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THE EXPLICIT PROBABILITY DISTRIBUTION OF A SIX-DIMENSIONAL RANDOM FLIGHT

We consider the symmetric random motion with finite speed $\mathbf{X}(t)$ in the Euclidean space $\mathbb{R}^6$ subject to the control of a homogeneous Poisson process. The explicit probability distribution of $\mathbf{X}(t)$, $t > 0$, is obtained.

1. Introduction

The multidimensional diffusion with finite speed of propagation is generated by the finite-velocity random motions of a particle that moves in the Euclidean space $\mathbb{R}^m$, $m \geq 2$, and whose evolution is driven by some stochastic process. The most studied model is performed by the symmetric random motion controlled by a homogeneous Poisson process with the uniform choice of directions. Such a type of motion is referred to as the random flight or, in a more general sense, random evolution. One of the most important features of such a motion is that it generates an isotropic transport process in the Euclidean space $\mathbb{R}^m$ (see, for instance, Tolubinsky (1969), Papanicolaou (1975), Pinsky (1976)).

Random flights in the Euclidean spaces of different dimensions have thoroughly been examined in a series of works. In the study of such processes, the most desirable goal is undoubtedly their explicit distributions in the cases (very few, indeed) where such distributions can be obtained. The explicit form of the distribution of a two-dimensional symmetric random motion with finite speed was derived (by different methods) by Stadje (1987), Masoliver *et al.* (1993), Kolesnik and Orsingher (2005), and Kolesnik (2007). The distribution of a random flight in $\mathbb{R}^3$ was given by Tolubinsky (1969, Chapter 2, pp. 35-60) and by Stadje (1989) in rather complicated integral forms. Finally, the explicit form of the distribution of a random flight in $\mathbb{R}^4$ was obtained by Kolesnik (2006). The random flights in spaces of arbitrary higher dimensions were examined by Kolesnik (2008a); however, no new distributions were obtained in this work for higher dimensions $m \geq 5$.

Since the exact probability laws of random flights for lower dimensions were derived by rather complicated and sometimes tricky methods, the possibility of obtaining the explicit form of the distributions seemed very doubtful for higher dimensions $m \geq 5$.

However, a general unified method of studying the random flights in spaces of arbitrary dimensions was suggested in Kolesnik (2008a) based on the analysis of the integral transforms of their distributions. This method applied to the six-dimensional random motion enables us, surprisingly, to obtain the explicit probability law of the process, and this result is the core of the present paper. Although this method works for any dimension, the derivation of the *explicit* probability law in the space of such high dimension $m = 6$ looks like a "lucky accident" which, apparently, cannot be extended to higher dimensions.

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The distribution derived has a considerably more complicated form in comparison with those obtained for the dimensions $m = 2$ and $m = 4$. It is presented as a series of the finite sums of Gauss hypergeometric functions which seemingly cannot be reduced to a more elegant formula. Nevertheless, this formula is of a great interest because it gives the *explicit* form of the distribution, and, on the other hand, it is a new step toward the most desirable goal, namely, constructing the general theory of distributions for random flights in the Euclidean spaces $\mathbb{R}^m$ of arbitrary dimension $m \geq 2$.

The main result of this paper was announced (without proof) in Kolesnik (2008b).

2. Description of motion and the distribution structure

We consider the stochastic motion performed by a particle starting its motion from the origin $\mathbf{0} = (0, 0, 0, 0, 0, 0)$ of the six-dimensional Euclidean space $\mathbb{R}^6$ at the time $t = 0$. The particle is endowed with constant finite speed c (note that c is treated as the constant norm of the velocity). The initial direction is a six-dimensional random vector with uniform distribution (Lebesgue probability measure) on the unit sphere

$$S_1 = \left\{ \mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^6 : \ \|\mathbf{x}\|^2 = \sum_{i=1}^{6} x_i^2 = 1 \right\}.$$

The particle changes direction at random instants which form a homogeneous Poisson process with rate $\lambda > 0$. At these moments, it instantaneously takes on a new direction with uniform distribution on S_1, independently of its previous motion.

Let $\mathbf{X}(t) = (X_1(t), X_2(t), X_3(t), X_4(t), X_5(t), X_6(t))$ be the position of the particle at an arbitrary time $t > 0$. Consider the conditional distributions

$$\Pr\{\mathbf{X}(t) \in d\mathbf{x} \mid N(t) = n\} = \Pr\left\{ \bigcap_{i=1}^{6} (X_i(t) \in dx_i) \ \mid N(t) = n \right\}, \qquad n \geq 1, \tag{1}$$

where $N(t)$ is the number of Poisson events that have occurred in the interval $(0, t)$, and $d\mathbf{x}$ is the infinitesimal element in the space $\mathbb{R}^6$ with the Lebesgue measure $\mu(d\mathbf{x}) = dx_1 dx_2 dx_3 dx_4 dx_5 dx_6$.

At any time $t > 0$, the particle is located with probability 1 in the six-dimensional ball of radius ct

$$\mathbf{B}_{ct} = \left\{ \mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^6 : \ \|\mathbf{x}\|^2 = \sum_{i=1}^{6} x_i^2 \leq c^2 t^2 \right\}.$$

The distribution $\Pr\{\mathbf{X}(t) \in d\mathbf{x}\}, \ \mathbf{x} \in B_{ct}, \ t \geq 0$, consists of two components. The singular component corresponds to the case where no Poisson event occurs in the interval $(0, t)$ and is concentrated on the sphere

$$S_{ct} = \partial \mathbf{B}_{ct} = \left\{ \mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6) \in R^6 : \ \|\mathbf{x}\|^2 = \sum_{i=1}^{6} x_i^2 = c^2 t^2 \right\}.$$

In this case, the particle is located on the sphere S_{ct}, and the probability of this event is

$$\Pr\{\mathbf{X}(t) \in S_{ct}\} = e^{-\lambda t}.$$

If one or more than one Poisson events occur, the particle is located strictly inside the ball $\mathbf{B}_{ct}$, and the probability of this event is

$$\Pr\{\mathbf{X}(t) \in \text{int } \mathbf{B}_{ct}\} = 1 - e^{-\lambda t}. \tag{2}$$

The part of the distribution $\Pr\{\mathbf{X}(t) \in d\mathbf{x}\}$ corresponding to this case is concentrated in the interior

$$\text{int } \mathbf{B}_{ct} = \left\{ \mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^6 : \ \|\mathbf{x}\|^2 = \sum_{i=1}^{6} x_i^2 < c^2t^2 \right\},$$

and forms its absolutely continuous component. Therefore, there exists the density $p(\mathbf{x}, t) = p(x_1, x_2, x_3, x_4, x_5, x_6; t), \quad \mathbf{x} \in \text{int } \mathbf{B}_{ct}, \ t > 0$, of the absolutely continuous component of the distribution function $\Pr\{\mathbf{X}(t) \in d\mathbf{x}\}$.

The derivation of the explicit form of the density $p(\mathbf{x}, t), \ t > 0$, is the main goal of our paper.

3. The density of the process

According to the total probability formula, we can represent the density $p(\mathbf{x}, t)$ in the form of the uniformly converging series

$$p(\mathbf{x}, t) = e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n!} \, p_n(\mathbf{x}, t), \tag{3}$$

where $p_n(\mathbf{x}, t), \ n \geq 1$, are the conditional densities of the conditional distributions (1).

Our principal result represents the explicit form of the density $p(\mathbf{x}, t)$.

Theorem. *For any $t > 0$, the density $p(\mathbf{x}, t)$ has the form*

$$\begin{aligned} p(\mathbf{x}, t) =& \frac{16\lambda t e^{-\lambda t}}{\pi^3 (ct)^6} \left(1 - \frac{5}{6} \frac{\|\mathbf{x}\|^2}{c^2t^2} \right) \\ &+ \frac{e^{-\lambda t}}{2\pi^3 (ct)^6} \sum_{n=2}^{\infty} (\lambda t)^n (n+1)! \sum_{k=0}^{n+1} \frac{(k+1)(k+2)(n+2k+1)}{3^k (n-k+1)! (n+k-2)!} \\ &\qquad \times F\left(-(n+k-2), k+3; 3; \frac{\|\mathbf{x}\|^2}{c^2t^2} \right), \end{aligned} \tag{4}$$

where $\|\mathbf{x}\|^2 = \sum_{i=1}^{6} x_i^2 < c^2t^2$, the function

$$F(\xi, \eta; \zeta; z) = {}_2F_1(\xi, \eta; \zeta; z) = \sum_{k=0}^{\infty} \frac{(\xi)_k (\eta)_k}{(\zeta)_k} \frac{z^k}{k!} \tag{5}$$

is the Gauss hypergeometric function and

$$(a)_k = a(a+1)\dots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}$$

is the Pochgammer symbol.

Proof. In view of formula (3), we should concentrate our efforts on finding the explicit forms of the conditional densities $p_n(\mathbf{x}, t), \ n \geq 1$. According to Kolesnik (2008a, formula (2.5)), the characteristic function (Fourier transform) of the uniform distribution on the surface of the sphere S_{ct} has the form

$$\varphi(t) = 8 \, \frac{J_2(ct\|\boldsymbol{\alpha}\|)}{(ct\|\boldsymbol{\alpha}\|)^2}, \tag{6}$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) \in \mathbb{R}^6$ is the 6-dimensional real vector of inversion parameters, $\|\boldsymbol{\alpha}\| = \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 + \alpha_5^2 + \alpha_6^2}$, and $J_2(z)$ is the Bessel function of order 2 with real argument.

In view of Kolesnik (2008a, formula (2.6)), the conditional characteristic functions $H_n(t)$ of the conditional densities $p_n(\mathbf{x},t), \quad n \ge 1$, are given by

$$H_n(t) = \frac{n!}{t^n}\, \mathcal{I}_n(t), \qquad n \ge 1, \tag{7}$$

where

$$\mathcal{I}_n(t) = \int_0^t d\tau_1 \int_{\tau_1}^t d\tau_2 \dots \int_{\tau_{n-1}}^t d\tau_n \left\{ \prod_{j=1}^{n+1} \varphi(\tau_j - \tau_{j-1}) \right\}.$$

Note that if $n = 0$ (this correspond to the case where no Poisson events occur in the time interval $(0,t)$), formula for the characteristic function $H_0(t)$ has the form:

$$H_0(t) = \mathcal{I}_0(t) = \varphi(t) = 8\, \frac{J_2(ct\|\boldsymbol{\alpha}\|)}{(ct\|\boldsymbol{\alpha}\|)^2}, \tag{8}$$

According to Kolesnik (2008a, formula (2.13)), the Laplace transform $\mathcal{L}$ of the function $\mathcal{I}_n(t)$ has the form

$$\mathcal{L}[\mathcal{I}_n(t)](s) = \left(\mathcal{L}[\varphi(t)](s)\right)^{n+1}, \qquad n \ge 1, \quad \operatorname{Re} s > 0, \tag{9}$$

where the function $\varphi(t)$ is given by (6).

In view of Bateman and Erdelyi (1954, table 4.14, formula 6), the Laplace transform of function (6) is

$$\begin{aligned} \mathcal{L}[\varphi(t)](s) &= \frac{8}{(c\|\boldsymbol{\alpha}\|)^2}\, \mathcal{L}\left[\frac{J_2(ct\|\boldsymbol{\alpha}\|)}{t^2}\right](s) \\ &= \frac{8}{(c\|\boldsymbol{\alpha}\|)^2}\, \frac{c\|\boldsymbol{\alpha}\|}{4} \left[\frac{c\|\boldsymbol{\alpha}\|}{s + \sqrt{s^2 + (c\|\boldsymbol{\alpha}\|)^2}} + \frac{1}{3}\left(\frac{c\|\boldsymbol{\alpha}\|}{s + \sqrt{s^2 + (c\|\boldsymbol{\alpha}\|)^2}} \right)^3 \right] \\ &= 2 \left[\left(s + \sqrt{s^2 + (c\|\boldsymbol{\alpha}\|)^2}\right)^{-1} + \frac{(c\|\boldsymbol{\alpha}\|)^2}{3} \left(s + \sqrt{s^2 + (c\|\boldsymbol{\alpha}\|)^2}\right)^{-3} \right]. \end{aligned}$$

By substituting this relation into (9) and applying the Newton binomial theorem, we obtain

$$\begin{aligned} \mathcal{L}[\mathcal{I}_n(t)](s) &= 2^{n+1} \left[\left(s + \sqrt{s^2 + (c\|\boldsymbol{\alpha}\|)^2}\right)^{-1} + \frac{(c\|\boldsymbol{\alpha}\|)^2}{3} \left(s + \sqrt{s^2 + (c\|\boldsymbol{\alpha}\|)^2}\right)^{-3} \right]^{n+1} \\ &= 2^{n+1} \sum_{k=0}^{n+1} C_{n+1}^k\, \frac{(c\|\boldsymbol{\alpha}\|)^{2k}}{3^k} \left(s + \sqrt{s^2 + (c\|\boldsymbol{\alpha}\|)^2}\right)^{-(n+2k+1)}. \end{aligned}$$

The inverse Laplace transformation of this expression yields

$$\begin{aligned} \mathcal{I}_n(t) &= 2^{n+1} \sum_{k=0}^{n+1} C_{n+1}^k\, \frac{(c\|\boldsymbol{\alpha}\|)^{2k}}{3^k}\, \mathcal{L}^{-1}\left[\left(s + \sqrt{s^2 + (c\|\boldsymbol{\alpha}\|)^2}\right)^{-(n+2k+1)} \right](t) \\ &\qquad \text{(see Bateman and Erdelyi (1954, table 5.3, formula 43))} \\ &= 2^{n+1} \sum_{k=0}^{n+1} C_{n+1}^k\, \frac{(c\|\boldsymbol{\alpha}\|)^{2k}}{3^k}\, \frac{n+2k+1}{t}\, (c\|\boldsymbol{\alpha}\|)^{-(n+2k+1)}\, J_{n+2k+1}(ct\|\boldsymbol{\alpha}\|) \\ &= \frac{2^{n+1}}{(c\|\boldsymbol{\alpha}\|)^{n+1}\, t} \sum_{k=0}^{n+1} C_{n+1}^k\, \frac{n+2k+1}{3^k}\, J_{n+2k+1}(ct\|\boldsymbol{\alpha}\|). \end{aligned}$$

Then, according to (7), the conditional characteristic functions have the form

$$H_n(t) = \frac{2^{n+1}\, n!}{(ct\|\boldsymbol{\alpha}\|)^{n+1}} \sum_{k=0}^{n+1} C_{n+1}^k\, \frac{n+2k+1}{3^k}\, J_{n+2k+1}(ct\|\boldsymbol{\alpha}\|), \qquad n \ge 1. \tag{10}$$

This is also valid for $n = 0$. Really, formula (10) transforms in this case into

$$H_0(t) = \frac{2}{ct\|\boldsymbol{\alpha}\|} \left[J_1(ct\|\boldsymbol{\alpha}\|) + J_3(ct\|\boldsymbol{\alpha}\|) \right] = \frac{2}{ct\|\boldsymbol{\alpha}\|} \, \frac{4}{ct\|\boldsymbol{\alpha}\|} \, J_2(ct\|\boldsymbol{\alpha}\|) = 8 \, \frac{J_2(ct\|\boldsymbol{\alpha}\|)}{(ct\|\boldsymbol{\alpha}\|)^2},$$

and this relation coincides with (8). Note that here we have used the well-known recurrent relation for the Bessel functions (see, for instance, Gradshtein and Ryzhik (1980, Formula 8.471(1))):

$$J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{2\nu}{z} \, J_\nu(z). \tag{11}$$

To obtain the conditional densities $p_n(\mathbf{x}, t)$, $n \geq 1$, we should evaluate the inverse Fourier transform $\mathcal{F}_{\boldsymbol{\alpha}}^{-1}$ of the characteristic functions (10) with respect to $\boldsymbol{\alpha}$, that is,

$$\begin{aligned} p_n(\mathbf{x}, t) &= \mathcal{F}_{\boldsymbol{\alpha}}^{-1} \left[H_n(t) \right] \\ &= \frac{2^{n+1} \, n!}{(ct)^{n+1}} \sum_{k=0}^{n+1} C_{n+1}^k \, \frac{n+2k+1}{3^k} \; \mathcal{F}_{\boldsymbol{\alpha}}^{-1} \left[\frac{J_{n+2k+1}(ct\|\boldsymbol{\alpha}\|)}{(\|\boldsymbol{\alpha}\|)^{n+1}} \right], \qquad n \geq 1. \end{aligned} \tag{12}$$

For the particular case $n = 1$, we do not need to invert (12) because the conditional density $p_1(\mathbf{x}, t)$ can be easily obtained by applying formula (7) of Kolesnik (2008c). Then, according to this formula, we immediately get

$$\begin{aligned} p_1(\mathbf{x}, t) &= \frac{16}{\pi^3 \, (ct)^6} \, F\left(\frac{5}{2}, -1; 3; \frac{\|\mathbf{x}\|^2}{c^2 t^2} \right) \\ &= \frac{16}{\pi^3 \, (ct)^6} \left(1 - \frac{5}{6} \, \frac{\|\mathbf{x}\|^2}{c^2 t^2} \right), \qquad \|\mathbf{x}\| < ct. \end{aligned} \tag{13}$$

Let now $n \geq 2$. Then, by applying the Hankel inversion formula (see Vladimirov (1981, Sect. 23, formula (43))) and using Gradshtein and Ryzhik (1980, Formula 6.574(1)), we can compute the inverse Fourier transforms in formula (12):

$$\begin{aligned} &\mathcal{F}_{\boldsymbol{\alpha}}^{-1} \left[\frac{J_{n+2k+1}(ct\|\boldsymbol{\alpha}\|)}{(\|\boldsymbol{\alpha}\|)^{n+1}} \right] \\ &= (2\pi)^{-3} \|\mathbf{x}\|^{-2} \int_0^\infty r^3 \, J_2(\|\mathbf{x}\| r) \, \frac{J_{n+2k+1}(ctr)}{r^{n+1}} \, dr \\ &= \frac{1}{(2\pi)^3 \|\mathbf{x}\|^2} \int_0^\infty r^{-(n-2)} \, J_2(\|\mathbf{x}\| r) \, J_{n+2k+1}(ctr) \, dr \\ &= \frac{1}{(2\pi)^3 \|\mathbf{x}\|^2} \, \frac{\|\mathbf{x}\|^2 \, \Gamma(k+3)}{2^{n-2} \, (ct)^{-n+5} \, \Gamma(n+k-1) \, \Gamma(3)} \, F\left(k+3, -(n+k-2); 3; \frac{\|\mathbf{x}\|^2}{c^2 t^2} \right) \\ &= \frac{(k+2)!}{\pi^3 \, 2^{n+2} \, (ct)^{-n+5} \, (n+k-2)!} \, F\left(-(n+k-2), k+3; 3; \frac{\|\mathbf{x}\|^2}{c^2 t^2} \right), \qquad \|\mathbf{x}\| < ct. \end{aligned}$$

Substituting this expression into (12), we obtain the conditional densities for arbitrary $n \geq 2$:

$$
\begin{aligned}
p_n(\mathbf{x},t) &= \frac{2^{n+1}\, n!}{(ct)^{n+1}} \sum_{k=0}^{n+1} C_{n+1}^k \, \frac{n+2k+1}{3^k} \\
&\quad \times \frac{(k+2)!}{\pi^3 \, 2^{n+2} \, (ct)^{-n+5} \, (n+k-2)!} \, F\left(-(n+k-2), k+3; 3; \frac{\|\mathbf{x}\|^2}{c^2t^2}\right) \\
&= \frac{n!}{2\pi^3 \, (ct)^6} \sum_{k=0}^{n+1} \frac{(n+1)!}{k! \, (n-k+1)!} \, \frac{(n+2k+1)\,(k+2)!}{3^k \, (n+k-2)!} \\
&\qquad \times F\left(-(n+k-2), k+3; 3; \frac{\|\mathbf{x}\|^2}{c^2t^2}\right) \\
&= \frac{n! \, (n+1)!}{2\pi^3 \, (ct)^6} \sum_{k=0}^{n+1} \frac{(k+1)(k+2)(n+2k+1)}{3^k \, (n-k+1)! \, (n+k-2)!} \\
&\qquad \times F\left(-(n+k-2), k+3; 3; \frac{\|\mathbf{x}\|^2}{c^2t^2}\right), \\
&\qquad \|\mathbf{x}\| < ct, \qquad n \geq 2.
\end{aligned} \tag{14}
$$

Substituting now the explicit forms of the conditional densities $p_n(\mathbf{x},t)$, $n \geq 1$, given by (13) and (14) into formula (3), we finally obtain (4).

The theorem is thus completely proved.

Remark 1. Formula (14) shows that the conditional densities $p_n(\mathbf{x},t)$ for $n \geq 2$ have very complicated forms in spaces of higher dimensions, and this contrasts with the two- and four-dimensional cases where such conditional densities have very simple forms for any $n \geq 1$ (see, for comparison, Kolesnik and Orsingher (2005, formula (11)) for the dimension $m = 2$, and Kolesnik (2006, formula (6)) for the dimension $m = 4$). For instance, in our six-dimensional case, the conditional density $p_2(\mathbf{x},t)$ corresponding to two changes of directions has the form:

$$
p_2(\mathbf{x},t) = \frac{4}{\pi^3 \, (ct)^6} \left(\frac{53}{3} - \frac{130}{3} \, \frac{\|\mathbf{x}\|^2}{(ct)^2} + 35 \, \frac{\|\mathbf{x}\|^4}{(ct)^4} - \frac{28}{3} \, \frac{\|\mathbf{x}\|^6}{(ct)^6} \right). \tag{15}
$$

This can be obtained from formula (14) for $n = 2$ after long computations. Clearly, the expressions for $n \geq 3$ are more complicated than (15).

Remark 2. Since the first coefficient of the hypergeometric function in formula (14) is negative for any n and k, the conditional densities $p_n(\mathbf{x},t)$ are, in fact, some polynomials of finite orders of the variable $\|\mathbf{x}\|^2/(ct)^2$. This is also clearly seen from formula (15). Therefore, the transition density $p(\mathbf{x},t)$ given by (4) represents a functional series composed of some polynomials. Such a structure of the density is the specific feature of random flights in spaces of even dimensions.

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References

Bateman H. and Erdelyi A. (1954), *Tables of Integral Transforms*, McGraw-Hill, New York.

Gradshtein I.S. and Ryzhik I.M. (1980), *Tables of Integrals, Sums, Series, and Products*, Academic Press, New York.

Kolesnik A.D. (2008a), *Random motions at finite speed in higher dimensions*, J. Statist. Phys., vol 131, 1039–1065.

Kolesnik A.D. (2008b), *Symmetric random evolution in the space* $\mathbb{R}^6$, Bull. Acad. Sci. Moldova, Ser. Math., vol 2(57), 114–117.

Kolesnik A.D. (2008c), *An asymptotic formula for the density of a multidimensional random evolution with rare Poissonian switchings*, Ukrain. Math. J., vol 60, no. 12, 1631–1641.

Kolesnik A.D. (2007), *A note on planar random motion at finite speed*, J. Appl. Probab., vol 44, 838–842.

Kolesnik A.D. (2006), *A four-dimensional random motion at finite speed*, J. Appl. Probab., vol 43, 1107–1118.

Kolesnik A.D. and Orsingher E. (2005), *A planar random motion with an infinite number of directions controlled by the damped wave equation*, J. Appl. Probab., vol 42, 1168–1182.

Masoliver J., Porrá J.M. and Weiss G.H. (1993), *Some two and three-dimensional persistent random walks*, Physica A, vol 193, 469–482.

Papanicolaou G. (1975), *Asymptotic analysis of transport processes*, Bull. Amer. Math. Soc., vol 81, 330–392.

Pinsky M. (1976), *Isotropic transport process on a Riemannian manifold*, Trans. Amer. Math. Soc., vol 218, 353–360.

Stadje W. (1989), *Exact probability distributions for non-correlated random walk models*, J. Statist. Phys., vol 56, 415–435.

Stadje W. (1987), *The exact probability distribution of a two-dimensional random walk*, J. Statist. Phys., vol 46, 207–216.

Tolubinsky E.V. (1969), *The Theory of Transfer Processes*, Naukova Dumka, Kiev.

Vladimirov V.S. (1981), *The Equations of Mathematical Physics*, Nauka, Moscow.

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