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THE LAPLACE TRANSFORM OF THE ERGODIC DISTRIBUTION OF THE PROCESS OF SEMI-MARKOVIAN RANDOM WALK WITH NEGATIVE DRIFT, NONNEGATIVE JUMPS, DELAYS, AND DELAYING SCREEN AT ZERO

The Laplace–Stieltjes transform with respect to x, the Laplace transform with respect to t, the conditional distribution, the unconditional distribution, and the Laplace transform of the ergodic distribution of the process of semi-Markovian random walk with negative drift, nonnegative jumps, delays, and delaying screen at zero are obtained.

Introduction

Let the sequence $\{\xi_k, \zeta_k\}_{k=1}^{\infty}$ be given on the probability space $(\Omega, \Im, P(\cdot))$, where the random variables ξ_k ($\xi_k > 0$) and ζ_k ($\zeta_k > 0$) are independent and identically distributed.

We construct the process

$$Y(t) = z - t + \sum_{i=1}^{k-1} \zeta_i, \quad \text{if} \quad \sum_{i=1}^{k-1} \xi_i \le t < \sum_{i=1}^{k} \xi_i, \quad \sum_{1}^{0} = 0, \; k = 1, 2, ..., \; t \ge 0.$$

Let us delay the process $Y(t)$ with a screen at zero. We denote it by $Y_*(t)$:

$$Y_*(t) = Y(t) - \inf_{0 \le s \le t}(0, Y(s)).$$

This process is called the virtual waiting time in the queuing theory. The ergodic distribution of the process $Y_*(t)$ can be found, e.g., in [2–4]. In [2], the ergodic distribution of the process $Y_*(t)$ was found in the case where the random variables ξ_k ($\xi_k > 0$) and ζ_k ($\zeta_k > 0$) have the arbitrary distributions. This solution cannot be used in practice in this form.

The process $Y(t)$ was generalized in two directions: the jumps of the process take place with probability ρ ($0 < \rho \le 1$), and there is the random number of delays between two positive jumps of the process (see Fig. 1).

We delayed the generalized process with a screen at zero. Then we obtained the Laplace transform of the ergodic distribution of the last process, by using another method.

1. The process construction

Let the sequences $\{\xi_k\}_{k=1}^{\infty}$, $\{\eta_k\}_{k=1}^{\infty}$, and $\{\zeta_k\}_{k=1}^{\infty}$ be given on the probability space $(\Omega, F, P(\cdot))$, where ξ_k, η_k, ζ_k, $k = \overline{1,\infty}$, are the random variables which are independent, identically distributed, and independent between themselves. We suppose that $\xi_k > 0$, $\eta_k > 0$, $\zeta_k \ge 0, 0 < E\xi_k < \infty, 0 < E\eta_k < \infty, 0 < E\zeta_k < \infty$ and $E\xi_k > E\zeta_k$, $k = \overline{1,\infty}$.

From the beginning, the system linearly decreases from the initial state z up to a moment t $\xi_1 > 0 : X(t) = z - t$ for $0 \le t < \xi_1$. The random variable ξ_1 is the duration of the drift of the system. When the drift ceases, the system stops in the state $z - \xi_1$ for the duration of the random variable η_1. We call the random time η_1 as a delay. The

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number of the successive alternations "negative drift and the delay" to the first jump of the size ζ_1 $(\zeta_1 > 0)$ is the random number. This random number is denoted by ν_1. Thus, we deal with the random variables ξ_1, η_1, ζ_1 and ν_1. We can define the next random variables similarly to ξ_2, η_2, ζ_2, ν_2; $\ldots$. We call this constructed process as the process of a semi-Markovian random walk with negative drift, nonnegative jumps, and delays.

By l $(l \geq 1)$, we denote the number of periods (the part of the process between two successive jumps is called a period), $k_l(t)$ is the number of the negative drifts to the moment t in the l-th period, ν_i is the number of the alternations "negative drift and the delay" to the i-th jump of the size ζ_i.

Let us construct the process

$$X(t) = \begin{cases} z - t + \sum\limits_{i=1}^{l-1} \zeta_i + \sum\limits_{i=1}^{\nu_1+\ldots+\nu_{l-1}+k_l(t)-1} \eta_i, \text{ if } \sum\limits_{i=1}^{\nu_1+\ldots+\nu_{l-1}+k_l(t)-1} (\xi_i + \eta_i) \leq \\ \qquad \leq t < \sum\limits_{i=1}^{\nu_1+\ldots+\nu_{l-1}+k_l(t)-1} (\xi_i + \eta_i) + \xi_{\nu_1+\ldots+\nu_{l-1}+k_l(t)}, \\ z + \sum\limits_{i=1}^{l-1} \zeta_i - \sum\limits_{i=1}^{\nu_1+\ldots+\nu_{l-1}+k_l(t)} \xi_i, \text{ if } \sum\limits_{i=1}^{\nu_1+\ldots+\nu_{l-1}+k_l(t)-1} (\xi_i + \eta_i) + \\ \qquad +\xi_{\nu_1+\ldots+\nu_{l-1}+k_l(t)} \leq t < \sum\limits_{i=1}^{\nu_1+\ldots+\nu_{l-1}+k_l(t)} (\xi_i + \eta_i), l \geq 1, \sum\limits_{1}^{0} = 0. \end{cases} \quad (1)$$

One of the realizations of the process $X(t)$ is depicted in Fig. 1:

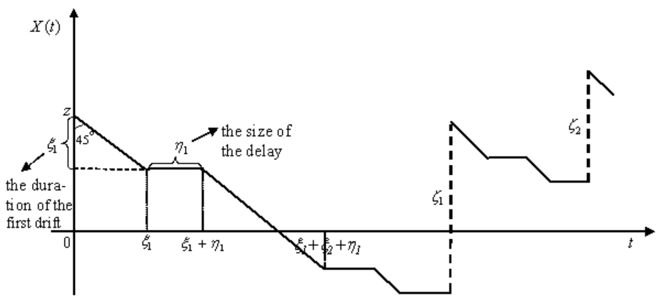

Figure 1

Let us delay the process $X(t)$ with a screen at zero:

$$X_*(t) = X(t) - \inf_{0 \leq s \leq t} (0, X(s)).$$

We call the process $X_*(t)$ as the process of a semi-Markovian random walk with negative drift, nonnegative jumps, delays, and delaying screen at zero.

One of the realizations of the process $X_*(t)$ is demonstrated in Fig. 2:

We denote

$$R(t, x | z) = P\{X_*(t) < x | X_*(0) = z\}, \quad x \geq 0.$$

$$\tilde{R}(\theta, x | z) = \int\limits_{t=0}^{\infty} e^{-\theta t} R(t, x | z) dt, \;\; \theta > 0,$$

$$\tilde{\tilde{R}}(\theta, \alpha | z) = \int\limits_{x=0}^{\infty} e^{-\alpha x} d_x \tilde{R}(\theta, x | z), \;\; \alpha > 0,$$

$$\tilde{\tilde{R}}(\theta, \alpha) = \int\limits_{z=0}^{\infty} \tilde{\tilde{R}}(\theta, \alpha | z) dP\{X_*(0) < z\},$$

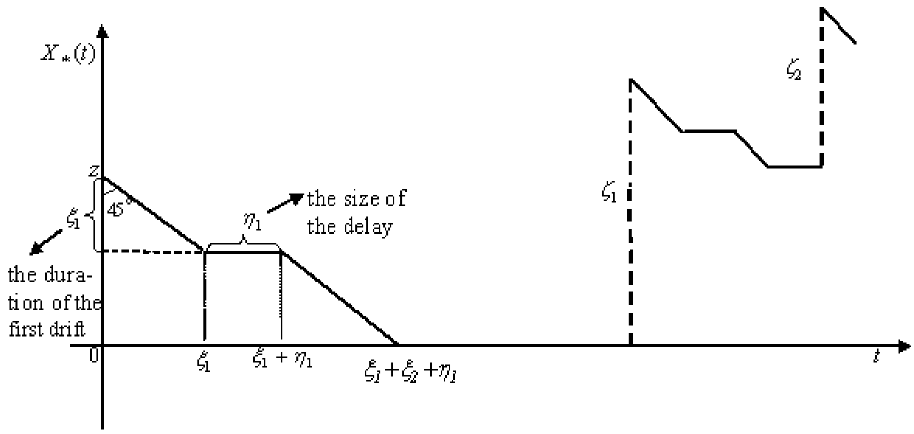

Figure 2

$\tilde{R}(\alpha)=\lim\limits_{\theta\to 0}\theta\,\tilde{\tilde{R}}(\theta,\,\alpha),\ \alpha>0$ is the Laplace–Stieltjes transform of the ergodic distribution of the process $X_*(t)$.

$$\varphi(\theta)=Ee^{-\theta\eta_k},\quad \theta>0,\quad k=\overline{1,\infty}.$$

$$\rho=P\left\{\zeta_k>0\right\},k=\overline{1,\infty}.$$

It is obvious that

$$P\left\{\nu_i=k\right\}=(1-\rho)^{k-1}\rho,\quad k=\overline{1,\infty},\ i=\overline{1,\infty}.$$

Let

$$\begin{cases} P\left\{\xi_1<t\right\}=\left[1-e^{-\mu t}\right]\epsilon(t),\ \mu>0, \\ P\left\{\zeta_1<t\right\}=\left[1-\rho e^{-\lambda t}\sum\limits_{i=0}^{m-1}\frac{(\lambda t)^i}{i!}\right]\epsilon(t),\quad \lambda>0,\ m=\overline{1,\infty}, \end{cases} \tag{2}$$

where

$$\epsilon(t)=\begin{cases}0, & t<0,\\ 1, & t>0.\end{cases}$$

It is clear that the distribution function $P\left\{\zeta_1<t\right\}$ has one jump at zero of the size $1-\rho$.

Our aim in this paper is to find the Laplace transform of the ergodic distribution of the process $X_*(t)$.

We will find $\tilde{R}(\alpha)$ in case (2).

2. The determination of $\tilde{R}(\alpha)$

To find $\tilde{R}(\alpha)$, we must find $\tilde{\tilde{R}}(\theta,\,\alpha)$. To find $\tilde{\tilde{R}}(\theta,\,\alpha)$, we must find $\tilde{\tilde{R}}(\theta,\,\alpha|\,z)$.

We prove the following theorem about the integral presentation of $\tilde{\tilde{R}}(\theta,\,\alpha|\,z)$.

Theorem. If $\left\{\xi_k,\eta_k,\zeta_k\right\},k=\overline{1,\infty}$, is the sequence of independent identically distributed and independent between themselves random variables $\xi_k,\ \eta_k,\ \zeta_k$, where $\xi_k>0$, $\eta_k>0,\ \zeta_k\geq 0,k=\overline{1,\infty}$. Then the Laplace transform of the conditional distribution $R(t,x|\,z)$ with respect to the time and to the phase satisfies the integral equation

$$\tilde{\tilde{R}}(\theta,\,\alpha|\,z)=e^{-\theta z}\int\limits_{x=0}^{z}e^{-(\alpha-\theta)x}P\left\{\xi_1>z-x\right\}dx+\int\limits_{t=z}^{\infty}e^{-\theta t}P\left\{\xi_1>t\right\}dt+$$

$$+\frac{1-\varphi(\theta)}{\theta}\left[-e^{-\theta z}\int\limits_{x=0}^{z}e^{-(\alpha-\theta)x}d_xP\left\{\xi_1<z-x\right\}+\int\limits_{t=z}^{\infty}e^{-\theta t}dP\left\{\xi_1<t\right\}\right]+$$

$$+(1-\rho)\varphi(\theta)\int\limits_{t=z}^{\infty}e^{-\theta t}dP\left\{\xi_1<t\right\}\tilde{\tilde{R}}(\theta,\alpha|\,0)+$$

$$+\rho\varphi(\theta)\int\limits_{t=z}^{\infty}e^{-\theta t}dP\left\{\xi_1<t\right\}\int\limits_{y=0}^{\infty}\tilde{\tilde{R}}(\theta,\,\alpha|\,y)dP\left\{\zeta_1<y\right\}-$$

$$-(1-\rho)\varphi(\theta)e^{-\theta z}\int\limits_{v=0}^{z}e^{\theta v}\tilde{\tilde{R}}(\theta,\alpha|\,v)d_vP\left\{\xi_1<z-v\right\}+$$

$$+\rho\varphi(\theta)\int\limits_{u=0}^{z}e^{-\theta u}\left[\int\limits_{y=z-u}^{\infty}\tilde{\tilde{R}}(\theta,\alpha|\,y)dyP\left\{\zeta_1<y-z+u\right\}\right]dP\left\{\xi_1<u\right\}. \qquad (3)$$

Proof. Using the form for the total probability, we have

$$R\left(t,x\,|\,z\right)=P\left\{X_*\left(t\right)<x\,|X_*\left(0\right)=z\right\}=$$
$$=P\left\{X_*\left(t\right)<x;z-\xi_1\le 0\,|X_*\left(0\right)=z\right\}+$$
$$+P\left\{X_*\left(t\right)<x;z-\xi_1>0\,|X_*\left(0\right)=z\right\}=$$
$$=P\left\{X_*\left(t\right)<x,z-\xi_1\le 0,\xi_1+\eta_1>t\,|X_*\left(0\right)=z\right\}+$$
$$+P\left\{X_*\left(t\right)<x,z-\xi_1>0,\xi_1+\eta_1>t\,|X_*\left(0\right)=z\right\}+$$
$$+P\left\{X_*\left(t\right)<x,z-\xi_1\le 0,\xi_1+\eta_1<t\,|X_*\left(0\right)=z\right\}+$$
$$+P\left\{X_*\left(t\right)<x,z-\xi_1>0,\xi_1+\eta_1<t\,|X_*\left(0\right)=z\right\}=$$
$$=P\left\{X_*\left(t\right)<x,z-\xi_1\le 0,\xi_1+\eta_1>t,\,0<t<z\,|X_*\left(0\right)=z\right\}+$$
$$+P\left\{X_*\left(t\right)<x,z-\xi_1\le 0,\xi_1+\eta_1>t,z<t<\xi_1+\eta_1\,|X_*\left(0\right)=z\right\}+$$
$$+P\left\{X_*\left(t\right)<x,z-\xi_1>0,\xi_1+\eta_1>t,0<t<\xi_1\,|X_*\left(0\right)=z\right\}+$$
$$+P\left\{X_*\left(t\right)<x,z-\xi_1>0,\xi_1+\eta_1>t,\xi_1<t<\xi_1+\eta_1\,|X_*\left(0\right)=z\right\}+$$
$$+P\left\{X_*\left(t\right)<x,z-\xi_1\le 0,\xi_1+\eta_1<t,\zeta_1=0\,|X_*\left(0\right)=z\right\}+$$
$$+P\left\{X_*\left(t\right)<x,z-\xi_1\le 0,\xi_1+\eta_1<t,\zeta_1>0\,|X_*\left(0\right)=z\right\}+$$
$$+P\left\{X_*\left(t\right)<x,z-\xi_1>0,\xi_1+\eta_1<t,\zeta_1=0\,|X_*\left(0\right)=z\right\}+$$
$$+P\left\{X_*\left(t\right)<x,z-\xi_1>0,\xi_1+\eta_1<t,\zeta_1>0\,|X_*\left(0\right)=z\right\}=$$
$$=P\left\{z-t<x,z-\xi_1\le 0,0<t<z\right\}+$$
$$+P\left\{0<x,z-\xi_1\le 0,z<t<\xi_1+\eta_1\,|X_*\left(0\right)=z\right\}+$$
$$+P\left\{z-t<x,z-\xi_1>0,0<t<\xi_1\right\}+$$
$$+P\left\{z-\xi_1<x,z-\xi_1>0,\xi_1<t<\xi_1+\eta_1\right\}+$$
$$+\left(1-\rho\right)\int\limits_{s=0}^{t}\int\limits_{y=0}^{\infty}P\left\{z-\xi_1\le 0,\xi_1+\eta_1\in ds,X_*\left(s\right)\in dy\right\}\times$$
$$\times P\left\{X\left(t-s\right)<x\,|X_*\left(0\right)=y\right\}+$$
$$+\rho\int\limits_{s=0}^{t}\int\limits_{y=0}^{\infty}P\left\{z-\xi_1\le 0,\xi_1+\eta_1\in ds,X_*\left(s\omega\right)\in dy\right\}\times$$
$$\times P\left\{X_*\left(t-s\right)<x\,|X_*\left(0\right)=y\right\}+$$

$$+\,(1-\rho)\int\limits_{s=0}^{t}\int\limits_{y=0}^{z} P\left\{z-\xi_1>0,\xi_1+\eta_1\in ds, z-\xi_1\in dy\right\}\times$$

$$\times P\left\{X_*\left(t-s\right)<x\left|X_*\left(0\right)=y\right.\right\}+$$

$$+\,\rho\int\limits_{s=0}^{t}\int\limits_{y=0}^{\infty} P\left\{z-\xi_1>0,\xi_1+\eta_1\in ds, z-\xi_1+\zeta_1\in dy\right\}\times$$

$$\times P\left\{X_*\left(t-s\right)<x\left|X_*\left(0\right)=y\right.\right\}=$$

$$=\epsilon(t-\max(0,z-x))\epsilon(z-t)P\left\{\xi_1>t\right\}+$$

$$\int\limits_{u=\max(0,z-x)}^{z} P\left\{\eta_1>t-u\right\}dP\left\{\xi_1<u\right\}+$$

$$+\epsilon\left(x\right)\left[P\left\{t>z\right\}P\left\{\xi_1>t\right\}+\int\limits_{u=z}^{t}P\left\{\eta_1>t-u\right\}dP\left\{\xi_1<u\right\}\right]+$$

$$+\,(1-\rho)\int\limits_{s=z}^{t}R\left(t-s,x\left|0\right.\right)d_s\int\limits_{u=z}^{s}P\left\{\eta_1<s-u\right\}dP\left\{\xi_1<u\right\}+$$

$$+\rho\int\limits_{s=z}^{t}\int\limits_{y=0}^{\infty}R\left(t-s,x\left|y\right.\right)dP\left\{\zeta_1<y\right\}d_s\int\limits_{u=z}^{s}P\left\{\eta_1<s-u\right\}dP\left\{\xi_1<u\right\}+$$

$$+\,(1-\rho)\int\limits_{s=0}^{t}\int\limits_{u=0}^{z}R\left(t-s,x\left|z-u\right.\right)d_sP\left\{\eta_1<s-u\right\}dP\left\{\xi_1<u\right\}+$$

$$+\rho\int\limits_{s=0}^{t}\int\limits_{u=0}^{z}d_sP\left\{\eta_1<s-u\right\}\int\limits_{y=z-u}^{\infty}R\left(t-s,x\left|y\right.\right)d_y\times$$

$$\times P\left\{\zeta_1<y-z+u\right\}dP\left\{\xi_1<u\right\}.$$

Thus, we obtained the integral equation for $R\left(t,x\left|z\right.\right)$:

$$R\left(t,x\left|z\right.\right)=\epsilon(t-\max(0,z-x))\epsilon(z-t)P\left\{\xi_1>t\right\}+$$

$$+\int\limits_{u=\max(0,z-x)}^{z} P\left\{\eta_1>t-u\right\}dP\left\{\xi_1<u\right\}+$$

$$+\epsilon\left(x\right)\left[P\left\{t>z\right\}P\left\{\xi_1>t\right\}+\int\limits_{u=z}^{t}P\left\{\eta_1>t-u\right\}dP\left\{\xi_1<u\right\}\right]+$$

$$+\,(1-\rho)\int\limits_{s=z}^{t}R\left(t-s,x\left|0\right.\right)d_s\int\limits_{u=z}^{s}P\left\{\eta_1<s-u\right\}dP\left\{\xi_1<u\right\}+$$

$$+\rho\int\limits_{s=z}^{t}\int\limits_{y=0}^{\infty}R\left(t-s,x\left|y\right.\right)dP\left\{\zeta_1<y\right\}d_s\int\limits_{u=z}^{s}P\left\{\eta_1<s-u\right\}dP\left\{\xi_1<u\right\}+$$

$$+\,(1-\rho)\int\limits_{s=0}^{t}\int\limits_{u=0}^{z}R\left(t-s,x\left|z-u\right.\right)d_sP\left\{\eta_1<s-u\right\}dP\left\{\xi_1<u\right\}+$$

$$+\rho \int\limits_{s=0}^{t} \int\limits_{u=0}^{z} d_s P\left\{\eta_1 < s-u\right\} \int\limits_{y=z-u}^{\infty} R\left(t-s, x\left|y\right.\right) d_y \times$$
$$\times P\left\{\zeta_1 < y-z+u\right\} dP\left\{\xi_1 < u\right\}.$$

By applying the Laplace transformation with respect to t and the Laplace–Stieltjes transformation with respect to x to both sides of the last equation and allowing for

$$\int\limits_{x=0}^{\infty} e^{-\alpha x} d_x \int\limits_{t=\max(0,z-x)}^{z} e^{-\theta t} P\left\{\xi_1 > t\right\} dt =$$
$$= \int\limits_{x=0}^{\infty} e^{-\alpha x} \epsilon\left(z-x\right) e^{-\theta \max(0,z-x)} P\left\{\xi_1 > \max\left(0, z-x\right)\right\} dx =$$
$$= e^{-\theta z} \int\limits_{x=0}^{z} e^{-(\alpha-\theta)x} P\left\{\xi_1 > z-x\right\} dx,$$

we prove the theorem.

We will solve Eq. (3) in case (2). In this case, Eq. (3) an be written in the form

$$\tilde{\tilde{R}}\left(\theta, \alpha\left|z\right.\right) = e^{-(\mu+\theta)z} \int\limits_{x=0}^{z} e^{-(\alpha-\mu-\theta)x} dx + \int\limits_{t=z}^{\infty} e^{-(\mu+\theta)t} dt +$$
$$\frac{\left[1-\varphi\left(\theta\right)\right]\mu}{\theta} e^{-(\mu+\theta)z} \int\limits_{x=0}^{z} e^{-(\alpha-\mu-\theta)x} dx +$$
$$+\frac{\left[1-\varphi\left(\theta\right)\right]\mu}{\theta} \int\limits_{t=z}^{\infty} e^{-(\mu+\theta)t} dt + \left(1-\rho\right)\mu\varphi\left(\theta\right) \int\limits_{t=z}^{\infty} e^{-(\mu+\theta)t} \tilde{\tilde{R}}\left(\theta, \alpha\left|0\right.\right) dt +$$
$$+\frac{\lambda^m \mu\rho\varphi\left(\theta\right)}{(m-1)!} \int\limits_{t=z}^{\infty} e^{-(\mu+\theta)t} \int\limits_{y=0}^{\infty} y^{m-1} e^{-\lambda y} \tilde{\tilde{R}}\left(\theta, \alpha/y\right) dy dt +$$
$$+\left(1-\rho\right)\mu\varphi\left(\theta\right) e^{-(\mu+\theta)z} \int\limits_{u=0}^{z} e^{(\mu+\theta)u} \tilde{\tilde{R}}\left(\theta, \alpha\left|u\right.\right) du +$$
$$+\frac{\lambda^m \mu\rho\varphi\left(\theta\right)}{(m-1)!} e^{\lambda z} \int\limits_{u=0}^{z} e^{-(\lambda+\mu+\theta)u} \int\limits_{y=z-u}^{\infty} \left(y+u-z\right)^{m-1} e^{-\lambda y} \tilde{\tilde{R}}\left(\theta, \alpha\left|y\right.\right) du dy.$$

From the last equation, we have

$$\tilde{\tilde{R}}\left(\theta, \alpha\left|z\right.\right) = \frac{\alpha\left\{\theta+\mu\left[1-\varphi\left(\theta\right)\right]\right\}}{\theta\left(\alpha-\mu-\theta\right)\left(\mu+\theta\right)} e^{-(\mu+\theta)z} - \frac{\theta+\left[1-\varphi\left(\theta\right)\right]\mu}{\theta\left(\alpha-\mu-\theta\right)} e^{-\alpha z} +$$
$$+\frac{\left(1-\rho\right)\mu\varphi\left(\theta\right)}{\mu+\theta} \tilde{\tilde{R}}\left(\theta, \alpha\left|0\right.\right) e^{-(\mu+\theta)z} +$$
$$+\frac{\lambda^m \mu\rho\varphi\left(\theta\right)}{(m-1)!\left(\mu+\theta\right)} e^{-(\mu+\theta)z} \int\limits_{y=0}^{\infty} y^{m-1} e^{-\lambda y} \tilde{\tilde{R}}\left(\theta, \alpha\left|y\right.\right) dy +$$
$$+\left(1-\rho\right)\mu\varphi\left(\theta\right) e^{-(\mu+\theta)z} \int\limits_{u=0}^{z} e^{(\mu+\theta)u} \tilde{\tilde{R}}\left(\theta, \alpha\left|u\right.\right) du + \frac{\lambda^m \mu\rho\varphi\left(\theta\right)}{(m-1)!} e^{\lambda z} \times$$

$$\times \int\limits_{u=0}^{z} e^{-(\lambda+\mu+\theta)u} \int\limits_{y=z-u}^{\infty} (y+u-z)^{m-1} e^{-\lambda y} \tilde{\tilde{R}}(\theta,\alpha\,|y)\,dudy. \tag{4}$$

Multiplying both sides of this equation by $e^{(\mu+\theta)z}$ and taking the derivative with respect to z, we obtain the equation

$$\left\{\tilde{\tilde{R}}'(\theta,\alpha\,|z) + [\mu+\theta-\mu(1-\rho)\varphi(\theta)]\,\tilde{\tilde{R}}(\theta,\alpha\,|z)\right\} e^{-\lambda z} =$$
$$= \frac{\{\theta+[1-\varphi(\theta)]\,\mu\}}{\theta} e^{-(\alpha+\lambda)z} + \frac{\lambda^m \mu\rho\varphi(\theta)(\lambda+\mu+\theta)}{(m-1)!}\times$$
$$\times \int\limits_{u=0}^{z} e^{-(\lambda+\mu+\theta)u} \int\limits_{y=z-u}^{\infty} (y+u-z)^{m-1} e^{-\lambda y} \tilde{\tilde{R}}(\theta,\alpha\,|y)\,dudy+$$
$$+\frac{\lambda^m \mu\rho\varphi(\theta)}{(m-1)!} e^{-(\lambda+\mu+\theta)z} \int\limits_{y=0}^{\infty} y^{m-1} e^{-\lambda y} \tilde{\tilde{R}}(\theta,\alpha\,|y)\,dy-$$
$$-\frac{\lambda^m \mu\rho\varphi(\theta)}{(m-2)!} \int\limits_{u=0}^{z} e^{-(\lambda+\mu+\theta)u} \int\limits_{y=z-u}^{\infty} (y+u-z)^{m-2} e^{-\lambda y} \tilde{\tilde{R}}(\theta,\alpha\,|y)\,dudy.$$

By multiplying both sides of this equation by $e^{-(\lambda+\mu+\theta)z}$, we obtain

$$\left\{\tilde{\tilde{R}}'(\theta,\alpha\,|z) + [\mu+\theta-\mu(1-\rho)\varphi(\theta)]\,\tilde{\tilde{R}}(\theta,\alpha\,|z)\right\} e^{-\lambda z} =$$
$$= \frac{\{\theta+[1-\varphi(\theta)]\,\mu\}}{\theta} e^{-(\alpha+\lambda)z} + \frac{\lambda^m \mu\rho\varphi(\theta)(\lambda+\mu+\theta)}{(m-1)!}\times$$
$$\times \int\limits_{u=0}^{z} e^{-(\lambda+\mu+\theta)u} \int\limits_{y=z-u}^{\infty} (y+u-z)^{m-1} e^{-\lambda y} \tilde{\tilde{R}}(\theta,\alpha\,|y)\,dudy+$$
$$+\frac{\lambda^m \mu\rho\varphi(\theta)}{(m-1)!} e^{-(\lambda+\mu+\theta)z} \int\limits_{y=0}^{\infty} y^{m-1} e^{-\lambda y} \tilde{\tilde{R}}(\theta,\alpha\,|y)\,dy-$$
$$-\frac{\lambda^m \mu\rho\varphi(\theta)}{(m-2)!} \int\limits_{u=0}^{z} e^{-(\lambda+\mu+\theta)u} \int\limits_{y=z-u}^{\infty} (y+u-z)^{m-2} e^{-\lambda y} \{\tilde{\tilde{R}}(\theta,\alpha\,|y)\,dudy.$$

Taking the derivative of the last equation m times with respect to z, we have the $(m+1)$-order inhomogeneous differential equation:

$$\sum_{i=0}^{m} C_m^i \left\{\tilde{\tilde{R}}^{(i+1)}(\theta,\alpha|\,z) + [\mu+\theta-\mu(1-\rho)\varphi(\theta)]\,\tilde{\tilde{R}}^{(i)}(\theta,\alpha|\,z)\right\}\times$$
$$\times(-1)^{m-i}\lambda^{m-i} - (-1)^m \lambda^m \mu\rho\varphi(\theta)\tilde{\tilde{R}}(\theta,\alpha|\,z) =$$
$$= \frac{(-1)^m\{\theta+\mu[1-\varphi(\theta)]\}(\alpha+\lambda)^m}{\theta} e^{-\alpha z}. \tag{5}$$

The general solution of this differential equation is

$$\tilde{\tilde{R}}(\theta,\alpha|\,z) = \sum_{i=1}^{m+1} C_i(\theta,\alpha) e^{K_1(\theta,\rho)z} +$$
$$+\frac{\{\theta+\mu[1-\varphi(\theta)]\}(\alpha+\lambda)^m}{\theta\left\{\sum\limits_{i=0}^{m} C_m^i \{\mu+\theta-\mu(1-\rho)\varphi(\theta)-\alpha\}\lambda^{m-i}\alpha^i - \lambda^{m+}\mu\rho\varphi(\theta)\right\}} e^{-\alpha z}, \tag{6}$$

where $K_i(\theta,\rho)$, $i=\overline{1,m+1}$, are the roots of the characteristic equation

$$\sum_{i=0}^{m} C_m^i \left\{K^{i+1}(\theta,\rho)+\left[\mu+\theta-\mu(1-\rho)\varphi(\theta)\right]K^i(\theta,\rho)\right\}\times$$

$$\times(-1)^{m-i}\lambda^{m-i}-(-1)^m\lambda^m\mu\rho\varphi(\theta)=0. \qquad (7)$$

of the differential equation (5).

Let us find the initial conditions of the differential equation (5). For this purpose, we differentiate Eq. (4) m times:

$$\sum_{i=0}^{l} C_l^i \left\{\tilde{\tilde{R}}^{(i+1)}(\theta,\alpha|\,z)+\left[\mu+\theta-\mu(1-\rho)\varphi(\theta)\right]\tilde{\tilde{R}}^{(i)}(\theta,\alpha|\,z)\right\}\times$$

$$\times(-1)^{l-i}\lambda^{l-i}e^{-\lambda z}=\frac{(-1)^l\left\{\theta+\left[1-\varphi(\theta)\right]\mu\right\}(\alpha+\lambda)}{\theta}e^{-(\alpha+\lambda)z}+$$

$$+\sum_{k=1}^{l}\frac{(-1)^{l+1}\lambda^m\mu\rho\varphi(\theta)(\lambda+\mu+\theta)^{l-(k-1)}}{(m-k)!}e^{-(\lambda+\mu+\theta)z}\times$$

$$\times\int_{y=0}^{\infty}y^{m-k}e^{-\lambda y}\tilde{\tilde{R}}(\theta,\alpha\,|y)\,dy+\frac{(-1)^l\lambda^m\mu\rho\varphi(\theta)(\lambda+\mu+\theta)}{(m-(l+1))!}\times$$

$$\times\int_{u=0}^{z}e^{-(\lambda+\mu+\theta)u}\int_{y=z-u}^{\infty}(y+u-z)^{m-(l+1)}e^{-\lambda y}\tilde{\tilde{R}}(\theta,\alpha/y)\,dudy+$$

$$+\frac{(-1)^l\lambda^m\mu\rho\varphi(\theta)(\lambda+\mu+\theta)^l}{(m-1)!}e^{-(\lambda+\mu+\theta)z}\int_{y=0}^{\infty}y^{m-1}e^{-\lambda y}\tilde{\tilde{R}}(\theta,\alpha\,|y)\,dy-$$

$$-\sum_{k=1}^{l}\frac{(-1)^{l+1}\lambda^m\mu\rho\varphi(\theta)(\lambda+\mu+\theta)^{l-k}}{(m-(k-1))!}e^{-(\lambda+\mu+\theta)z}\times$$

$$\times\int_{y=0}^{\infty}y^{m-(k-1)}e^{-\lambda y}\tilde{\tilde{R}}(\theta,\alpha\,|y)\,dy-\frac{(-1)^l\lambda^m\mu\rho\varphi(\theta)}{(m-(l+2))!}\int_{u=0}^{z}e^{-(\lambda+\mu+\theta)u}\times$$

$$\times\int_{y=z-u}^{\infty}(y+u-z)^{m-(l+2)}e^{-\lambda y}\{\tilde{\tilde{R}}(\theta,\alpha\,|y)\,dudy,\ l=\overline{0,m-2}. \qquad (8)$$

Putting $z=0$ in the integral equation (4) and in expressions (8), we obtain the initial conditions for the differential equation (5):

$$\begin{cases}
\tilde{\tilde{R}}(\theta,\alpha|\,0)=\frac{\theta+\mu[1-\varphi(\theta)]}{\theta(\mu+\theta)}+\frac{(1-\rho)\mu\varphi(\theta)}{\mu+\theta}\tilde{\tilde{R}}(\theta,\alpha|\,0)+\\
+\frac{\lambda^m\mu\rho\varphi(\theta)}{(m-1)!(\mu+\theta)}\int\limits_{y=0}^{\infty}y^{m-1}e^{-\lambda y}\tilde{\tilde{R}}(\theta,\alpha\,|y)\,dy,\\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
\sum\limits_{i=0}^{l}C_l^i\left\{\tilde{\tilde{R}}^{(i+1)}(\theta,\alpha|\,0)+[\mu+\theta-\mu(1-\rho)\varphi(\theta)]\,\tilde{\tilde{R}}^{(i)}(\theta,\alpha|\,0)\right\}\times\\
\times(-1)^{l-i}\lambda^{l-i}=\frac{(-1)^l\{\theta+[1-\varphi(\theta)]\mu\}(\alpha+\lambda)^l}{\theta}+\\
+\sum\limits_{k=1}^{l}\frac{(-1)^{l+1}\lambda^m\mu\rho\varphi(\theta)(\lambda+\mu+\theta)^{l-(k-1)}}{(m-k)!}\int\limits_{y=0}^{\infty}y^{m-k}e^{-\lambda y}\tilde{\tilde{R}}(\theta,\alpha\,|y)\,dy+\\
+\frac{(-1)^l\lambda^m\mu\rho\varphi(\theta)(\lambda+\mu+\theta)^l}{(m-1)!}\int\limits_{y=0}^{\infty}y^{m-1}e^{-\lambda y}\tilde{\tilde{R}}(\theta,\alpha\,|y)\,dy+\\
+\frac{(-1)^l\lambda^m\mu\rho\varphi(\theta)(\lambda+\mu+\theta)^l}{(m-1)!}\int\limits_{y=0}^{\infty}y^{m-1}e^{-\lambda y}\tilde{\tilde{R}}(\theta,\alpha\,|y)\,dy,\quad l=\overline{0,m-2},\\
\sum\limits_{i=0}^{m-1}C_{m-1}^i\left\{\tilde{\tilde{R}}^{(i+1)}(\theta,\alpha|\,0)+[\mu+\theta-\mu(1-\rho)\varphi(\theta)]\,\tilde{\tilde{R}}^{(i)}(\theta,\alpha|\,0)\right\}\times\\
\times(-1)^{m-(i+1)}\lambda^{m-(i+1)}=\frac{(-1)^{m-1}\{\theta+[1-\varphi(\theta)]\mu\}(\alpha+\lambda)^{m-1}}{\theta}+\\
+\sum\limits_{k=1}^{m-2}\frac{(-1)^m\lambda^m\mu\rho\varphi(\theta)(\lambda+\mu+\theta)^{m-k}}{(m-k)!}\int\limits_{y=0}^{\infty}y^{m-k}e^{-\lambda y}\tilde{\tilde{R}}(\theta,\alpha\,|y)\,dy+\\
+(-1)^{m-2}\lambda^m\mu\,(\lambda+\mu+\theta)\,\rho\varphi\,(\theta)\int\limits_{y=0}^{\infty}ye^{-\lambda y}\tilde{\tilde{R}}(\theta,\alpha\,|y)\,dy+\\
+\sum\limits_{k=1}^{m-2}\frac{(-1)^m\lambda^m\mu\rho\varphi(\theta)(\lambda+\mu+\theta)^{m-(k+1)}}{(m-1)!}\int\limits_{y=0}^{\infty}y^{m-(k-1)}e^{-\lambda y}\tilde{\tilde{R}}(\theta,\alpha\,|y)\,dy-\\
-(-1)^{m-2}\lambda^m\mu\rho\varphi\,(\theta)\int\limits_{y=0}^{\infty}e^{-\lambda y}\tilde{\tilde{R}}(\theta,\alpha\,|y)\,dy.
\end{cases}\qquad(9)$$

Writing the expression for

$$\tilde{\tilde{R}}^{(i)}(\theta,\alpha|\,0),\qquad i=\overline{0,m},$$

from system (8) and the expression for $\tilde{\tilde{R}}(\theta, \alpha|\, y)$ from (6), we get the following system of linear algebraic equations for $C_i(\theta), i = \overline{1, m+1}$:

$$
\begin{cases}
\sum\limits_{i=1}^{m+1} \left\{ 1 - \frac{(1-\rho)\mu\varphi(\theta)}{(\mu+\theta)} - \frac{\lambda^m \mu\rho\varphi(\theta)}{(\mu+\theta)[\lambda - K_i(\theta,\rho)]^m} \right\} C_i(\theta, \alpha) = - \\
- \dfrac{\alpha(\alpha+\lambda)^m \{\theta + [1-\varphi(\theta)]\mu\}}{\theta(\mu+\theta)\{[\mu+\theta-\mu(1-\rho)\varphi(\theta) - \alpha](\alpha+\lambda)^m - \lambda^m \mu\rho\varphi(\theta)\}}, \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
\sum\limits_{j=1}^{m+1} \left\{ \sum\limits_{i=0}^{l} C_l^i \left\{ K_j^{i+1}(\theta,\rho) + [\mu+\theta-\mu(1-\rho)\varphi(\theta)]\, K_j^i(\theta,\rho) \right\} \times \right. \\
\times (-1)^{l-i} \lambda^{l-i} - \sum\limits_{k=1}^{l} \dfrac{(-1)^{l+1} \lambda^m \mu\rho\varphi(\theta)(\lambda+\mu+\theta)^{l-(k-1)}}{[\lambda - K_j(\theta,\rho)]^{m-(k-1)}} - \\
- \dfrac{(-1)^{l+1} \lambda^m \mu\rho\varphi(\theta)(\lambda+\mu+\theta)^{l-(k-1)}}{[\lambda - K_j(\theta,\rho)]^{m}} + \\
\left. + \sum\limits_{k=1}^{l} \dfrac{(-1)^{l+1} \lambda^m \mu\rho\varphi(\theta)(\lambda+\mu+\theta)^{l-k}}{[\lambda - K_j(\theta,\rho)]^{m-k}} \right\} C_j(\theta, \alpha) = 0, \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
\sum\limits_{j=1}^{m+1} \left\{ \sum\limits_{i=0}^{m-1} C_{m^+-1}^i \left\{ K_j^{i+1}(\theta,\rho) + [\mu+\theta-\mu(1-\rho)\varphi(\theta)]\, K_j^i(\theta,\rho) \right\} \times \right. \\
\times (-1)^{m-(i+1)} \lambda^{m-(i+1)} - \sum\limits_{k=1}^{m^+-2} \dfrac{(-1)^{m} \lambda^m \mu\rho\varphi(\theta)(\lambda+\mu+\theta)^{m-k}}{[\lambda - K_j(\theta,\rho)]^{m-(k-1)}} - \\
- \dfrac{(-1)^{m-2} \lambda^m \mu\rho\varphi(\theta)(\lambda+\mu+\theta)}{[\lambda - K_j(\theta,\rho)]^{2}} + \\
+ \dfrac{(-1)^{m-1} \lambda^m \mu\rho\varphi(\theta)(\lambda+\mu+\theta)^{m-1}}{[\lambda - K_j(\theta,\rho)]^{m}} - \\
- \sum\limits_{k=1}^{m-2} \dfrac{(-1)^{m} \lambda^m \mu\rho\varphi(\theta)(\lambda+\mu+\theta)^{m-(k+1)}}{[\lambda - K_j(\theta,\rho)]^{m-k}} - \\
\left. - \dfrac{(-1)^{m-2} \lambda^m \mu\rho\varphi(\theta)}{\lambda - K_j(\theta,\rho)} \right\} C_j(\theta, \alpha) = 0.
\end{cases}
\tag{10}
$$

Using the Vieta conditions for the characteristic equation (7), we have

$$
\prod_{i=1}^{m+1} [\lambda - K_i(\theta,\rho)] = (-1)^{m+1} \lambda^m \mu\rho\varphi(\theta),
$$

$$
\prod_{\substack{i=1 \\ \overrightarrow{i \neq j}}}^{m+1} [\lambda - K_i(\theta,\rho)] =
$$

$$= (-1)^{m-1}\left[\lambda - K_j\left(\theta,\rho\right)\right]^{m-1}\left[K_j\left(\theta,\rho\right) + \mu + \theta - \mu(1-\rho)\varphi(\theta)\right].$$

In view of these forms, system (10) yields the equation

$$\sum_{j=1}^{m+1} K_j\left(\theta,\rho\right) C_j\left(\theta,\alpha\right) =$$

$$= \frac{\alpha\left\{\theta + \mu\left[1-\varphi(\theta)\right]\right\}(\alpha+\lambda)^m}{\theta\left\{\sum\limits_{i=0}^{m} C_m^i\left\{\mu + \theta - \mu(1-\rho)\varphi(\theta) - \alpha\right\}\lambda^{m-i}\alpha^i - \lambda^m\mu\rho\varphi(\theta)\right\}}.$$

This linear algebraic equation has infinite number of solutions. We take the solution

$$C_1\left(\theta,\alpha\right) =$$

$$= \frac{\alpha\left\{\theta + \mu\left[1-\varphi(\theta)\right]\right\}(\alpha+\lambda)^m}{\theta\left\{\sum\limits_{i=0}^{m} C_m^i\left\{\mu + \theta - \mu(1-\rho)\varphi(\theta) - \alpha\right\}\lambda^{m-i}\alpha^i - \lambda^m\mu\rho\varphi(\theta)\right\} K_1(\theta,\rho)},$$

$$C_j\left(\theta,\alpha\right) = 0, \overline{j = 2, m+1}.$$

In this case, the general solution (6) of the differential equation will take the following form:

$$\tilde{\tilde{R}}(\theta,\alpha|\, z) =$$

$$= \frac{\alpha\left\{\theta + \mu\left[1-\varphi(\theta)\right]\right\}(\alpha+\lambda)^m}{\theta\left\{\sum\limits_{i=0}^{m} C_m^i\left\{\mu + \theta - \mu(1-\rho)\varphi(\theta) - \alpha\right\}\lambda^{m-i}\alpha^i - \lambda^m\mu\rho\varphi(\theta)\right\} K_1(\theta,\rho)} e^{K_1(\theta,\rho)z} +$$

$$+ \frac{\left\{\theta + \mu\left[1-\varphi(\theta)\right]\right\}(\alpha+\lambda)^m}{\theta\left\{\sum\limits_{i=0}^{m} C_m^i\left\{\mu + \theta - \mu(1-\rho)\varphi(\theta) - \alpha\right\}\lambda^{m-i}\alpha^i - \lambda^{m+}\mu\rho\varphi(\theta)\right\}} e^{-\alpha z},$$

where $K_1(0,\rho) = 0$. This solution satisfies the differential equation (5), initial conditions (9), and integral equation (4).

Let us find $\tilde{\tilde{R}}(\theta,\alpha)$. Since the random variable $X_*(0)$ has the m-order Erlang distribution, we can write

$$\tilde{\tilde{R}}(\theta,\alpha) = \int_0^\infty \tilde{\tilde{R}}(\theta,\alpha|\, z) dP\left\{X_*(0) < z\right\} = \int_0^\infty \tilde{\tilde{R}}(\theta,\alpha|\, z) dP\left\{\zeta_1 < z\right\}$$

$$= \frac{\lambda^m}{\left[\lambda - K_1(\theta,\rho)\right]^m}$$

$$\times \frac{\alpha\left\{\theta + \mu\left[1-\varphi(\theta)\right]\right\}(\alpha+\lambda)^m}{\theta\left\{\sum\limits_{i=0}^{m} C_m^i\left\{\mu + \theta - \mu(1-\rho)\varphi(\theta) - \alpha\right\}\lambda^{m-i}\alpha^i - \lambda^{m+}\mu\rho\varphi(\theta)\right\} K_1(\theta,\rho)}$$

$$+ \frac{\lambda^m\left\{\theta + \mu\left[1-\varphi(\theta)\right]\right\}}{\theta\left\{\sum\limits_{i=0}^{m} C_m^i\left\{\mu + \theta - \mu(1-\rho)\varphi(\theta) - \alpha\right\}\lambda^{m-i}\alpha^i - \lambda^m\mu\rho\varphi(\theta)\right\}}.$$

Remark. If $\lambda - m\mu\rho > 0$, the process $X_*(t)$ will be ergodic.
The proof of this remark is obtained from Theorem 2 in [1].

Therefore, using the Tauberian theorem, we can find the Laplace transform of the ergodic distribution of the process $X_*(t)$:

$$\tilde{R}(\alpha) = \lim_{\theta\to 0} \theta\, \tilde{\tilde{R}}(\theta,\, \alpha) = \frac{\lambda - m\mu\rho}{\lambda}\times$$
$$\times\frac{(\alpha+\lambda)^m}{m(\alpha-\mu\rho)\lambda^{m-1} - \frac{m(m-1)}{2}\alpha(\alpha-\mu\rho)\lambda^{m-2} + \lambda^m + \sum_{i=3}^{m} C_m^i(\alpha-\mu\rho)\alpha^{i-1}\lambda^{m-i}}. \tag{11}$$

By X_*, we denote the random variable satisfying the equality

$$\lim_{t\to\infty} P\left\{X_*(t) < x\right\} = P\left\{X_* < x\right\},\, x \in R^+.$$

From (12), we find the expectation and variance of the ergodic distribution of the process $X_*(t)$:

$$EX = \frac{m(m+1)\mu\rho}{2\lambda(\lambda - m\mu\rho)} = \frac{(m+1)\left[E\zeta_1\right]^2}{2m\rho\left[E\xi_1 - E\zeta_1\right]},$$
$$DX = \frac{m(m+1)\mu\rho\left[4\,(m+2)\,\lambda - m(m+5)\mu\rho\right]}{12\lambda^2(\lambda-m\mu\rho)^2} =$$
$$\frac{(m+1)\ \left[4\,(m+2)\,E\xi_1 - (m+5)E\zeta_1\right]}{12(m\rho)^2\left[E\xi_1 - E\zeta_1\right]^2}\left[E\zeta_1\right]^3.$$

In particular, for $m = 1$ and $\rho = 1$, the result

$$EX = \frac{\mu}{\lambda(\lambda-\mu)} = \frac{\left[E\zeta_1\right]^2}{\left[E\xi_1 - E\zeta_1\right]},$$
$$DX = \frac{\mu(2\lambda-\mu)}{\lambda^2(\lambda-\mu)^2} = \frac{\left[2E\xi_1 - E\zeta_1\right]}{\left[E\xi_1 - E\zeta_1\right]^2}\left[E\zeta_1\right]$$

follows from [3, p. 121].

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