\Lambda\backslash x} z^{|Y\cup x|-1}\int P(d\omega_Y)e^{-\beta U(\omega_Y)} = \chi_\Lambda(x)(1 - \int P(d\omega_x)\rho^\Lambda(\omega_x)).$$

It is equivalent to

$$\Xi_\Lambda^{-1}\chi_\Lambda(x)\sum_{x\in Y\subseteq\Lambda} z^{|Y|}\int P(d\omega_Y)e^{-\beta U(\omega_Y)} = \int P(d\omega_x)\rho^\Lambda(\omega_x).$$

Let $\alpha(\omega_X)=\delta_{|X|,1}$. Let, also, the KS operator K be given for $\Lambda=\mathbb{Z}^d$ by the right-hand side in (3.3′), if $|X|>1$, and by the right-hand side of the next equality without unity, if $X=x$. As a result, the finite-volume and infinite-volume KS equations in the abstract look like

(3.4) $$\rho_\Lambda = zK_\Lambda\rho_\Lambda + z\chi_\Lambda\alpha, \qquad \rho = zK\rho + z\alpha,$$

where $K_\Lambda=\chi_\Lambda K\chi_\Lambda$, χ_Λ is the operator of multiplication by the characteristic function of Λ: $(\chi_\Lambda F)_X(\omega_X)=\chi_\Lambda(X)F_X(\omega_X)$.

In order to treat the case of non-positive potentials, one has to symmetrize the KS equations with the help of the superstability condition (see [10]) which follows from (2.12),

$$\sum_{x\neq y\in X} u^1_{x,y}(\omega_x,\omega_y) \geq -\bar{J}\sum_{x\in X} v^1(\omega_x), \qquad |X|\geq 2.$$

where $\bar{J} = ||J||_1$. This means that there is a non-trivial set on which the following inequality holds:

$$\sum_{y\in X, y\neq x} u^1_{x,y}(\omega_x,\omega_y) \geq -\bar{J}v^1(\omega_x). \tag{3.4'}$$

Let $\chi_x(\omega_X)$ be its characteristic (indication) function. Then

$$\sum_{x\in X}\chi^*_x(\omega_X)=1, \qquad \chi^*_x(\omega_X)=(\sum_{y\in X}\chi_y(\omega_X))^{-1}\chi_x(\omega_X). \tag{3.5}$$

The symmetrized KS operator $\tilde{K}$ is given, for $|X|\geq 2$, by

$$(\tilde{K}F)(\omega_X)=\sum_{x\in X}\chi^*_x(\omega_X)\sum_{Z\subseteq X^c}\int K(\omega_x|_{X\backslash x};\omega_Z)[F(\omega_{X\backslash x\cup Z}) - \int P(d\omega_x)F(\omega_{X\cup Z})]P(d\omega_Z).$$

The symmetrized finite-volume and infinite-volume KS equations are given, respectively, by

$$\rho_\Lambda = z\tilde{K}_\Lambda\rho_\Lambda + z\chi_\Lambda\alpha, \qquad \tilde{K}_\Lambda=\chi_\Lambda\tilde{K}\chi_\Lambda,$$

and

$$\rho = z\tilde{K}\rho + z\alpha. \tag{3.6}$$

Proposition 3.1. *Let all the potentials be finite-range except the pair one and have the range R. Then the following equality holds for $X\cap Y=\emptyset$, $x\in X$:*

$$K(\omega_x|\omega_{X\backslash x};\omega_Y)=\sum_{S'\subseteq Y}K(\omega_x|\omega_{X\backslash x};\omega_{S'})\chi_{B_x(R)}(S')G(\omega_x|\omega_{Y\backslash S'})\chi_{B^c_x(R)}(Y\backslash S'). \tag{3.7}$$

Here, $B_x(R)$ is a hyperball with radius R centered at x, $B^c_x(R)=\mathbb{Z}^d\backslash B_x(R)$,

$$G(\omega_x|\omega_S)=\sum_{S'\subseteq S}(-1)^{|S\backslash S'|}e^{-\beta W_2(\omega_x|\omega_{S'})}=\prod_{y\in S}(e^{-\beta u_{(x,y)}(\omega_x,\omega_y)}-1),$$

and $W_2(\omega_x|\omega_{S'})=\sum_{y\in S'}u_{x,y}(\omega_x,\omega_y)$.

Proof. The many-body potentials have finite range R, that is, for an arbitrary $x\in X, |X|>2$, the equality $u_X(\omega_X)=0 holds, |x-x'|\geq R, x'\in X\backslash x$, and $|x-x'|$ is the Euclidean distance between two lattice sites. This means that

$$W(\omega_x|_{X\backslash x},\omega_S)=W(\omega_x|\omega_{X\backslash x},\omega_{S\backslash S_2})+W_2(\omega_x|\omega_{S_2}), \qquad y\notin B_x(R)\to y\in S_2. \tag{3.8}$$

Here, one has to take the following equality into account:

$$W_2(\omega_x|\omega_S)=W_2(\omega_x|q_{S_2})+W_2(\omega_x|\omega_{S\backslash S_2}).$$

Let us substitute the equality

$$1=\prod_{y\in Y}(\chi_{B^c_x(R)}(y)+\chi_{B_x(R)}(y))=\sum_{S'\subseteq Y}\chi_{B_x(R)}(S')\chi_{B^c_x(R)}(Y\backslash S')$$

into the expression for the KS kernel and apply (3.8). This results in

$$\sum_{S\subseteq Y}(-1)^{|Y\backslash S|}e^{-\beta W(\omega_x|\omega_{X\backslash x},\omega_S)}\sum_{S'\subseteq Y}\chi_{B_x(R)}(S')\chi_{B^c_x(R)}(Y\backslash S')=$$

$$= \sum_{S'\subseteq Y}\sum_{S\subseteq Y}(-1)^{|Y\backslash S|}e^{-\beta W(\omega_x|\omega_{X\backslash x},\omega_S)}\chi_{B_x(R)}(S')\chi_{B^c_x(R)}(Y\backslash S') =$$

$$= \sum_{S'\subseteq Y}\sum_{S_2\subseteq Y\backslash S'}\sum_{S_1\subseteq S'}(-1)^{(|Y|-|S_1|-|S_2|)}\times$$

$$\times e^{-\beta[W(\omega_x|\omega_{X\backslash x},\omega_{S_1})+W_2(x|\omega_{S_2})]}\chi_{B_x(R)}(S')\chi_{B^c_x(R)}(Y\backslash S') =$$

$$= \sum_{S'\subseteq Y}\chi_{B_x(R)}(S')\chi_{B^c_x(R)}(Y\backslash S')\sum_{S_1\subseteq S'}(-1)^{(|S'|-|S_1|)}e^{-\beta W(\omega_x|\omega_{X\backslash x},\omega_{S_1})}\times$$

$$\times \sum_{S_2\subseteq Y\backslash S'}(-1)^{(Y-|S'|-|S_2|)}e^{-\beta W_2(\omega_x|\omega_{S_2})}.$$

This proves the proposition.

4. Norm of the KS operator

For the norm of the symmetrized KS operator, we have

$$||\tilde{K}||_{\xi,f} \le (\xi^{-1}+N_0)ess\sup_{X,\omega_X} e^{-f(\omega_x)}$$

(4.1)
$$\times\sum_{x\in X}\sum_{Y\subseteq X^c}\xi^{|Y|}\chi^*_x(\omega_X)\int|K(\omega_x|\omega_{X\backslash x},\omega_Y)|e^{\sum\limits_{y\in Y}f(\omega_y)}P(d\omega_Y).$$

From the positivity of the many-body potentials and (2.12), one derives

$$|K(\omega_x|\omega_{X\backslash x};\omega_Y)| \le \sum_{S\subseteq Y}e^{-\beta W_2(\omega_x|\omega_{X\backslash x},\omega_S)} =$$

$$= e^{-\beta W_2(\omega_x|\omega_{X\backslash x})}\sum_{S\subseteq Y}e^{-\beta W_2(\omega_x|\omega_S)} \le$$

$$\le e^{-\beta W_2(\omega_x|\omega_{X\backslash x})}e^{\frac{\beta}{2}\bar{J}v^1(\omega_x)}\sum_{S\subseteq Y}e^{\frac{\beta}{2}\sum\limits_{z\in S}J_{x-z}v^1(\omega_z)} =$$

$$= e^{-\beta W_2(\omega_x|\omega_{X\backslash x})}e^{\frac{\beta}{2}\bar{J}v^1(\omega_x)}\prod_{z\in Y}(1+e^{\frac{\beta}{2}J_{x-z}v^1(\omega_z)}).$$

That is, the last inequality and (3.4′) lead to

(4.2)
$$\chi^*_x(\omega_X)|K(\omega_x|\omega_{X\backslash x};\omega_Y)| \le e^{\frac{3\beta}{2}\bar{J}v^1(\omega_x)}\prod_{z\in Y}(1+e^{\frac{\beta}{2}J_{x-z}v^1(\omega_z)})\chi^*_x(\omega_X).$$

It follows from the recursion relation (3.7), the definition of KS kernels, and the last inequality that

$$\chi^*_x(\omega_X)|K(q_x|\omega_{X\backslash x},\omega_Y)| \le$$

$$\le e^{\frac{3\beta}{2}\bar{J}v^1(\omega_x)}\sum_{S\subseteq Y}\prod_{z\in S}(1+e^{\frac{\beta}{2}J_{x-z}v^1(\omega_z)})\chi_{B_x(R)}(S)G(\omega_x|\omega_{Y\backslash S'})\chi^*_x(\omega_X) =$$

$$= e^{\frac{3\beta}{2}\bar{J}v^1(\omega_x)}\prod_{y\in Y}[G(\omega_x|\omega_y)+\chi_{B_x(R)}(y)(1+e^{\frac{\beta}{2}J_{x-y}v^1(\omega_y)})]\chi^*_x(\omega_X)$$

and then

(4.3)
$$\chi^*_x(\omega_X)|K(\omega_x|\omega_{X\backslash x},\omega_Y)| \le e^{\frac{3\beta}{2}\bar{J}v^1(\omega_x)}\prod_{y\in Y}[G(\omega_x|\omega_y)+2\chi_{B_x(R)}(y)e^{\frac{\beta}{2}\bar{J}v^1(\omega_y)}]\chi^*_x(\omega_X).$$

The last inequality yields

$$\sum_{x\in X}\sum_{Y\subseteq X^c}\xi^{|Y|}\chi^*_x(q_X))\int|K(\omega_x|\omega_{X\backslash x},\omega_Y)e^{\sum\limits_{y\in Y}f(\omega_y)}|P(d\omega_Y) \le$$

$$\leq \sum_{x\in X}\chi_x^*(\omega_X)e^{\frac{3\beta}{2}\bar{J}v^1(\omega_x)}\prod_{y\neq x}[1+\xi(K_{x,y}+2\chi_{B_x(R)}(y)N_1)] =$$

$$= \max_x e^{\frac{3\beta}{2}\bar{J}v^1(\omega_x)}\prod_{y\neq x}[1+\xi(K_{x,y}+2\chi_{B_x(R)}(y)N_1)], \tag{4.4}$$

where

$$K_{x,y} = \int |e^{-\beta u_{x,y}(\omega_x,\omega_y)}-1|e^{f(\omega_y)}P(d\omega_y),$$

$N_1 = ||e^{f+\frac{\beta}{2}\bar{J}v^1}||_1$. Here, we used (3.5). Let us estimate $K_{x,y}$. From

$$|e^{-\beta u_{x,y}(\omega_x,\omega_y)}-1| \leq |e^{-\beta u^1_{x,y}(\omega_x,\omega_y)}-1| + |e^{-i\beta u_{*(x,y)}(\omega,\omega')}-1| \leq$$
$$\leq |e^{-\beta u^1_{x,y}(\omega_x,\omega_y)}-1| + 2\beta|u_{*(x,y)}(\omega_x,\omega_y)|,$$

it follows that

$$|e^{-\beta u_{x,y}(\omega_x,\omega_y)}-1| \leq \beta[|u^1_{x,y}(\omega_x,\omega_y)|e^{\beta|u^1_{x,y}(\omega_x,\omega_y)|}+2|u_{*(x,y)}(\omega_x,\omega_y)|]. \tag{4.5}$$

From the Schwartz inequality and (2.12), one deduces

$$\int |u_{*(x,y)}(\omega_x,\omega_y)|e^{f(\omega_y)}P(d\omega_y) \leq J^0_{x-y}v_*(\omega_x)N_2, \qquad N_2^2 = ||e^{2f}||_1,$$

where $v_*^2(\omega) = \int |u_*(\omega,\omega')|^2P(d\omega')$(see the Remark at the end of the present paper). From (2.12), it follows also that

$$\int |u^1_{x,y}(\omega_x,\omega_y)e^{\beta|u^1_{x,y}(\omega_x,\omega_y)|}e^{f(\omega_y)}|P(d\omega_y) \leq 2^{-1}J_{x-y}(N_1v^1(w_x)+N_3)e^{\frac{\beta}{2}J_{x-y}v^1(\omega_x)},$$

where $N_3 = ||v^1e^{f+\frac{\beta}{2}\bar{J}v^1}||_1$. Two last inequalities show that

$$K_{x,y} \leq \beta[2^{-1}J_{x-y}(N_1v^1(w_x)+N_3)+J^0_{x-y}v_*(\omega_x)N_2]e^{\frac{\beta}{2}J_{x-y}v^1(\omega_x)}. \tag{4.6}$$

Two last inequalities imply that the expression under the sign of the product in (4.4) is less than the exponent of

$$\xi[2\chi_{B_x(R)}(y)N_1+\frac{\beta}{2}J_{x-y}((N_1+1)v^1(\omega_x)+N_3)+\xi N_2\sqrt{\beta J^0_{x-y}}+\sqrt{\beta J^0_{x-y}}v_*(\omega_x)]. \tag{4.6'}$$

Here, we applied the formulas $1+a+be^c \leq e^{a+b+c}, a,b,c \geq 0,\ bc \leq be^c$, where a, bc correspond to the first and second terms under the square bracket on the right-hand side of (4.6). As a result, we have

(4.7)

$$||\tilde{K}||_{\xi,f} \leq (\xi^{-1}+N_0)e^{\xi g_1}ess\sup_\omega \exp\{-f(\omega)+(\xi g_2+\frac{3}{2}\bar{J}\beta)v^1(\omega_x)+\sqrt{\beta}||\sqrt{J^0}||_1v_*(\omega_x)\},$$

where

$$g_1 = 2|B_0(R)|N_1+\frac{\beta}{2}||J||_1N_3+\sqrt{\beta}N_2||\sqrt{J^0}||_1, \quad g_2 = \frac{\beta}{2}||J||_1(N_1+1). \tag{4.8}$$

To make the KS norm finite, we have to choose

$$f(\omega) = \beta(1+\frac{3}{2}\bar{J})f_0(w)+\gamma_*\sqrt{\beta}v_*(\omega), \qquad f_0(w) = (2^{-1}v_0(q))^{1+\zeta_0}+v_1^{1+\zeta_1}(w(t))+v_2^{1+\zeta_2}(w),$$

where

$$\gamma_* > 0, \quad 1 < 1+\zeta_0 \leq \frac{n-1}{l_0}, \quad 1 < 1+\zeta_1 \leq \frac{n_1-1}{l_1}, \quad 1 < 1+\zeta_2 \leq \frac{2(n-1)}{l_0+n-1}.$$

For $\gamma_* \geq ||\sqrt{J^0}||_1$, the estimation of the norm of the KS equation is reduced to the estimation of $ess\sup_\omega \exp\{-\beta f_0(\omega)+\xi g_2v^1(\omega_x)\}$, since $v^1 \leq f_0+3$. Since

$$\max_{v\geq 0} e^{-v^{1+\zeta}+av} = \exp\{\zeta(\frac{a}{1+\zeta})^{\frac{1+\zeta}{\zeta}}\},$$

we obtain

$$||\tilde{K}||_{\xi,f} \le (\xi^{-1}+N_0)e^{\xi g_1+g_0(\xi)+\frac{9}{2}\bar{J}\beta}, \tag{4.9}$$

where

$$g_0(\xi)=\sum_{l=0}^{2} g^0(\zeta_l,\xi), \qquad g^0(\zeta,\xi)=\zeta(\frac{\beta^{-\frac{1}{\zeta+1}}g_2\xi}{1+\zeta})^{\frac{1+\zeta}{\zeta}}.$$

Lemma 4.1. *Let $\Lambda\subseteq\Lambda'\subseteq\Lambda''$, and let δ be the distance of Λ to the boundary of Λ'. Then*

$$||\chi_\Lambda\tilde{K}(\chi_{\Lambda''}-\chi_{\Lambda'})||_{\xi,f}\le C_\delta(\xi),$$

where the positive $C_\delta(\xi)$ tends to zero if δ tends to infinity.

Proof We have to bound the right-hand side of the inequality

$$||\chi_\Lambda\tilde{K}(\chi_{\Lambda''}-\chi_{\Lambda'})||_{\xi,f}\le(\xi^{-1}+N_0)\max_X\sup_{\omega_X}\sum_{x\in X}\chi^*_x(\omega_X)[\sum_{Y\in X^c}\xi^{|Y|}\chi_\Lambda(X)(\chi_{\Lambda''}(Y)-$$

$$-\chi_{\Lambda'}(Y))e^{-f(\omega_x)}\int|K(\omega_x|\omega_{X\backslash x},\omega_Y)|e^{\sum\limits_{y\in Y}f(\omega_y)}P(d\omega_Y)]. \tag{4.10}$$

In the derivation of this inequality, we employed the equalities

$$\chi_\Lambda(X)(\chi_{\Lambda''}(X\cup Y)-\chi_{\Lambda'}(X\cup Y))=\chi_\Lambda(X)(\chi_{\Lambda''}(Y)-\chi_{\Lambda'}(Y))],$$

$$\chi_\Lambda(X)(\chi_{\Lambda''}((X\backslash x)\cup Y)-\chi_{\Lambda'}((X\backslash x)\cup Y))=\chi_\Lambda(X)(\chi_{\Lambda''}(Y)-\chi_{\Lambda'}(Y)).$$

For $Y=\emptyset$, the right-hand sides are equal to zero. The first and the second equalities correspond to the first and the second terms on the right-hand side of the KS equation for ρ^Λ. From the formulas

$$0\le\chi_{\Lambda''}(Y)-\chi_{\Lambda'}(Y)\le\sum_{y\in Y}(1-\chi_{\Lambda'}(y)),$$

$$\sum_{Y\subseteq X^c}\sum_{y\in Y}(1-\chi_{\Lambda'}(y))F(Y)=\sum_{z\in X^c}(1-\chi_{\Lambda'}(z))\sum_{Y'\subseteq X^c\backslash z}F(z\cup Y') \tag{4.11}$$

(here, $X^c\backslash z=(X\cup z)^c$), (4.3) and (4.4), where the summation in Y is performed over $(X\cup z)^c$, one sees that the expression in the square brackets on the right-hand side of (4.10) is less than

$$\max_x\chi_\Lambda(x)\sum_z(1-\chi_{\Lambda'}(z))e^{-f(\omega_x)+\frac{3\beta}{2}\bar{J}v^1(\omega_x)}\xi K^*_{x,z}\prod_{y\ne x,z}(1+\xi K^*_{x,y}),$$

where $K^*_{x,y}=K_{x,y}+2\chi_{B_x(R)}(y)N_1$. Then this expression is less than [due to (4.6)]

$$\max_x\xi\chi_\Lambda(x)\sum_z(1-\chi_{\Lambda'}(z))$$
$$\times\left[\frac{\beta}{2}\{J_{x-z}(v^1(\omega_x)N_1+N_3)+J^0_{x-z}v_*(\omega_x)N_2\}\right.$$
$$\left.\times e^{-\frac{\beta}{2}f_0(\omega_x)-2^{-1}\sqrt{\beta}||\sqrt{J^0}||_1v_*(\omega_x)}+2N_1\chi_{B_x(R)}(z)\right]$$
$$\times e^{\frac{\beta}{2}J_{x-z}v^1(\omega_x)-f(\omega_x)+\frac{\beta}{2}f_0(\omega_x)+\frac{3\beta}{2}\bar{J}v^1(\omega_x)+2^{-1}\sqrt{\beta}||\sqrt{J^0}||_1v_*(\omega_x)}$$
$$\times\prod_{y\ne x,z}(1+\xi K^*_{x,y}).$$

The first term in the braces is less than $J_{x-z}g_3$ (we put $c_l = \max_{a\geq 0} ae^{-a^l}$), where

$$g_3 = \frac{\beta}{2}[N_3 + N_1\sum_{l=0}^{2}(\beta^{-1}2)^{\frac{1}{1+\zeta_l}}c_{1+\zeta_l} + 2(\gamma_*\sqrt{\beta})^{-1}N_2].$$

In addition, we have [due to $(4.6-6')$]

$$e^{\frac{\beta}{2}J_{x-z}v^1(q_x)}\prod_{y\neq x,z}(1+\xi K^*_{x,y}) \leq \exp\{\xi g_2 v^1(\omega_x) + \sqrt{\beta}||\sqrt{J^0}||_1 v_*(\omega_x) + \xi g_1\},$$

$$\chi_\Lambda(x)\sum_z(1-\chi_{\Lambda'}(z))J_{x-z} \leq \sum_{|z|\geq\delta}J_z = ||J\chi_{B_0^c(\delta)}||_1.$$

From (4.10) and two last inequalities, one deduces the needed equality with

$$C_\delta(\xi) = c(\delta)\xi(\xi^{-1}+N_0)e^{g_0(2\xi)}, \qquad \gamma_* \geq 2||\sqrt{J^0}||_1, \tag{4.12}$$

where

$$c(\delta) = g_3||J\chi_{B_0^c(\delta)}||_1 + 2N_1||\chi_{B_0(R)}\chi_{B_0^c(\delta)}||_1.$$

The value of $c(\delta)$ tends to zero, when δ tends to infinity, since $||J||_1 < \infty$. The lemma is proven.

The optimal choice of ξ is

$$\xi = g^{-1}, \qquad g = g_1 + [\beta^{-\frac{1}{\zeta_-+1}}(1-\chi^+(\beta-1)) + \chi^+(\beta-1)]g_2, \tag{4.13}$$

where $\zeta_- = \min\zeta_l$ and χ^+ is the characteristic function of $\mathbb{R}^+$ and $\chi^+(0) = 1$. These conditions, the expression for g_0, and (4.9) lead to the most simple bound

$$||\tilde{K}||_{g^{-1},f} \leq (g+N_0)e^{4+\frac{9}{2}\bar{J}\beta} = g_*. \tag{4.14}$$

The dependence of g and the norms N_s on t,β will be only rarely indicated.

5. Proof of Theorem 1.1

To prove that the symmetrized KS operator is bounded, we have to show that the norms N_s are finite. This will de done with the help of the following lemma.

Lemma 5.1. *For arbitrary positive γ_0, γ_1, the following inequality holds:*

$$\int e^{\gamma_0 v_*(w,w^*)}P_0(dw^*) \leq \kappa(\gamma_0\gamma_1^{-1}t^{1-n'}, z_0)[\sqrt{I_P} + e^{\frac{\gamma_1}{2}\int_0^t w^{2(2n-1)}(\tau)d\tau}], \quad n' = \frac{2l_0-1}{2(2n-1)}.$$

Here,

$$I_P = \int P'(dw)\exp\{\gamma_1\int_0^t w^{2(2n-1)}(\tau)d\tau\},$$

and $\kappa(a, z_0)$ does not depend on w and is an entire function of a, z_0.

Proof. From the Schwartz and Hölder inequalities, it follows that [v_* is determined below in(4.8)]

$$\int v_*^m(\omega)P_0(dw^*) \leq [\int v_*^{2m}(\omega)P_0(dw^*)]^{\frac{1}{2}}$$

$$\leq (\sqrt{z_0})^{m-1}[\int u_*^{2m}(\omega,\omega')P_0(dw^*)P(d\omega')]^{\frac{1}{2}}, \qquad \omega = (w, w^*).$$

Here we took the equality $P(d\omega) = P_0(dw^*)P'(dw)$ and the probability character of P_0:

$$\int P(d\omega) = \int P'(dw) = z_0$$

into account. The last inequality yields

$$\int e^{\gamma_0 v_*(\omega)} P_0(dw^*) \le 1 + (\sqrt{z_0})^{-1} \sum_{m\ge 1} \frac{(\sqrt{z_0}\gamma_0)^m}{m!} [\int u_*^{2m}(\omega,\omega') P_0(dw^*) P(d\omega')]^{\frac{1}{2}}. \tag{5.1}$$

In view of the relation

$$u_*^{2m}(\omega,\omega') \le 2^{2m}[f_*^{2m}(w',w|w'^*) + f_*^{2m}(w,w'|w^*)],$$

$$\int (\int_0^t f(\tau) dw^*(\tau))^{2m} P_0(dw^*) = \frac{(2m)!}{m!} [\int_0^t f^2(\tau) d\tau]^m,$$

where either $f(\tau) = \partial u^0(w(\tau), w'(\tau))$ or $f(\tau) = \partial' u^0(w'(\tau), w(\tau))$ (the Wiener integral of an odd power of the stochastic integral is zero), one derives

$$\int u_*^{2m}(\omega,\omega') P_0(dw^*) P_0(dw'^*) \le 2^{2m} \frac{(2m)!}{m!} [f_0^m(w,w') + f_0^m(w',w)], \tag{5.2}$$

where

$$f_0(w,w') = \int_0^t (\partial u^0(w(\tau), w'(\tau)))^2 d\tau.$$

It follows from (2.19), (5.2), and the inequalities

$$f_0^m(w,w') \le v'^m(w) + v'^m(w'),$$

$$v'(w) = a_1 \int_0^t |w(\tau)|^{2l_0 - 1} d\tau \le a_1 t^{1-n'} [\int_0^t w^{2(2n-1)}(\tau) d\tau]^{n'}$$

that (the last inequality is derived from the Hölder inequality)

$$\int f_0^m(w,w') P'(dw') \le (n' m e^{-1})^{mn'} (2\gamma_1^{-1} a_1 t^{1-n'})^m [\int P'(dw') e^{\gamma_1 \int_0^t w'^{2(2n-1)}(\tau) d\tau} + e^{\gamma_1 \int_0^t w^{2(2n-1)}(\tau) d\tau}]. \tag{5.3}$$

Here, one has to consider that

$$\max_{a\ge 0} a^m e^{-a} \le m^m e^{-m}.$$

Relations (5.1) and (5.3) lead to the inequality announced in the lemma with

$$\kappa(a, z_0) = 1 + 2(\sqrt{z_0})^{-1} \sum_{m\ge 1} \frac{(2\sqrt{a z_0 a_1})^m}{m!} (\frac{(2m)!}{m!} (n' m e^{-1})^{mn'})^{\frac{1}{2}}.$$

The series on the right-hand side converges since $l_0 < n, n' < 1$ and $\frac{(1+n')}{2} < 1$. Here, we took the inequality $e^{-m} m^m \le m! \le m^m$ into account. The Lemma is proven.

Consider the norms N_s. Let

$$u^0(q) = \sum_{j=1}^n \eta_j q^{2j}.$$

We now put

$$\gamma_1 = \frac{\beta^2}{9}\eta_n^2, \qquad \gamma_0 = \sqrt{\beta}\gamma_*, \qquad \gamma_* = 2||\sqrt{J^0}||_1$$

[such a choice is motivated by (4.12)] in Lemma 5.1. Using the inequality $|q|^k \le \varepsilon |q|^l - c_\varepsilon$, where $k < l$, ε is arbitrary small, and $c_\varepsilon \ge 0$, we find

$$u^0(q) \ge 2^{-1}\eta_n q^{2n} - \eta^0, \quad (\partial u^0(q))^2 \ge 2^{-1}\eta_n^2 q^{2(2n-1)} - \eta',$$
$$|\partial^2 u^0(q)| \le 2^{-1}\eta_n q^{2(n-1)} + \eta'',$$

where $\eta^0, \eta', \eta'' \ge 0$. As a result ($u_1$ is a bounded from below polynomial), we have

$$u(w) \ge u(w) - \gamma_1 \beta^{-1} \int_0^t w^{2(2n-1)}(\tau) d\tau \ge \frac{1}{4}\eta_n q^{2n} - c^+,$$

where c^+ is a linear positive polynomial in β and t (we omit the dependence of c^+ on t, β in our notations). Then

$$z_0 \le I_P \le e^{\beta c^+} I_0, \qquad I_0 = \int e^{-\frac{\beta}{8}\eta_n^2 q^{2n}} dq.$$

Let us put

$$\kappa_*(t,\beta) = (e^{\frac{\beta c^+}{2}}\sqrt{I_0} + 1)\kappa(18||\sqrt{J^0}||_1 \beta^{-\frac{3}{2}} \eta_n^{-1} t^{1-n'}, e^{\beta c^+} I_0).$$

Then Lemma 5.1 yields

$$N_0 \le N_1 \le N_3 \le \kappa_*(t,\beta) \int dq P_q(dw) e^{-\bar f(w)}, \quad N_2^2 \le \kappa_*(t,\beta) \int dq P_q(dw) e^{-\bar f(w)}, \tag{5.4}$$

where

$$\bar f(w) = \beta u(w) - (\frac{\beta}{2}\bar J + 1) v^1(w) - \frac{\gamma_1}{2} \int_0^t w^{2(2n-1)}(\tau) d\tau - \beta(1 + \frac{3}{2}\bar J) f_0(w).$$

Here, we used the inequality $v^1 \le e^{v^1}$, considering N_3. The polynomials in $q, w(t)$ in the expression for the function $\bar f(w) - u(w)$ have degrees less than those of the corresponding polynomials in the expression for $u(w)$. This fact yields the inequality

$$\bar f(w) \ge \frac{\beta}{8}\eta_n q^{2n} - \bar c,$$

where $\bar c$ is a positive quadratic polynomial in β and a linear one in t. Then relation (5.4) and this inequality show that the integral in (5.4) is less than $I_0 e^{\bar c}$ and that

$$N_0 \le N_1 \le N_3 \le \kappa_*(t,\beta) I_0 e^{\bar c}, \quad N_2^2 \le \kappa_*(t,\beta) I_0 e^{\bar c}. \tag{5.5}$$

The last inequality and Lemma 4.1 prove the following theorem.

Theorem 5.1. *Let $\gamma_* = 2||\sqrt{J^0}||_1$. Then the norm $||\tilde K||_{\xi,f}$ of the symmetrized KS operator is finite in $\mathbb{E}_{\xi,f}$, and the series*

$$\rho^\Lambda = \sum_{n\ge 0} z^{n+1} \tilde K_\Lambda^n \alpha, \quad \rho = \sum_{n \ge 0} z^{n+1} \tilde K^n \alpha \tag{5.6}$$

determine the unique solutions of the finite-volume and infinite-volume symmetrized KS equations in $\mathbb{E}_{f,\xi}$, respectively, if $|z| \cdot ||\tilde K||_{\xi,f} < 1$. Moreover, there exists a positive continuous function $\epsilon(\lambda,\xi)$ decreasing to zero at infinity in λ such that

$$||\chi_{\Lambda(\lambda)}(\rho - \rho^\Lambda)||_{f,\xi} \le \epsilon(\lambda, \xi), \tag{5.7}$$

where $\Lambda(\lambda)$ is the set of lattice sites located in Λ, whose distance from the boundary of Λ is greater than λ.

The proof of inequality (5.7) is given with the help of the Lemma 4.1 and standard arguments [see formulas (2.39), (2.45), and (2.48) in [11], from which we derive the formula

$$\epsilon(l\delta,\xi)=2|z|\xi(\frac{(|z|a(\xi))^{l+1}}{1-|z|a(\xi)}+\frac{|z|C_{\delta}(\xi)}{(1-|z|a(\xi))^2}),\qquad |z|a(\xi)<1,$$

where $a(\xi)$ coincides with the right-hand side of the inequality for the norm of the symmetrized KS operator and l is a positive integer. A general formula looks like

$$\epsilon(\lambda,\xi)=2|z|\xi(\frac{(|z|a(\xi))^{\sqrt{\lambda}}}{1-|z|a(\xi)}+\frac{|z|C_{\sqrt{\lambda}}(\xi)}{(1-|z|a(\xi))^2}).$$

To derive it from the previous formula, one has to use the inequalities

$$\lambda\ge[\sqrt{\lambda}]_-\sqrt{\lambda}=\lambda_-,\qquad \chi_{\Lambda(\lambda)}\le\chi_{\Lambda(\lambda_-)},\qquad g^{[\sqrt{\lambda}]_-+1}\le g^{\sqrt{\lambda}},\qquad g\le 1,$$

where $[a]_-$ is the integer part of a positive number a. Note that the multiplier $|z|\xi$ on the right-hand side of the last formula coincides with $||z\alpha||_{\xi,f}$.

Proof of Theorem 1.1. We determine $\rho(q_X;t\beta)$ by (1.8), in which the sequence of $\{\rho(\omega_X),X\subset\mathbb{Z}^d\}$ satisfy the symmetrized KS equation. Then Theorem 5.1 and relations (4.13-14) establish that its series in powers of z converges in a disc $|z|<g_*$ in the complex plane. Theorem 5.1 and relations (4.13-14) result also in

$$|\rho(q_X;t\beta)|\le g^{-|X|}\int\exp\{-\beta\sum_{x\in X}(u(w_x)-f(\omega_x))\}P_0(dw^*_X)P_{q_X}(dw_X)||\rho||_{g^{-1},f}.$$

Using Lemma 5.1 with $\gamma_1=\frac{\eta_n^2\beta^2}{9},\gamma_0=2\sqrt{\beta}||\sqrt{J^0}||_1$, we establish that

(5.8)

$$|\rho(q_X;t\beta)|\le\exp\{-\beta\sum_{x\in X}(2^{-1}u^0(q_x)-(1+\frac{3}{2}\bar{J})(2^{-1}v_0(q_x))^{1+\zeta_0})-\ln I_-(q_x)\}||\rho||_{g^{-1},f},$$

where

$$I_-(q)=\int e^{-\beta f_-(w)}P_q(dw)$$

and

$$f_-(w)=u(w)-\frac{1}{2}u_0(q)-(1+\frac{3}{2}\bar{J})(v_1^{1+\zeta_1}(w(t))+v_2^{1+\zeta_2}(w))-\frac{\gamma_1}{2}\beta^{-1}\int_0^t w^{2(2n-1)}(\tau)d\tau.$$

Taking the conditions on ζ_l into account, one sees that $I_-(q)$ is finite since $f_-(w)\ge -c_-$, where c_- is a linear positive polynomial in β and t, and $I_-(q)\le e^{\beta c_-}$. Here, one has to apply the Hölder inequality to the term with $v_2^{1+\zeta_2}$ and use the above inequality $|q|^k\le\varepsilon|q|^l-c_\varepsilon$, $k<l$. Moreover, we have the inequality

$$(2^{-1}v_0(q))^{1+\zeta_0}\le q^{2(n-1)}+1.$$

Then (1.4) follows from (5.8) if one puts

$$N_*=|z|(1-|z|g_*)^{-1}g,\qquad \bar{\gamma}=c_-+(1+\frac{3}{2}\bar{J}),\qquad \gamma=\frac{3}{2}\bar{J}.$$

Let $X\in\Lambda(\lambda)$. Then (1.3) is a result of (1.7-8), Theorem 5.1, the inequality

$$|\rho(q_X;t\beta)-\rho^\Lambda(q_X;t\beta)|\le$$

$$\le\int\exp\{-\beta\sum_{x\in X}u(\omega_x)\}|\rho(\omega_X)-\rho^\Lambda(\omega_X)|P_0(dw^*_X)P_{q_X}(dw_X)\le$$

$$\le\epsilon(\lambda,\xi)\int\exp\{-\beta\sum_{x\in X}(u(\omega_x)-f(\omega_x))\}P_0(dw^*_X)P_{q_X}(dw_X),$$

the bound for $I_-(q)$, and the equality $\epsilon_*(\lambda)=\epsilon(\lambda,g^{-1})$.

We now prove (1.2′). At first, we present a formal derivation of the finite-volume diffusion hierarchy

$$\frac{\partial}{\partial t}\rho^\Lambda(q_X;t)=\sum_{x\in X}\partial_x\{\beta^{-1}\partial_x\rho^\Lambda(q_X;t)+\rho^\Lambda(q_X;t)\partial_xU^0(q_X)+$$

$$+\sum_{y\in\Lambda\setminus X}\int(\partial_xu^0_{x,y})(q_x,q_y)\rho^\Lambda(q_{X\cup Y};t)dq_Y\}. \tag{5.9}$$

We take into account that the grand partition function does not depend on time due to the gradient character of the Smoluchowski equation. Let us differentiate the first equality in (1.2) under the sign of the integral and take the Smoluchowski equation for $\rho^0(q_X;t)$. This results in

$$\Xi_\Lambda\frac{\partial}{\partial t}\rho^\Lambda(q_X;t)= \tag{5.9′}$$

$$=\sum_{Y\subseteq\Lambda\setminus X}z^{|X|+|Y|}\int\sum_{y\in X\cup Y}\partial_y\{\beta^{-1}\partial_y\rho^0(q_{X\cup Y};t)+\rho^0(q_{X\cup Y};t)\partial_yU^0(q_{X\cup Y})\}dq_Y.$$

The terms corresponding to $y\in Y$ are equal to zero, and this equality gives rise to

$$\frac{\partial}{\partial t}\rho^\Lambda(q_X;t)=\sum_{x\in X}\{\partial_x[\beta^{-1}\partial_x\rho^\Lambda(q_X;t)+\rho^\Lambda(q_X;t)\partial_xU^0(q_X)]+$$

$$+\sum_{Y\subseteq\Lambda\setminus X}z^{|X|+|Y|}\int\sum_{y\in Y}[(\partial_xu^0_{x,y})(q_x,q_y)\partial_x\rho^0(q_{X\cup Y};t)+\rho^0(q_{X\cup Y};t)(\partial^2_xu^0_{x,y})(q_x,q_y)]dq_Y\}.$$

Here, we used the equality

$$\partial_xU^0(q_{X\cup Y})=\partial_xU^0(q_X)+\partial_x(U^0(q_{X\cup Y})-U^0(q_X)-U^0(q_Y))=$$
$$=\partial_xU^0(q_X)+\sum_{y\in Y}\partial_xu^0_{x,y}(q_x,q_y)$$

and its derivative in ∂_x. Let us utilized the equality

$$\sum_{Y\subseteq\Lambda\setminus X}\sum_{y\in Y}F(Y;y)=\sum_{y\in\Lambda\setminus X}\sum_{Y\subseteq\Lambda\setminus(X\cup y)}F(Y\cup y;y).$$

As a result, the last term in the equality for the time derivative of ρ^Λ is equal to

$$\sum_{y\in\Lambda\setminus X}[\int(\partial_xu^0_{x,y})(q_x,q_y)dq_y\sum_{Y\subseteq\Lambda\setminus(y\cup X)}z^{|Y|+|X|+|y|}\int\partial_x\rho^0(q_{X\cup Y\cup y};t)dq_Y+$$

$$+\int(\partial^2_xu^0_{x,y})(q_x,q_y))dq_y\sum_{Y\subseteq\Lambda\setminus(y\cup X)}z^{|Y|+|X|+|y|}\int\rho^0(q_{X\cup Y\cup y};t)dq_y]=$$

$$=\sum_{y\in\Lambda\setminus X}[\int(\partial_xu^0_{x,y})(q_x,q_y)\partial_x\rho^\Lambda(q_{X\cup y};t)dq_y+\int(\partial^2_xu^0_{x,y})(q_y,q_y)\rho^\Lambda(q_{X\cup y};t)dq_y]=$$

$$=\sum_{y\in\Lambda\setminus X}\partial_x\int(\partial_xu^0_{x,y})(q_x,q_y)\rho^\Lambda(q_{X\cup y};t)dq_y.$$

This proves (5.9). We can apply all these arguments in a rigorous fashion to prove that the sequence of the finite-volume correlation functions is a weak solution of the diffusion hierarchy. We do it with the help of the following lemma which will be proved in the appendix.

Lemma 5.2. *Let ψ_t be given by (2.3) with the Gibbsian initial state*

$$\psi_0(q_X) = e^{-\beta U_1(q_X)}.$$

Then ψ_t is a twice differentiable function in $q_x, x \in X$ and $|\partial_x^s \psi_t| \le C_s, s = 0, 1, 2,$ where C_s are positive constants.

Let us take a function f from $L^2(\mathbb{R}^{|X|})$. Then

$$\int f(q_X)\rho^\Lambda(q_X;t)dq_X$$

is represented as a finite sum in Y of the integrals

$$\int f'_Y(q_Y)\psi_t(q_Y;t)dq_Y,$$

where $f'_Y \in L^2(\mathbb{R}^{|Y|})$, since $e^{-\beta U_0(q_X)} \in L^p(\mathbb{R}^{|X|}), p \ge 1$. Since the time derivative is a strong L^2-derivative, we obtain

$$\begin{aligned}\frac{d}{dt}&\int f(q_X)\rho^\Lambda(q_X;t)dq_X \\ &= \int f(q_X)dq_X \sum_{Y\subseteq\Lambda\backslash X} z^{|X|+|Y|} \\ &\quad\times \int \sum_{y\in X\cup Y} \partial_y\{\beta^{-1}\partial_y\rho^0(q_{X\cup Y};t) + \rho^0(q_{X\cup Y};t)\partial_y U^0(q_{X\cup Y})\}dq_Y.\end{aligned}$$

We can use the previous arguments rigorously applying Lemma 2.1 and the facts that $\rho^0(q_X;t)$ tends to zero exponentially fast in oscillator variables and all the potentials are polynomials. So one can put the terms in (5.9′) with $y \in Y$ equal to zero, split the integral in $dq_{X\cup Z}$ into the multiple integral (due to the Fubini theorem), and change the order of differentiation in q_x and integration in q_y. All these arguments prove that ρ^Λ is a weak solution of the finite-volume lattice diffusion hierarchy, that is,

$$\begin{aligned}\frac{d}{dt}\int f(q_X)\rho^\Lambda(q_X;t)dq_X &= \sum_{x\in X}\int\{\rho^\Lambda(q_X;t)[\beta^{-1}\partial_x^2 f(q_X) - (\partial_x U^0)(q_X)\partial_x f(q_X)] - \\ &-(\partial_x f)(q_X)\sum_{y\in\Lambda\backslash X}\int(\partial_x u^0_{x,y})(q_x,q_y)\rho^\Lambda(q_{X\cup y};t)dq_y\}dq_X. \qquad (5.10)\end{aligned}$$

Now we establish that every term of the last equality converges to the corresponding term in (1.2′) in the limit $\Lambda \to \mathbb{Z}^d$. It is not difficult to see that the diagonal terms in the first line of (5.10) converge to the corresponding terms in the first line of (1.2′) due to (1.3). Let $X \subseteq B_0(r)$, and let the distance of $B_0(r)$ from Λ^c be equal to λ, where $B_x(r)$ is a hyperball with radius r centered at the lattice site x. In order to prove the convergence of the last term on the right-hand side of (5.10), we estimate the following integrals for $X \in B_0(r)$ using the decomposition $X^c = (\Lambda\backslash X) \cup \Lambda^c$:

$$\sum_{y\in\Lambda\backslash X}\int(\partial_x f)(q_X)dq_X\int(\partial_x u^0_{x,y})(q_x,q_y)(\rho(q_{X\cup y};t) - \rho^\Lambda(q_{X\cup y};t))dq_y,$$

$$\sum_{y\in\Lambda^c}\int(\partial_x f)(q_X)dq_X\int(\partial_x u^0_{x,y})(q_x,q_y)\rho(q_{X\cup y};t)dq_y.$$

The sum of these integrals equals the difference of the non-diagonal last terms in (1.2′) and (5.10). The last integral is less than [due to (1.4)]

$$\tilde{N}\tilde{g}^{-|X|-1}\sum_{y\in\Lambda^c}\int|(\partial_x f)(q_X)|dq_X\int|\partial_x u^0_{x,y}(q_x,q_y)|e^{-\beta\sum_{x\in X\cup y}\bar{u}(q_x)}dq_y \le$$

$$\leq \tilde{N}\tilde{g}^{-|X|-1}C_X(f)\max_{x\in B_0(r)}\sum_{y\in\Lambda^c}J^0_{x-y},$$

where

$$C_X(f)=2^{-1}a_1\max_{x\in X}\int e^{-\beta\sum\limits_{x\in X}\bar{u}(q_x)}|(\partial_x f)(q_X)|dq_X\int e^{-\beta\bar{u}(q)}(|q_x|^{2l_0-1}+|q|^{2l_0-1}+2)dq.$$

Here, we used (2.10) and put

$$\tilde{N}=N_*(t\beta^{-1},\beta),\qquad \tilde{g}=g(t\beta^{-1},\beta).$$

The considered integral converges to zero with increase in λ due to the inequality

$$\max_{x\subseteq B_0(r)}\sum_{y\in\Lambda^c}J^0_{x-y}\leq ||J^0\chi_{B^c_0(\lambda)}||_1=\epsilon_0(\lambda).$$

Let us decompose the sum before the integral with the difference $\rho-\rho^\Lambda$ into the sets $B_0(r+2^{-1}\lambda)\backslash X$ and $\Lambda\backslash B_0(r+2^{-1}\lambda)$. These sums are bounded by two finite expressions proportional to $\tilde{\varepsilon}(2^{-1}\lambda)=\varepsilon_*(2^{-1}\lambda,t\beta^{-1},\beta)$ and $\varepsilon_0(2^{-1}\lambda)$, respectively, due to the last bound and (1.3). The first sum is estimated as follows:

$$\max_{X\subseteq B_0(r)}\sum_{y\in B_0(r+2^{-1}\lambda)\backslash X}\int|\partial_x f(q_X)|dq_X\int|\partial_x u^0_{x,y}(q_x,q_y)||\rho(q_{X\cup y};t)-\rho^\Lambda(q_{X\cup y};t)|dq_y\leq$$

$$\leq \tilde{g}^{-|X|-1}\tilde{\varepsilon}(2^{-1}\lambda)\sum_y\int|(\partial_x f)(q_X)|dq_X\int|\partial_x u^0_{x,y}(q_x,q_y)|e^{-\sum\limits_{x\in X\cup y}\bar{u}(q_x)}dq_y\leq$$

$$\leq \tilde{g}^{-|X|-1}\tilde{\epsilon}(2^{-1}\lambda)C_X(f)||J^0||_1.$$

For the second sum, we have the bound

$$2\tilde{N}\tilde{g}^{-|X|-1}\max_{X\subseteq B_0(r)}\sum_{y\in\Lambda\backslash B_0(r+2^{-1}\lambda)}\int|\partial_x f)(q_X)|dq_X\int|\partial_x u^0_{x,y}(q_x,q_y)||dq_y\leq$$

$$\leq 2\tilde{N}\tilde{g}^{-|X|-1}C_X(f)\max_{x\in B_0(r)}\sum_{y\subseteq B^c_0(r+2^{-1}\lambda)}J^0_{x-y}\leq 2\tilde{N}\tilde{g}^{-|X|-1}C_X(f)\varepsilon_0(2^{-1}\lambda).$$

Here, we used the fact that ρ and ρ^Λ satisfy the same bound given in (1.4). Hence, the two considered integrals containing the difference $\rho-\rho^\Lambda$ converge to zero for increasing λ which takes integer values. The convergence of the right-hand side of (5.9) is uniform on the time interval, for which $|z|<\tilde{g}_*$, where $\tilde{g}_*=g_*(t\beta^{-1},\beta)$, since $\tilde{g}$ is uniformly bounded from below, and $\tilde{N}$ and $\tilde{\epsilon}(2^{-1}\lambda)$ are uniformly bounded from above on this time interval. The expression

$$\int f(q_X)\rho^\Lambda(q_X;t)dq_X$$

converges to

$$\int f(q_X)\rho(q_X;t)dq_X$$

also uniformly on this time interval. It is well known that the derivative of a limit of a convergent sequence of functions on a bounded interval coincides with a limit of the sequence of their derivatives if the sequence of their derivatives converges uniformly to its limit. Hence, (1.2′) is true and the Theorem is proven.

6. Appendix

Proof of Lemma 5.2. From the translation invariance of the Wiener measure, one derives

$$\psi_t(q_X) = \int P_0(dw_X)\Psi_t(w_X + q_X), \qquad \Psi_t(w_X + q_X) = e^{-\beta U_3(w_X+q_X)},$$

where $\omega_X + q_X = (\omega_x + q_x, x \in X)$ and

$$U_3(w_X) = \tilde{U}(w_X) - \frac{1}{2}U^0(w_X(0)) = \int_0^t U_2(w_X(\tau))d\tau + U_1(w_X(t)).$$

For U_1, we have the superstability bound

$$U_1(q_X) \geq \sum_{x\in X} u_1^-(q_x) \geq -|X|\bar{u}_1, \qquad u_1^-(q_x) = u_1(q_x) - ||J^1||_1 q_x^{2(n_1-1)} - 2||J^1||_1,$$

where $\bar{u}_1 \geq 0$. Here, we used the inequality $q^{2l_1} \leq q^{2(n_1-1)} + 1$. From (2.5) and the positivity of U_2' and (2.11′), one derives the superstability bound for U_2,

$$U_2(q_X) \geq \sum_{x\in X} u_2^-(q_x) \geq -\bar{u}_2|X|, \qquad u_2^-(q_x) = \tilde{u}(q_x) - ||J^0||_1\tilde{v}(q_x),$$

where $\bar{u}_2$ is a non-negative constant. The superstability bounds for U_2, U_1 yield the bound

$$\psi_t(q_X) \leq e^{\beta|X|(t\bar{u}_2+\bar{u}_1)}$$

which follows from the superstability bound for U_3. We have

$$U_3(w_X) \geq \sum_{x\in X} u_3^-(w_x) \geq -(\bar{u}_2 t + \bar{u}_1)|X|, \qquad u_3^-(w_x) = \int_0^t u_2^-(w_x(\tau))d\tau - u_1^-(w_x(t)).$$

Using (2.10), applying the inequalities

$$U_2'(q_X) \leq 2^{|X|}\sum_{x\neq y\in X}(\partial_x u^0_{x,y})^2(q_x,q_y) \leq 2^{|X|-1}\sum_{x\neq y\in X}(J^0_{x-y})^2(v^2_{0,1}(q_x) + v^2_{0,1}(q_y)) \leq$$

$$\leq 2^{|X|}a_1^2||(J^0)^2||_1\sum_{x\in X}(q_x^{2(2l_0-1)} + 1),$$

and the similar bounds for the term with the second derivatives in the expression for U_2, one sees that there exists the positive number $\bar{J}_2$ such that

$$U_2(q_X) \leq \sum_{x\in X}(|\tilde{u}(q_x)| + \bar{J}_2(|q_x|^{2(2n-1)-1} + 1)).$$

It is possible to derive an analog of the above inequality for $\partial_x U_2'(q_X)$ and the bound

$$|\partial_x U_2(q_X)| \leq \bar{J}_3\sum_{x\in X}(|q_x|^{2(2n-1)-1} + 1).$$

To prove the Lemma, one has to show that it is possible to differentiate twice the right-hand side of (2.3) under the sign of the Wiener integral. Let

$$T_{x;r}f(w_X) = f(w_{X\backslash x}, w_x + r).$$

Then one has to show that, for an arbitrary sequence of positive numbers r_n such that $\lim_{n\to\infty} r_n = 0$, the following equality holds:

$$\int P_0(dw_X)\partial_x\Psi_t(w_X + q_X) = \lim_{n\to\infty}\int P_{q_X}(dw_X)r_n^{-1}(T_{x,r_n}\Psi_t(w_X) - \Psi_t(w_X)) =$$

$$= \partial_x \int P_{q_X}(dw_X)\Psi_t(w_X). \tag{6.1}$$

It is clear that one has to rely on the Lebesgue dominated convergence theorem and show that the function under the sign of the first integral is uniformly bounded by an integrable function. The function under the sign of the second integral is less than

$$\exp\{-\beta U_3(w_X)\} r_n^{-1} |T_{x,r_n} U_3(w_X) - U_3(w_X)| e^{|T_{x,r_n} U_3(w_X) - U_3(w_X)|}.$$

Here, we used the inequality $|e^a - 1| \le |a|e^{|a|}$. Derivatives in q_x of $U_3(\omega_X + q_X)$ are calculated in an obvious way, since one deals with polynomials in Wiener paths. It follows from the mean-value theorem, the above inequality for $|\partial_x U_2|$, and similar inequalities for the derivatives of U_1 (the inequality for the partial derivatives of $u_{1;Y}$ mentioned in Introduction has to be applied) that there exists the positive number $\bar{J}_3$ such that

$$0 \le r'_n \le r_n, \qquad r_n^{-1} |T_{x,r_n} U_3(w_X) - U_3(w_X)| = |\partial_x U_3(w_{X\setminus x}, w_x + r'_n)| \le$$

$$\le \bar{J}_3 \sum_{x \in X} [\int_0^t |w_x(\tau)|^{2(2n-1)-1} d\tau + |w_x(t)|^{2n_1-1} + 2)]. \tag{6.2}$$

The last inequality, the previous one, and the superstability bound for U_3 establish that the function under the sign of the second integral in (6.1) is bounded by an integrable function independent of n, since the degree of the polynomial $\tilde{u}$ is greater than

$$2(2n-1) - 1,$$

and the degree of the polynomial u_1 is greater than $2n_1 - 1$. This proves (6.1). The equality

$$\partial_x \Psi_t(w_X + q_X) = -\beta \exp\{-\beta U_3(w_X + q_X)\} \partial_x U_3(w_X + q_X)$$

and the superstability bound for U_3 prove that $|\partial_x \psi_t| \le C_1$. To prove that ψ_t admits a bounded second derivative in q_x, one has to prove (6.1), where $\partial_x \Psi_t$ is substituted for Ψ_t. Using the formulas

$$ae^a - a'e^{a'} = a(e^a - e^{a'}) + e^{a'}(a - a') = e^a[a(1 - e^{a'-a}) + e^{a'-a}(a - a')],$$

$$|ae^a - a'e^{a'}| \le e^a(|a||a' - a|e^{|a'-a|} + e^{|a'-a|}|a' - a|^2) \le 2e^{a+|a'-a|}|a' - a|(|a'| + |a|)$$

for

$$a' = -\beta T_{x,r_n} U_3(w_X), \qquad a = -\beta U_3(w_X),$$

the superstability bound for U_3, and (6.2), we verify without any difficulty that $\psi_t(q_X)$ is a twice differentiable function, and there exists a positive constant C_2 such that

$$|\partial_x^2 \psi_t| \le C_2.$$

Lemma is proven.

Remark. *From (5.2-3) for $m = 1$ and the statement inverse to the Fubini theorem* [6]*, it follows that the function $|u_*(\omega, \omega')|^2$ is integrable by the measure $P(d\omega)P(d\omega')$. From the Fubini theorem, it follows also that the function $v_*^2(\omega)$ is integrable by $P(d\omega)$. We have utilized the statement inverse to the Fubini theorem in the proof of Lemma 5.1.*

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