Theory of Stochastic Processes
Vol. 15 (31), no. 1, 2009, pp. 15–32

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WEAK CONVERGENCE OF ADDITIVE FUNCTIONALS OF A SEQUENCE OF MARKOV CHAINS

We consider additive functionals $\phi_n, n \geq 1$ defined on a sequence of Markov chains that weakly converges to a Markov process. We give sufficient condition for $\phi_n, n \geq 1$ to converge in distribution, formulated in the terms of their characteristics (i.e. expectations). This condition generalizes Dynkin's theorem on convergence of W-functionals of a time homogeneous Markov process.

1. Introduction

Consider a sequence of processes $X_n = X_n(\cdot), n \geq 1$ that converge weakly, in an appropriate sense, to a process $X = X(\cdot)$. Consider also a family $\phi_n = \{\phi_n^{s,t}(X_n), s \leq t\}, n \geq 1$ of a functionals of the processes $X_n, n \geq 1$ and assume that the functionals are additive with respect to time variables. The general question, discussed in the present paper, is which information about the limit behavior of the distributions of functionals ϕ_n can be obtained in a situation where the processes X_n, X possess certain Markov properties. The starting point in our considerations is provided by the important particular case of the problem outlined above, in which all the processes X_n coincide with X. The well-known theorem by E.B. Dynkin ([1], Theorem 6.4) states that if X is a time homogeneous Markov process and ϕ_n are *W-functionals* of X (see [1], Chapter 6), then their limit behavior is completely determined by the limit behavior of their *characteristics* (that is, their expectations).

In the present paper, we consider the processes X_n that depend on n substantially. The class of a sequences $\{X_n\}$, considered in the framework of our approach, contains both sequences of Markov processes and sequences of Markov chains with appropriately re-scaled time, weakly convergent to Markov process X. An important partial case is provided by random broken lines corresponding to a random walk in $\mathbb{R}^d$ and weakly convergent to a time homogenous stable process X (particularly, to a Brownian motion).

We introduce a specific structural assumption on the sequence $\{X_n\}$ to provide *Markov approximation* for the process X. We show that, under this assumption, a full analogue of Dynkin's theorem holds: if the characteristics of a functional ϕ_n converge uniformly to the characteristic of a W-functional ϕ of the limit process X, then the distributions of ϕ_n converge weakly to the distribution of ϕ. Our method of proof is based on estimates for the L_2-distance between additive functionals, similar to those given in Lemma 6.5 [1]. These estimates are combined with the *coupling* technique, i.e. with a preliminary construction of processes X_n, X on one probability space in such a way that the functionals ϕ_n, ϕ, associated initially to different processes, are interpreted as functionals of a two-component process. The Markov property of the two-component process is essential for the estimates analogous to those given in Lemma 6.5 [1]; the assumption on the Markov approximation mentioned above is just the claim for such a property to hold true in an appropriate form.

2000 *Mathematics Subject Classification.* Primary 60J55; Secondary 60F17.
Key words and phrases. Additive functional, characteristic of additive functional, W-functional, local time, Markov approximation.

The method introduced in the present paper allows one to reduce the problem of studying the asymptotic behavior of the laws of additive functionals to an *a priori* simpler problem of studying their means. It provides a good addition to the available methods of investigation of the limit behavior of additive functionals both for the important particular case of random walks (we do not give the detailed overview here, referring the reader to monographs [3],[4],[5], papers [6],[7] and references therein) and for general Markov chains. As for the latter, it is necessary to mention the method based on the passing to the limit in the difference equations for characteristic functions of additive functionals of Markov chains, which ascends to the works by I.I. Gikhman in the 1950s ([8],[9], see also [10] and the survey paper [11]).

The structure of the article is following. In Section 2, we introduce the notion of Markov approximation and give examples that illustrate it. In Section 3, the main theorem of the article is introduced and proved. In Section 4, two examples of application of this theorem are given.

2. Markov approximation

We assume the processes X_n, X to be defined on $\mathbb{R}^+$ and to have a locally compact metric phase space $(\mathbb{X}, \rho)$. We say that the process X possesses the Markov property at the time moment $s \in \mathbb{R}^+$ w.r.t. a filtration $\{\mathcal{G}_t, t \in \mathbb{R}^+\}$, if X is adapted to this filtration and, for each $k \in \mathbb{N}, t_1, \dots, t_k > s$, there exists a stochastic kernel $\{P_{st_1\dots t_k}(x, A), x \in \mathbb{X}, A \in \mathcal{B}(\mathbb{X}^k)\}$ such that

$$E[\mathbf{I}_A((X(t_1), \dots, X(t_k)))|\mathcal{G}_s] = P_{st_1\dots t_k}(X(s), A) \quad \text{a.s.}, \quad A \in \mathcal{B}(\mathbb{X}^k). \tag{2.1}$$

The measure $P_{st_1\dots t_k}(x, \cdot)$ has a natural interpretation as the finite-dimensional distribution of X at the points $t_1, \dots, t_k$, conditioned by $\{X(s) = x\}$; in what follows, we denote $P_{st_1\dots t_k}(x, \cdot) = P((X(t_1), \dots, X(t_k)) \in \cdot | X(s) = x)$.

Remark 1. In some cases, (2.1) can be written in a stronger functional form

$$E[\mathbf{I}_C(X|_s^\infty)|\mathcal{G}_s] = E[\mathbf{I}_C(X|_s^\infty)|X(s)], \quad C \in \mathcal{C}, \tag{2.2}$$

where $X|_s^\infty$ denotes the trajectory of the process X on the time interval $[s, +\infty)$, considered as an element of an appropriate functional space, and $\mathcal{C}$ is some σ-algebra of subsets of this space. For instance, if the Kolmogorov's sufficient condition for the existence of a continuous modification holds true both for unconditional and conditional finite dimensional distributions of X, then (2.2) holds with $X|_s^\infty$ considered as an element of $C([s, +\infty), \mathbb{X})$.

We introduce a notational convention. Let a process X possess the Markov property w.r.t. its natural filtration at the point $s = \frac{i}{n}, i \in \mathbb{Z}_+$, and let ξ be a *cylindrical* functional of $X|_s^\infty$ (i.e., a random variable being a function of the finite set of the values of X at the time moments $t_1, \dots, t_k > s$). Then the expectation of ξ w.r.t. the family of conditional finite-dimensional distributions $\{P_{st_1\dots t_k}(x, \cdot), t_1, \dots, t_k > s, k \in \mathbb{N}, x \in \mathbb{X}\}$ is well defined. We denote this expectation by $E[\xi | X(s) = x], s \in \frac{1}{n}\mathbb{Z}_+,\ x \in \mathbb{X}$.

Further on, we assume that the process X possesses the Markov property w.r.t. its canonic filtration at every point $s \in \mathbb{R}^+$, and, for $n \in \mathbb{N}$, the process X_n possesses the same property at every point of the type $\frac{i}{n}, i \in \mathbb{Z}_+$. The choice of the denominator here is quite arbitrary; it is possible to put any expression $N(n) \to \infty, n \to \infty$ instead of n, but we avoid to do this in order to shorten the notation.

The next definition is introduced in [12].

Definition 1. The sequence $\{X_n\}$ provides the Markov approximation for the process X, if, for arbitrary $\gamma > 0, T < +\infty$, there exist $K(\gamma, T) \in \mathbb{N}$ and a sequence of two-component processes $\{\hat{Y}_n = (\hat{X}_n, \hat{X}^n)\}$ defined on another probability space such that

(i) $\hat{X}_n \stackrel{d}{=} X_n, \hat{X}^n \stackrel{d}{=} X$;

(ii) the processes $\hat{Y}_n$, $\hat{X}_n$, and $\hat{X}^n$ possess the Markov property at the points $\frac{iK(\gamma,T)}{n}$, $i \in \mathbb{N}$ w.r.t. the filtration $\{\hat{\mathcal{F}}_t^n = \sigma(\hat{Y}_n(s), s \le t)\}$;

(iii) $\limsup_{n\to+\infty} P\left(\sup_{i\le \frac{Tn}{K(\gamma,T)}} \rho\left(\hat{X}_n\left(\frac{iK(\gamma,T)}{n}\right), \hat{X}^n\left(\frac{iK(\gamma,T)}{n}\right)\right) > \gamma\right) < \gamma.$

The following examples illustrate Definition 1.

Example 1. Let $\{\xi_k\}$ be a sequence of i.i.d. random vectors in $\mathbb{R}^d$ with $E\|\xi_k\|_{\mathbb{R}^d}^{2+\delta} < +\infty$ for some $\delta > 0$. Assume $\{\xi_k\}$ to have zero mean and identity covariance matrix. Let the sequence of processes X_n ("random broken lines") on $\mathbb{R}^+$ be defined by

$$X_n(t) = \frac{S_{k-1}}{\sqrt{n}} + (nt-k+1)\left[\frac{S_k}{\sqrt{n}} - \frac{S_{k-1}}{\sqrt{n}}\right], \quad t \in \left[\frac{k-1}{n}, \frac{k}{n}\right), \quad k \in \mathbb{N}, \tag{2.3}$$

where $S_n = \sum_{k=1}^n \xi_k$. Then X_n converge by distribution in $C(\mathbb{R}^+, \mathbb{R}^d)$ to a Brownian motion X in $\mathbb{R}^d$.

It is shown in [12] that the sequence $\{X_n\}$ provides the Markov approximation for the process X (part I of Theorem 1 [12]). On the other hand, the following effect is revealed in the same paper (part II of the same Theorem). By $\mathbf{K}(\gamma, T)$, we denote the minimal constant $K(\gamma, T)$ such that there exists a process $\hat{Y}_n$ satisfying conditions (i)-(iii) of Definition 1. Then, in all the cases except one trivial case $\xi_k \sim \mathcal{N}(0, I)$, one has $\mathbf{K}(\gamma, T) \to +\infty$ as $\gamma \to 0+$ for every fixed $T > 0$. In other words, while the accuracy of the approximation of a Brownian motion X by the random walk X_n becomes better (this accuracy is described by the parameter γ), the Markov properties of the pair of processes (X, X_n) necessarily become worse (these properties are characterized by $\mathbf{K}(\gamma, T)$).

Example 2. Let $\{\xi_k\}$ be i.i.d. random variables that belong to the normal domain of attraction of an α-stable distribution $\mathcal{L}$, $\alpha \in (0, 2)$. By definition, this means that

$$n^{-\frac{1}{\alpha}}[S_n - a_n] \Rightarrow \mathcal{L}, \quad a_n = \begin{cases} 0, & \alpha \in (0,1) \\ nE\xi_1, & \alpha \in (1,2) \\ n^2 E \sin\frac{\xi_1}{n}, & \alpha = 1 \end{cases}$$

([13], Chapter XVII.5). Assume that $a_n \equiv 0$ and consider processes X_n of the type

$$X_n(t) = n^{-\frac{1}{\alpha}} S_{k-1} + (nt-k+1)\left[n^{-\frac{1}{\alpha}} S_k - n^{-\frac{1}{\alpha}} S_{k-1}\right], \quad t \in \left[\frac{k-1}{n}, \frac{k}{n}\right), \quad k \in \mathbb{N} \tag{2.4}$$

on $\mathbb{R}^+$. Then X_n weakly converge in $\mathbb{D}(\mathbb{R}^+)$ to the time homogeneous process with independent increments X in $\mathbb{R}$ with $X(1) - X(0) \stackrel{d}{=} \mathcal{L}$ (an *α-stable* process).

It is shown in [12] (Theorem 2) that the sequence $\{X_n\}$ provides the Markov approximation for the process X. Furthermore, in this situation, on the contrary to the previous example, $\mathbf{K}(\gamma, T) = 1$ for all γ, T. This means that, in this case, the Markov properties do not become worse while the accuracy of approximation improves.

Remark 2. The last example shows that the property of $\{X_n\}$ to possess the Markov approximation for X does not imply, in general, the weak convergence of the processes X_n to X in $\mathbb{C} = C(\mathbb{R}^+, \mathbb{X})$ even if $X_n, n \ge 1$ have continuous trajectories.

We remark that the notion of Markov approximation is closely related to Skorokhod's method of embedding of a random walk into a Wiener process by means of the appropriate sequence of stopping times ([14]), widely used in the literature. The basic idea is the same: to construct two processes on the same probability space, with the pair keeping Markov or martingale properties. However, Skorokhod's method, while being quite efficient for one-dimensional random walks that approximate the Wiener process, is much less appropriate in a multidimensional situation or for a stable domain of attraction. Examples 1 and 2 show that the claim for the Markov approximation to hold true is not restrictive, at least for all basic classes of random walks with no regard to the dimension of the phase space or to the type of the limit distribution.

Moreover, the following example shows that the property of the Markov approximation is "stable", in a sense. Namely, this property is preserved under construction of a new pair (Z_n, Z) from the pair (X_n, X), that possesses this property, in some regular way (e.g. as a solution of a family of stochastic equations).

Example 3. Consider X_n, X defined in Example 1. Let functions $a : \mathbb{R}^m \to \mathbb{R}^m, b : \mathbb{R}^m \to \mathbb{R}^{m\times d}$ be Lipschitz. Define

(2.5)
$$Z_n\left(\frac{k+1}{n}\right) = Z_n\left(\frac{k}{n}\right) + a\left(Z_n\left(\frac{k}{n}\right)\right)\frac{1}{n} + b\left(Z_n\left(\frac{k}{n}\right)\right)\Delta X_n\left(\frac{k}{n}\right), \quad Z_n(0) = z,$$

$\Delta X_n(\frac{k}{n}) \equiv [X_n(\frac{k+1}{n}) - X_n(\frac{k}{n})] = \frac{\xi_{k+1}}{\sqrt{n}}$. Then ([15], [16]) Z_n converge by distribution in $C(\mathbb{R}^+, \mathbb{R}^m)$ to the process Z defined by SDE

$$dZ(t) = a(Z(t))dt + b(Z(t))dX(t), \quad Z(0) = z \tag{2.6}$$

(recall that X is a Brownian motion in $\mathbb{R}^d$). It is natural to call the sequence $\{Z_n\}$ the *difference approximation* for the diffusion process Z.

Let us show that the sequence $\{Z_n\}$ provides the Markov approximation for the process Z. For arbitrary γ, T, we construct a pair $(\hat{X}_n, \hat{X}^n)$ that corresponds to processes X_n, X and satisfies conditions of Definition 1 (such a construction is possible, see Example 1 and reference therein).

We construct the processes $\hat{Z}_n, \hat{Z}^n$ as the functionals of the processes $\hat{X}_n, \hat{X}^n$ by equalities (2.5),(2.6) with X_n replaced by $\hat{X}_n$ and X replaced by $\hat{X}^n$ (we remark that (2.6) has unique strong solution, hence this procedure is correct). By construction, the pair $(\hat{Z}_n, \hat{Z}^n)$ satisfies condition (i) of Definition 1. It is easy to verify that the Markov condition (ii) for the pair $(\hat{X}_n, \hat{X}^n)$ holds in the functional form (2.2) with $\hat{Y}_n|_s^\infty$ considered as an element of $C([s, +\infty), \mathbb{R}^d\times\mathbb{R}^d)$ (see Remark 1). Hence, the pair $(\hat{Z}_n, \hat{Z}^n)$ also satisfies condition (ii) of Definition 1. We write

$$\Delta(\gamma) = \limsup_{n\to+\infty} P\left(\sup_{i\le \frac{Tn}{K(\gamma,T)}} \rho\left(\hat{Z}_n\left(\frac{iK(\gamma,T)}{n}\right), \hat{Z}^n\left(\frac{iK(\gamma,T)}{n}\right)\right) > \gamma\right)$$

and show that

$$\Delta(\gamma) \to 0+, \quad \gamma \to 0+. \tag{2.7}$$

Note that (2.7) immediately implies the Markov approximation property: for arbitrary $\delta > 0$, we chose, using (2.7), $\gamma = \gamma(\delta)$ such that the inequalities $\gamma < \delta$ and $\Delta(\gamma) < \delta$ hold. Then the pair $(\hat{Z}_n, \hat{Z}^n)$, constructed in the way described above, satisfy Definition 1 with the constant γ replaced by δ. Under this construction, the constant $K(\delta, T) \equiv K_Z(\delta, T)$ for the pair $(\hat{Z}_n, \hat{Z}^n)$ can be expressed through the analogous constant for the pair $(\hat{X}_n, \hat{X}^n)$ by the relation $K_Z(\delta, T) = K_X(\gamma(\delta), T)$.

Now assume that (2.7) does not hold. Then there exist constant $c > 0$ and sequences $\gamma_k \to 0+, n_k \to +\infty$ such that

(2.8)
$$\frac{K(\gamma_k, T)}{n_k} \to 0, \quad P\left(\sup_{i\le \frac{Tn_k}{K(\gamma_k,T)}} \rho\left(\hat{Z}_n\left(\frac{iK(\gamma_k,T)}{n_k}\right), \hat{Z}^n\left(\frac{iK(\gamma_k,T)}{n_k}\right)\right) > \gamma_k\right) > c.$$

Consider the sequence of four-component processes $(\hat X_{n_k}, \hat X^{n_k}, \hat Z_{n_k}, \hat Z^{n_k})$. Every component of this sequence is weakly compact in $C(\mathbb{R}^+,\mathbb{R}^d)$ or $C(\mathbb{R}^+,\mathbb{R}^m)$; hence, the whole sequence is weakly compact in $C(\mathbb{R}^+,\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^m\times\mathbb{R}^m)$. Consider an arbitrary weak limit point $(\hat X_*, \hat X^*, \hat Z_*, \hat Z^*)$ of this sequence. From (2.8), we have

$$P(Z_* \neq Z^*) > 0. \tag{2.9}$$

It follows from Theorem 2.2 [16] (see also Chapter 9.5 [15]) that the processes Z_* and Z^* satisfy SDE (2.6) with X replaced by X_* and X^*, respectively. However, SDE (2.6) possesses the pathwise uniqueness property (see [17]), and the property (iii) of the pair $(X_{n_k}, \hat X^{n_k})$ implies that the processes X_*, X^* coincide a.s. Therefore, the processes Z_*, Z^* also coincide a.s., which contradicts (2.9) and show that our assumption $\Delta(\gamma)\not\to 0+, \gamma\to 0+$ fails.

The examples given above show that the claim for the Markov approximation to hold is not too restrictive and is provided in a typical situations. On the other hand, this claim is strong enough to provide one the opportunity to obtain an analog of Dynkin's theorem; this will be shown in the next section.

3. The main theorem

We consider functionals of the type

(3.1)
$$\phi_n^{s,t}(X_n) \stackrel{\text{def}}{=} \sum_{k:s\le k/n<t} F_n\left(X_n\left(\frac{k}{n}\right), X_n\left(\frac{k+1}{n}\right),\dots, X_n\left(\frac{k+L-1}{n}\right)\right),\quad 0\le s<t,$$

where the functions $F_n(\cdot)$ are nonnegative, and L is a fixed integer. Together with the functionals ϕ_n that are "stepwise" functions w.r.t. every time variable, we consider the random broken lines related to these functions:

$$\psi_n^{s,t} = \phi_n^{\frac{j-1}{n},\frac{k-1}{n}} - (ns-j+1)\phi_n^{\frac{j-1}{n},\frac{j}{n}} + (nt-k+1)\phi_n^{\frac{k-1}{n},\frac{k}{n}},$$
$$s\in\left[\frac{j-1}{n},\frac{j}{n}\right), t\in\left[\frac{k-1}{n},\frac{k}{n}\right).$$

We interpret the random broken lines ψ_n as a random elements in the space $C(\mathbb{T},\mathbb{R}^+)$, where $\mathbb{T}\stackrel{\text{def}}{=}\{(s,t)|0\le s\le t\}$.

We recall (see [1], Chapter 6) that, for a time homogeneous Markov process X, a functional $\phi(X)=\{\phi^{s,t}(X), 0\le s\le t\}$ is called a W-functional if it is additive, nonnegative, continuous, and almost homogeneous and satisfy the moment condition

$$\sup_{x\in\mathbb{X}} E_x\phi^{0,t} < +\infty,\quad t\in\mathbb{R}^+.$$

Here and below, we use standard notation E_x for the expectation w.r.t. probability measure P_x from the definition of the Markov process (see [1], Chapter 3 §1). For a W-functional $\phi=\phi(X)$, its *characteristic* f_t is defined by the relation

$$f_t(x) = E_x\phi^{0,t}(X),\quad t>0, x\in\mathbb{X}. \tag{3.2}$$

We define the characteristic f_n of the functional ϕ_n analogously to (3.2):

$$f_n^{s,t}(x) \stackrel{\text{def}}{=} E[\phi_n^{s,t}(X_n)|X_n(s)=x],\quad s=\frac{i}{n}, i\in\mathbb{Z}_+, t>s, x\in\mathbb{X}. \tag{3.3}$$

We use here the notational convention introduced in the previous section: we recall that the process X_n is supposed to possess the Markov property at points of the type $s=\frac{i}{n}, i\in\mathbb{Z}_+$ and the functional $\phi_n^{s,t}$ is a cylindrical one. Therefore, (3.3) is formally correct. In order to adjust notation, we write $f^{s,t}\equiv f_{t-s}$ for the characteristic of a W-functional ϕ.

The main result of the paper is given in the following theorem.

Theorem 1. *Let the sequence X_n provide the Markov approximation for the time homogeneous Markov process X. Consider a sequence $\{\phi_n \equiv \phi_n(X_n)\}$ of functionals of the type (3.1). Let the following conditions hold true:*

(1) *The functions $F_n(\cdot)$ are non-negative, bounded, and tend to zero uniformly:*

$$\delta(F_n) \stackrel{\text{def}}{=} \sup_{x_1,\dots,x_L\in\mathbb{X}} F_n(x_1,\dots,x_L) \to 0, \quad n\to\infty.$$

(2) *There exists a W-functional $\phi = \phi(X)$ of the limiting Markov process X such that, for every T,*

$$\sup_{s=\frac{i}{n},t\in(s,T)} \left\| f_n^{s,t}(\cdot) - f^{s,t}(\cdot)\right\| \to 0, \quad n\to\infty,$$

where $\|g(\cdot)\| \stackrel{\text{def}}{=} \sup_{x\in\mathbb{X}}|g(x)|$ and f is the characteristic of the functional ϕ.

(3) *The characteristic f is uniformly continuous with respect to the variable x; i.e., for every T*

$$\sup_{0\le s\le t<T} \left|f^{s,t}(x') - f^{s,t}(x'')\right| \to 0, \quad \rho(x',x'')\to 0.$$

Then

$$\psi_n(X_n) \Rightarrow \phi(X) \equiv \{\phi^{s,t}(X),(s,t)\in\mathbb{T}\}$$

in $C(\mathbb{T},\mathbb{R}^+)$, where $\psi_n(X_n), n\ge 1$ are the random broken lines corresponding to the functionals $\phi_n(X_n), n\ge 1$.

Remark 3. Conditions 1 and 2 are analogous to those of Dynkin's theorem: condition 2 is exactly the condition for the characteristics to converge, condition 1 controls the sizes of the jumps of ϕ_n and corresponds to the condition of Dynkin's theorem for ϕ_n to be continuous w.r.t. time variables. Condition 3, though not very restrictive, is specific and is caused by the necessity to consider functionals defined on different processes.

Remark 4. If $X_n \Rightarrow X$ in $\mathbb{C}$ or in $\mathbb{D}$ (this condition is not provided by the conditions of the theorem, see Remark 2), then $(X_n,\psi_n(X_n)) \Rightarrow (X,\phi(X))$ in $\mathbb{C}\times C(\mathbb{T},\mathbb{R}^+)$ or in $\mathbb{D}\times C(\mathbb{T},\mathbb{R}^+)$, respectively. One can easily see this from the proof.

Proof of the theorem. Let us show first that the finite-dimensional distributions of ϕ_n converge to the corresponding distributions of ϕ. We fix constants γ, T and consider the processes $\widehat{X}_n, \widehat{X}^n$ satisfying conditions (i)-(iii) of Definition 1 with these constants. For these processes, we consider the functionals $\phi_n(\hat{X}_n), \phi(\hat{X}^n)$; obviously, their distributions and characteristics coincide with those for $\phi_n(X_n), \phi(X)$. In order to shorten the notation, we denote $\phi_n = \phi_n(\hat{X}_n), \phi = \phi(\hat{X}^n), K = K(\gamma,T), \mathcal{F}_t = \hat{\mathcal{F}}^n_t \equiv \sigma(\hat{X}_n(s),\hat{X}^n(s), s\le t)$.

Let us prove that, for an arbitrary $t\in\left(\frac{iK}{n},T\right]$,

$$E\left[\phi^{\frac{Ki}{n},t}|\mathcal{F}_{\frac{Ki}{n}}\right] = f^{\frac{Ki}{n},t}\left(\widehat{X}^n\left(\frac{Ki}{n}\right)\right), \quad E\left[\phi_n^{\frac{Ki}{n},t}|\mathcal{F}_{\frac{Ki}{n}}\right] = f_n^{\frac{Ki}{n},t}\left(\widehat{X}_n\left(\frac{Ki}{n}\right)\right) \tag{3.4}$$

almost surely.

Let $s\in\frac{1}{n}\mathbb{Z}_+, t_1,\dots,t_r > s$ be fixed, and let $G:\mathbb{X}^r\mapsto\mathbb{R}$ be a bounded measurable function. By $M_{s,\bar{t},n}:\mathbb{X}\to\mathbb{R}$, we denote such a measurable function that

$$E[G(X_n(t_1),\dots,X_n(t_r))|\sigma(X(r), r\in[0,s])] = M_{s,\bar{t},n}(X_n(s))$$

a.s. This function exists due to the Markov property of X_n. Denote also, by $\hat{M}_{s,\bar{t},n}:\mathbb{X}\to\mathbb{R}$, such a measurable function that

$$E[G(\hat{X}_n(t_1)\dots,\hat{X}_n(t_r))|\hat{\mathcal{F}}^n_s] = \hat{M}_{s,\bar{t},n}(\hat{X}_n(s))$$

a.s., this function exists due to condition (ii) of Definition 1. Denote, by $\nu_{s,n}$, the distribution of $X_n(s)$. It is equal to the distribution of $\hat{X}_n(s)$ by condition (i) of Definition 1. The same condition implies that, for every bounded measurable function $Q : \mathbb{X} \to \mathbb{R}$,

$$\int_{\mathbb{X}} M_{s,\bar{t},n}(y)Q(y)\nu_{s,n}(dy) = EM_{s,\bar{t},n}(X_n(s))Q(X_n(s)) =$$
$$= EG(X_n(t_1)\dots,X_n(t_r))Q(X_n(s)) = EG(\hat{X}_n(t_1)\dots,\hat{X}_n(t_r))Q(\hat{X}_n(s)) =$$
$$= E\hat{M}_{s,\bar{t},n}(\hat{X}_n(s))Q(\hat{X}_n(s)) = \int_{\mathbb{X}} \hat{M}_{s,\bar{t},n}(y)Q(y)\nu_{s,n}(dy).$$

Since Q is arbitrary, we conclude that $\hat{M}_{s,\bar{t},n} = M_{s,\bar{t},n}$ $\nu_{s,n}$-a.s., and therefore

$$E[G(X_n(t_1)\dots,X_n(t_r))|\hat{\mathcal{F}}^n_s] = M_{s,\bar{t},n}(\hat{X}_n(s)) \quad \text{a.s.} \tag{3.5}$$

We apply (3.5) to every summand in formula (3.1) and get the second equality in (3.4).

The W-functional ϕ can be represented as the mean square limit

$$\phi^{s,t} = \lim_{\operatorname{diam} S\to 0} \sum_{k=0}^{n-1} f^{s_k,s_{k+1}}(\hat{X}^n(s_k)),$$

where $S \overset{d}{=} \{s = s_0 < s_1 < \dots < s_n = t\}, \operatorname{diam} S \overset{d}{=} \max_{k=0,\dots,n-1}(s_{k+1} - s_k)$ ([1], Theorem 6.3). The arguments analogous to those made above, being combined with an appropriate limit procedure, yield the first equality in (3.4).

The main step in the proof of the theorem is given by the following lemma.

Lemma 1. *For $0 \le s \le t \le T$, the following estimate holds:*

$$\limsup_{n\to\infty} E\left(\phi^{s,t}_n(\widehat{X}_n) - \phi^{s,t}(\widehat{X})\right)^2 \le 4\left\|f^{0,T}\right\| G(f,\gamma,T) + 4\sqrt{2\gamma}\left\|f^{0,T}\right\|^2,$$

where $G(f,\gamma,T) = \sup_{0\le s\le t\le T, \rho(x',x'')<\gamma} |f^{s,t}(x') - f^{s,t}(x'')|.$

Proof. For a notational convenience, we prove the statement of the lemma for $s = 0, t = T$ only. For other values of s, t, the proof is exactly the same. Consider the partition of the axis $\mathbb{R}^+$ by points of the type $\frac{Ki}{n}, i \in \mathbb{N}$. Denote $I_n = [\frac{nT}{K}] + 1$,

$$\Delta^n_i = \phi_n^{\frac{(i-1)K}{n},(\frac{iK}{n})\wedge T}, \quad \widetilde{\Delta}^n_i = \phi^{\frac{(i-1)K}{n},\frac{iK}{n}\wedge T}, \quad i = 1,\dots,I_n.$$

We have

$$\left(\phi^{0,T}_n - \phi^{0,T}\right)^2 = \left(\sum_{i=1}^{I_n} \Delta^n_i - \widetilde{\Delta}^n_i\right)^2 =$$
$$= \left(\sum_{i=1}^{I_n} \Delta^n_i\right)^2 + \left(\sum_{i=1}^{I_n} \widetilde{\Delta}^n_i\right)^2 - 2\sum_{i=1}^{I_n}\sum_{j=1}^{I_n} \Delta^n_i\widetilde{\Delta}^n_j = \Sigma^n_1 + 2\Sigma^n_2,$$

where

$$\Sigma^n_1 \overset{\text{def}}{=} \sum_{i=1}^{I_n}(\Delta^n_i)^2 + \sum_{i=1}^{I_n}(\widetilde{\Delta}^n_i)^2 - 2\sum_{i=1}^{I_n}\Delta^n_i\widetilde{\Delta}^n_i,$$

$$\Sigma^n_2 \overset{\text{def}}{=} \left[\sum_{1\le i<l\le I_n}\Delta^n_i\Delta^n_l - \sum_{1\le i<j\le I_n}\Delta^n_i\widetilde{\Delta}^n_j\right] + \left[\sum_{1\le j<k\le I_n}\widetilde{\Delta}^n_j\widetilde{\Delta}^n_k - \sum_{1\le j<i\le I_n}\Delta^n_i\widetilde{\Delta}^n_j\right].$$

We estimate the expectations of Σ^n_1, Σ^n_2 separately. Since the increments $\Delta^n_i, \widetilde{\Delta}^n_i$ are non-negative, the first sum can be estimated by the sum of two first terms:

$$\Sigma^n_1 \le \sum_{i=1}^{I_n}(\Delta^n_i)^2 + \sum_{i=1}^{I_n}(\widetilde{\Delta}^n_i)^2. \tag{3.6}$$

The expectation of the first term in (3.6) can be estimated via the definition of ϕ_n:

$$E\sum_{i=1}^{I_n}(\Delta_i^n)^2 \le E\left(\sup_{i=1,I_n}\Delta_i\right)\sum_{i=1}^{I_n}\Delta_i^n \le K\delta_n f_n^{0,T}\left(\hat{X}_n(0)\right) \le K\delta_n\left\|f_n^{0,T}\right\| \to 0,\ n\to+\infty,$$

where $\delta_n \stackrel{\text{def}}{=} \delta(F_n)$. Convergence of the expectation of the second term in (3.6) to zero is provided by the arguments analogous to those used in Chapter 6 [1]: on the one hand, by continuity of the functional ϕ, $\sum_{i=1}^{I_n}(\widetilde{\Delta}_i^n)^2 \to 0$ by probability; on the other hand, $\sum_{i=1}^{I_n}(\widetilde{\Delta}_i^n)^2$ is dominated by the variable $(\phi^{0,T})^2$; the expectation of this variable, by Lemma 6.4 [1], does not exceed $2\left\|f^{0,T}\right\|^2 < \infty$. Therefore, $E\sum_{i=1}^{I_n}(\widetilde{\Delta}_i^n)^2 \to 0$ due to the Lebesgue theorem on dominated convergence. Hence, $\limsup_{n\to\infty} E\Sigma_1^n \le 0$.

The expectation of Σ_2^n is equal to

$$E\Sigma_2^n = E\left[\sum_{1\le i<l\le I_n}\Delta_i^n\Delta_l^n - \sum_{1\le i<j\le I_n}\Delta_i^n\widetilde{\Delta}_j^n\right] + E\left[\sum_{1\le j<k\le I_n}\widetilde{\Delta}_j^n\widetilde{\Delta}_k^n - \sum_{1\le j<i\le I_n}\Delta_i^n\widetilde{\Delta}_j^n\right]$$

$$(3.7)\qquad = E\sum_{i=1}^{I_n-1}\Delta_i^n\left[\phi_n^{\frac{Ki}{n},T} - \phi^{\frac{Ki}{n},T}\right] - E\sum_{i=1}^{I_n-1}\widetilde{\Delta}_i^n\left[\phi_n^{\frac{Ki}{n},T} - \phi^{\frac{Ki}{n},T}\right].$$

Let us estimate the second term in (3.7). The variable $\tilde{\Delta}_i^n$ is measurable w.r.t. $\mathcal{F}_{\frac{Ki}{n}}$; therefore, we can use (3.4) and get the estimate

$$-E\sum_{i=1}^{I_n-1}\widetilde{\Delta}_i^n\left[\phi_n^{\frac{Ki}{n},T} - \phi^{\frac{Ki}{n},T}\right] = -E\sum_{i=1}^{I_n-1}\widetilde{\Delta}_i^n E\left[\left(\phi_n^{\frac{Ki}{n},T} - \phi^{\frac{Ki}{n},T}\right)|\mathcal{F}_{\frac{Ki}{n}}\right] =$$

$$= E\sum_{i=1}^{I_n-1}\widetilde{\Delta}_i^n\left(f_n^{\frac{Ki}{n},T}\left(\hat{X}_n\left(\frac{Ki}{n}\right)\right) - f^{\frac{Ki}{n},T}\left(\hat{X}^n\left(\frac{Ki}{n}\right)\right)\right) \le$$

$$\le E\sum_{i=1}^{I_n-1}\widetilde{\Delta}_i^n\left|f_n^{\frac{Ki}{n},T}\left(\hat{X}_n\left(\frac{Ki}{n}\right)\right) - f^{\frac{Ki}{n},T}\left(\hat{X}_n\left(\frac{Ki}{n}\right)\right)\right| +$$

$$+E\sum_{i=1}^{I_n-1}\widetilde{\Delta}_i^n\left|f^{\frac{Ki}{n},T}\left(\hat{X}_n\left(\frac{Ki}{n}\right)\right) - f^{\frac{Ki}{n},T}\left(\hat{X}^n\left(\frac{Ki}{n}\right)\right)\right| \le$$

$$\le \|f^{0,T}\|\sup_{s=\frac{i}{n},t\in(s,T)}\left\|f_n^{s,t}(\cdot) - f^{s,t}(\cdot)\right\| +$$

$$(3.8)\qquad +E\sum_{i=1}^{I_n-1}\widetilde{\Delta}_i^n\left|f^{\frac{Ki}{n},T}\left(\hat{X}_n\left(\frac{Ki}{n}\right)\right) - f^{\frac{Ki}{n},T}\left(\hat{X}^n\left(\frac{Ki}{n}\right)\right)\right|.$$

In the last inequality, we have used that $\sum_{i=1}^{I_n-1}\widetilde{\Delta}_i^n \le \phi^{0,T}$ and $E\phi^{0,T} \le \|f^{0,T}\|$. The first term in (3.8) tends to zero. In order to estimate the second term, we denote

$$\Omega_{\gamma,T} = \left\{\sup_{i\le\frac{Tn}{K}}\rho\left(\hat{X}_n\left(\frac{iK}{n}\right),\hat{X}^n\left(\frac{iK}{n}\right)\right) > \gamma\right\};$$

we recall that $P(\Omega_{\gamma,T}) < \gamma$ due to claim (iii) of Definition 1. We have

$$E\sum_{i=1}^{I_n-1}\widetilde{\Delta}_i^n\left|f^{\frac{Ki}{n},T}\left(\hat{X}_n\left(\frac{Ki}{n}\right)\right) - f^{\frac{Ki}{n},T}\left(\hat{X}^n\left(\frac{Ki}{n}\right)\right)\right| \le E\phi^{0,T}G(f,\gamma,T)\mathbb{1}_{\Omega\setminus\Omega_{\gamma,T}} +$$

$$(3.9)\qquad +E\sum_{i=1}^{I_n-1}\widetilde{\Delta}_i^n\left|f^{\frac{Ki}{n},T}\left(\hat{X}_n\left(\frac{Ki}{n}\right)\right) - f^{\frac{Ki}{n},T}\left(\hat{X}^n\left(\frac{Ki}{n}\right)\right)\right|\mathbb{1}_{\Omega_{\gamma,T}}.$$

The first term in (3.9) can be estimated by $\|f^{0,T}\|G(f,\gamma,T)$. The second term is estimated by the Cauchy inequality:

$$E\sum_{i=1}^{I_n-1}\widetilde{\Delta}_i^n\left|f^{\frac{Ki}{n},T}\left(\hat{X}_n\left(\frac{Ki}{n}\right)\right)-f^{\frac{Ki}{n},T}\left(\hat{X}^n\left(\frac{Ki}{n}\right)\right)\right|\mathbb{I}_{\Omega_{\gamma,T}}\leq$$

$$\leq\left\|f^{0,T}\right\|E\phi^{0,T}\mathbb{I}_{\Omega_{\gamma,T}}\leq\left\|f^{0,T}\right\|\left[E(\phi^{0,T})^2\right]^{\frac{1}{2}}\left[P(\Omega_{\gamma,T})\right]^{\frac{1}{2}}\leq\left\|f^{0,T}\right\|^2\sqrt{2\gamma}.$$

Here, we have used Lemma 6.4 [1] in order to estimate $E(\phi^{0,T})^2$. Summing up the above relations, we deduce

$$\limsup_{n\to\infty}\left\{-E\sum_{i=1}^{I_n-1}\widetilde{\Delta}_i^n\left[\phi_n^{\frac{Ki}{n},T}-\phi^{\frac{Ki}{n},T}\right]\right\}\leq\left\|f^{0,T}\right\|G(f,\gamma,T)+\left\|f^{0,T}\right\|^2\sqrt{2\gamma}. \tag{3.10}$$

Let us proceed with the estimation of the first term in (3.7). Here, it is impossible to use (3.4) straightforwardly, since the variable Δ_i^n is a functional of the values of the process $\hat{X}_n$ at the points $\frac{Ki}{n}, \frac{Ki+1}{n}, \dots \frac{Ki+L}{n}$; that is, it is not measurable with respect to $\mathcal{F}_{\frac{Ki}{n}}$. Without loss of generality, we can assume that $K\geq L$ (otherwise, one can make the same procedure with the constant K replaced by $K\cdot L$). Then the variable Δ_i^n is measurable with respect to $\mathcal{F}_{\frac{K(i+1)}{n}}$. The functionals ϕ_n,ϕ are additive at points of the type $\frac{j}{n}, j\geq 1$. From (3.4) and condition 1 of the Theorem, we get the relation

$$E\sum_{i=1}^{I_n-1}\Delta_i^n\left[\phi_n^{\frac{Ki}{n},T}-\phi^{\frac{Ki}{n},T}\right]=E\sum_{i=1}^{I_n-1}\Delta_i^n\left[\phi_n^{\frac{Ki}{n},\frac{K(i+1)}{n}}-\phi^{\frac{Ki}{n},\frac{K(i+1)}{n}}\right]+$$

$$+E\sum_{i=1}^{I_n-1}\Delta_i^n\left[f^{\frac{K(i+1)}{n},T}\left(\hat{X}_n\left(\frac{K(i+1)}{n}\right)\right)-f^{\frac{K(i+1)}{n},T}\left(\hat{X}^n\left(\frac{K(i+1)}{n}\right)\right)\right]\leq$$

$$\leq K\delta_n\left|f_n^{0,T}\right| + E\sum_{i=1}^{I_n-1}\Delta_i^n\left[f^{\frac{K(i+1)}{n},T}\left(\hat{X}_n\left(\frac{K(i+1)}{n}\right)\right)-f^{\frac{K(i+1)}{n},T}\left(\hat{X}^n\left(\frac{K(i+1)}{n}\right)\right)\right]. \tag{3.11}$$

The first term in (3.11) tends to zero. The second term in (3.11) can be estimated in the same way as the second term in (3.7) with one necessary change. We cannot apply Lemma 6.4 [1] in order to estimate the second moment of $\phi_n^{0,T}$; therefore, this estimate must be obtained separately. This can be done in a following way:

$$E(\phi_n^{0,T})^2=E\sum_{i=1}^{I_n}(\Delta_i^n)^2+2E\sum_{1\leq i<j\leq I_n}\Delta_i^n\Delta_j^n=E\sum_{i=1}^{I_n}(\Delta_i^n)^2+2E\sum_{1\leq i\leq I_n}\Delta_i^n\phi_n^{\frac{iK}{n},T}=$$

$$=E\sum_{i=1}^{I_n}(\Delta_i^n)^2+2E\sum_{1\leq i\leq I_n}\Delta_i^n[\phi_n^{iK/n,(i+1)K/n}+\phi_n^{(i+1)K/n,T}]\leq$$

$$=E\sum_{i=1}^{I_n}(\Delta_i^n)^2+2K\delta_nE\sum_{1\leq i\leq I_n}\Delta_i^n+2E\sum_{1\leq i\leq I_n}\Delta_i^n\left\|f_n^{0,T}\right\|\leq$$

$$\leq\left\{(2K+1)\delta_n+2\|f_n^{0,T}\|\right\}E\phi_n^{0,T}\leq(2K+1)\delta_n\left\|f_n^{0,T}\right\|+2\left\|f_n^{0,T}\right\|^2, \tag{3.12}$$

all transitions here are analogous to those introduced above and thus are not discussed in details. Repeating literally the estimates for the second term in (3.7), we obtain the estimate

$$\limsup_{n\to\infty} E\sum_{i=1}^{I_n-1}\Delta_i^n\left[\phi_n^{\frac{Ki}{n},T}-\phi^{\frac{Ki}{n},T}\right]\leq \left\|f^{0,T}\right\|G(f,\gamma,T)+\left\|f^{0,T}\right\|^2\sqrt{2\gamma}. \tag{3.13}$$

It follows from (3.10),(3.13) that $\limsup_{n\to\infty}[2\Sigma_2^n]\leq 4\left\|f^{0,T}\right\|G(f,\gamma,T)+4\sqrt{2\gamma}\left\|f^{0,T}\right\|^2$. This, combined with the estimate $\limsup_{n\to\infty}[\Sigma_1^n]\leq 0$ proved before, provides the required statement. The lemma is proved.

Now, we can complete the proof of the convergence of the finite-dimensional distributions of ϕ_n to those of ϕ. In order to shorten notation, we consider one-dimensional distributions only. In the general case, considerations are completely the same.

Take arbitrary $s,t,s<t$. In order to prove the weak convergence of $\phi_n^{s,t}(X_n)$ to $\phi^{s,t}(X)$, it is sufficient to show that, for an arbitrary bounded Lipschitz function g,

$$\limsup_{n\to\infty}\left|Eg(\phi_n^{s,t}(X_n))-Eg(\phi^{s,t}(X))\right|=0. \tag{3.14}$$

Let g be fixed. Consider a pair of processes $\hat X_n,\hat X^n$, corresponding (in a sense of Definition 1) to $T=t$ and a given positive γ. By construction,

$$\phi_n^{s,t}(X_n)\stackrel{d}{=}\phi_n^{s,t}(\hat X_n),\phi^{s,t}(X)\stackrel{d}{=}\phi^{s,t}(\hat X^n).$$

From Lemma 1, we get

$$\limsup_{n\to\infty}\left|Eg(\phi_n^{s,t}(X_n))-Eg(\phi^{s,t}(X))\right|\leq\limsup_{n\to\infty}E\left|g(\phi_n^{s,t}(\hat X_n))-\phi^{s,t}(\hat X^n)\right|\leq$$

$$\leq \mathrm{Lip}(g)\limsup_{n\to\infty}E\left|\phi_n^{s,t}\left(\hat X_n\right)-\phi^{s,t}\left(\hat X^n\right)\right|\leq 2\mathrm{Lip}(g)\sqrt{\|f^{0,t}\|G(f,\gamma,t)+\sqrt{2\gamma}\left\|f^{0,t}\right\|^2},$$

where $\mathrm{Lip}(g)$ denotes the Lipschitz constant for g. Condition 3 of the Theorem provides that $G(f,\gamma,t)\to 0$ as $\gamma\to 0+$. Since $\gamma>0$ is arbitrary, the above-given estimate yields (3.14).

Since $\sup_{s,t}|\psi_n^{s,t}-\phi_n^{s,t}|\leq\delta_n\to 0$, all finite-dimensional distributions of ψ_n converge to the corresponding distributions of ϕ. Thus, the only thing left to show in the proof of the Theorem is that the family of distributions of ψ_n is dense in $C(\mathbb{T},\mathbb{R}^+)$. By the construction of ψ_n,

$$\psi_n^{s,t}=\psi_n^{0,t}-\psi_n^{0,s},\quad 0\leq s\leq t,$$

and thus it is enough to verify that the distributions of $\{\psi_n^{0,\cdot}\}$ are tight in $C(\mathbb{R}^+,\mathbb{R})$.

The value of the function $\psi_n^{0,\cdot}$ at the point t differs from the value of this function at the closest knot $t_*\in\frac{1}{n}\mathbb{Z}_+$ at most on δ_n, and ψ_n is monotone. Hence, in order to prove the required statement, it is sufficient to show that, for an arbitrary sequence of partitions $\left\{S_n=\{s_0^n=0<s_1^n<\dots<s_k^n<\dots\}\subset\frac{1}{n}\mathbb{Z}_+,n\in\mathbb{N}\right\}$ with $\sigma_n\equiv\max_k(s_k^n-s_{k-1}^n)\to 0,n\to+\infty$ and arbitrary $T\in\mathbb{R}^+$,

$$E\sum_{k:s_k^n\leq T}\left[\psi_n^{s_{k-1}^n,s_k^n}\right]^2\to 0,\quad n\to+\infty.$$

We set $\gamma_{n,T}=\sup_{0<t-s<\sigma_n,t<T}\|f_n^{s,t}\|$ and remark that $\gamma_{n,T}\to 0,n\to+\infty$ due to the continuity of the limit characteristic f and the uniform convergence $f_n\rightrightarrows f$. In the same way as in (3.12), we obtain the estimate

$$E\left[\phi_n^{s_{k-1}^n,s_k^n}\right]^2\leq\{(2K+1)\delta_n+2\gamma_{n,T}\}E\phi_n^{s_{k-1}^n,s_k^n}. \tag{3.15}$$

Summing up the estimates (3.15) w.r.t. k, we get

$$E\sum_{k:s_k^n\leq T}\left[\psi_n^{s_{k-1}^n,s_k^n}\right]^2\leq\{(2K+1)\delta_n+2\gamma_{n,T}\}\|f_n^{0,T}\|\to 0,\quad n\to+\infty$$

(recall that $\phi_n^{s,t} = \psi_n^{s,t}$ when $s, t \in \frac{1}{n}\mathbb{Z}_+$). The Theorem is proved.

4. Applications

4.1. **Local time of a random walk.** Consider a sequence $\{\xi_k\}$ of i.i.d. random variables with the common law that belongs to the normal domain of attraction of an α-stable law, $\alpha \in (1, 2]$. We assume that $E\xi_1 = 0$ and define the random broken lines X_n by equality (2.4) (see Examples 1 and 2).

Consider, for a fixed $z_* \in \mathbb{R}$, functionals $\phi_n = \phi_n(X_n)$ of the type (3.1) with $L = 2$,

$$F_n(x, y) = \frac{1}{n}\frac{1}{|y - x|}\left[\mathbf{1}_{(x-z_*)(y-z_*)<0} + \frac{1}{2}(\mathbf{1}_{x\neq z_*, y=z_*} + \mathbf{1}_{x=z_*, y\neq z_*})\right].$$

One can verify straightforwardly that, for every $s < t, s, t \in \{\frac{j}{n}, j \in \mathbb{Z}_+\}$,

$$\phi_n^{s,t}(X_n) = \lim_{\varepsilon\to 0+}\frac{1}{2\varepsilon}\int_s^t \mathbf{1}_{X_n(r)\in(z_*-\varepsilon, z_*+\varepsilon)\setminus\{z_*\}}\, dr \tag{4.1}$$

almost surely. Therefore, the functionals ϕ_n can be naturally interpreted as the *censored* local times for the broken lines X_n at the point z_*. Here, the operation of censoring consists in the removal of horizontal parts of the broken lines.

Theorem 1 yields the following statement.

Proposition 1. *Let the distribution of the jump ξ_1 of the random walk be concentrated on $\mathbb{Z}$ and aperiodic. Then $\phi_n^{s,t}(X_n)$ converge by distribution to*

$$\phi^{s,t}(X) = P(\xi_1 \neq 0) \cdot L^{s,t}(X, z_*),$$

where $L(X, z_*)$ is the local time of the limit α-stable process X at the point z_*.

Proof. Condition on $\{X_n\}$ to provide the Markov approximation for X holds true (see Examples 1 and 2 and references there). By construction, if the increment of the process X_n at a pair of neighboring knots $s = \frac{i}{n}, t = \frac{i+1}{n}$ is non-zero, then the absolute value of this increment is at least $n^{-\frac{1}{\alpha}}$. Therefore, condition 1 of Theorem 1 holds with $\delta_n = 2n^{\frac{1}{\alpha}-1}$.

Let us show that the characteristics f_n of the functionals ϕ_n converge uniformly to the function

$$f^{s,t}(x) = P(\xi_1 \neq 0)\int_0^{t-s} p_r(z_* - x)\, dr, \tag{4.2}$$

where $p_r(\cdot)$ is the density of the distribution $X(r)$ under condition $X(0) = 0$. This will provide condition 2 of Theorem 1, since the function

$$f_t(x) = \int_0^t p_r(z_* - x)\, dr$$

is the characteristic of the local time $L(X, z_*)$ of the α-stable process X ([10]).

In order to shorten notation, we take $z_* = 0$. This obviously does not restrict generality. Denote $P_i^k = P(S_k = i), P_j = P_j^1 = P(\xi_1 = j), i, j \in \mathbb{Z}$. We have

$$f_n^{s,t}(x) = n^{\frac{1}{\alpha}-1}\sum_{\frac{k}{n}<t-s}\left[\sum_{j\neq 0}\frac{P_j}{|j|}\left(\sum_{i\in(xn^{\frac{1}{\alpha}}-j, xn^{\frac{1}{\alpha}})} P_i^k + \frac{1}{2}\mathbf{1}_{xn^{\frac{1}{\alpha}}\in\mathbb{Z}}\left(P^k_{xn^{\frac{1}{\alpha}}} + P^k_{xn^{\frac{1}{\alpha}}-j}\right)\right)\right].$$

Here and below, the notation $i \in (a, b)$ in the case $a > b$ means that $b < i < a$. Gnedenko's local limit theorem for lattice random variables (see [18], Theorem 4.2.1) states that

$$\varepsilon_k \stackrel{\text{def}}{=} \sup_{i\in\mathbb{Z}}\left|k^{\frac{1}{\alpha}}P_i^k - p_1\left(\frac{i}{k^{\frac{1}{\alpha}}}\right)\right| \to 0, \quad k \to +\infty. \tag{4.3}$$

Hence, for $0 \le s \le t \le T$,

$$f_n^{s,t}(x) = \frac{1}{n} \sum_{\frac{k}{n} < t-s} \left[\sum_{j \neq 0} \frac{P_j}{|j|} \Bigg(\sum_{i \in (xn^{\frac{1}{\alpha}} - j, xn^{\frac{1}{\alpha}})} \left(\frac{n}{k}\right)^{\frac{1}{\alpha}} p_1 \left(\frac{i}{k^{\frac{1}{\alpha}}} \right) + \right.$$

$$\left. + \frac{n^{\frac{1}{\alpha}}}{2k^{\frac{1}{\alpha}}} \mathbb{1}_{xn^{\frac{1}{\alpha}} \in \mathbb{Z}} \left\{ p_1 \left(\frac{xn^{\frac{1}{\alpha}}}{k^{\frac{1}{\alpha}}} \right) + p_1 \left(\frac{xn^{\frac{1}{\alpha}} - j}{k^{\frac{1}{\alpha}}} \right) \right\} \Bigg) \right] + \Xi_n^T(x), \tag{4.4}$$

where

$$|\Xi_n^T(x)| \le \frac{1}{n} \sum_{k=1}^{[nt]} \left(\frac{n}{k}\right)^{\frac{1}{\alpha}} \varepsilon_k, \quad x \in \mathbb{R}. \tag{4.5}$$

The latter estimate and Toeplitz's theorem provide that $\Xi_n^T \rightrightarrows 0, n \to +\infty$.

The density p_1 is uniformly continuous over $\mathbb{R}$. Hence, using similar estimates, one can show that, up to a summand that uniformly converges to zero, the value of $f^{s,t}(x)$ equals

$$\frac{1}{n} \sum_{\frac{k}{n} < t-s} \left[\sum_{j \neq 0} \frac{P_j}{|j|} \left(\sum_{i \in (xn^{\frac{1}{\alpha}} - j, xn^{\frac{1}{\alpha}})} \left(\frac{n}{k}\right)^{\frac{1}{\alpha}} p_1 \left(\frac{xn^{\frac{1}{\alpha}}}{k^{\frac{1}{\alpha}}} \right) + \left(\frac{n}{k}\right)^{\frac{1}{\alpha}} \mathbb{1}_{xn^{\frac{1}{\alpha}} \in \mathbb{Z}} p_1 \left(\frac{xn^{\frac{1}{\alpha}}}{k^{\frac{1}{\alpha}}} \right) \right) \right].$$

The interval $(xn^{\frac{1}{\alpha}} - j, xn^{\frac{1}{\alpha}})$ contains $|j|$ integer points if $xn^{\frac{1}{\alpha}} \notin \mathbb{Z}$, and $|j| - 1$ integer points otherwise. Therefore, up to a summand that uniformly converges to zero, the value of $f^{s,t}(x)$ equals

$$\frac{1}{n} \sum_{\frac{k}{n} < t-s} \left[\sum_{j \neq 0} \frac{P_j}{|j|} \cdot |j| \cdot p_1 \left(\frac{xn^{\frac{1}{\alpha}}}{k^{\frac{1}{\alpha}}} \right) \right] =$$

$$= \frac{P(\xi_1 \neq 0)}{n} \sum_{\frac{k}{n} < t-s} \left(\frac{n}{k}\right)^{\frac{1}{\alpha}} p_1 \left(xn^{\frac{1}{k^{\frac{1}{\alpha}}}} \right) = \frac{P(\xi_1 \neq 0)}{n} \sum_{\frac{k}{n} < t-s} p_{\frac{k}{n}}(x). \tag{4.6}$$

Here, we have used that the process X is self-similar, that is, $p_r(x) = r^{-\frac{1}{\alpha}} p_1(r^{-\frac{1}{\alpha}} x), r > 0$. The sum on the right-hand side of (4.6) is exactly the integral sum for the integral on the right-hand side of (4.2), the functions $\{p_r(\cdot), r \ge r_0\}$ are uniformly continuous for arbitrary $r_0 > 0$ and $\sup_x p_r(x) \le Cr^{-\frac{1}{\alpha}}$. This immediately provides the required uniform convergence of f_n to f.

Condition 3 of Theorem 1 holds trivially due to the above-mentioned properties of the densities $p_r(\cdot), r \ge 0$.

We have verified all conditions of Theorem 1. From this theorem, we get the required statement. The proposition is proved.

Condition $\alpha \in (1, 2]$ is essential in our estimation: if $\alpha \le 1$, inequality (4.5) does not imply $\Xi_n^T \rightrightarrows 0$. This condition is precise and cannot be weakened: it is well known that an α-stable process possesses the local time at the point 0 if, and only if, $\alpha > 1$ ([10]).

The result similar to the one of Proposition 1 can be proved for essentially non-lattice random walks, i.e. when there exists n_0 such that S_{n_0} possesses a bounded distribution density. The statement of Proposition 1 and its analog for essentially non-lattice random walks are not essentially new. One can obtain them by applying first Theorem 1, Chapter 5.3 [3] and then either the technique exposed in §§III.2, III.3 [4] or the reasonings similar to those used in the proof of Theorem 3 [10]. Our reason to expose this example here is two-fold. On the one hand, we would like to emphasize the following interesting fact which has not been mentioned in the references available to us. For a "good" random walks (either lattice or essentially non-lattice), their local times at the point defined by the natural relation (4.1) converge by distribution exactly to the local time of the limit process at the same point, as soon as the broken lines corresponding to the random walk do not contain horizontal segments. On the other hand, this example illustrates, in the simplest situation, the way condition 2 of Theorem 1 can be provided via a local limit theorem.

Local limit theorems appear to be a powerful tool that allows one to provide the convergence of characteristics of additive functionals in various quite complicated cases. This allows one to apply Theorem 1 for a wide variety of sequences $\{X_n\}$, including the difference approximations for the multidimensional diffusion and random broken lines convergent to an α-stable process with $\alpha \le 1$. We do not give a detailed discussion of these topics here, referring the reader to papers [19],[20] and Section 2.2 of paper [21].

4.2. **Difference approximations for the local time of one-dimensional diffusion.** In this subsection, we demonstrate one more trick that, in some cases, allows one to prove the convergence of characteristics of additive functionals of the type (3.1) under a mild conditions on the sequence $\{X_n\}$. Let us begin our consideration from the case where $X_n, n \ge 1$ are the random broken lines generated by Bernoulli's random walk, i.e. X_n are defined by (2.3) with $P(\xi_k = \pm 1) = \frac{1}{2}$. It is well known (see [2]) that the functionals

$$\phi_n^{s,t} = \frac{1}{\sqrt{n}} \sum_{k \in [sn, tn)} \mathbf{I}_{X_n(\frac{k}{n})=0} \tag{4.7}$$

(the normalized numbers of visits of X_n to 0) converge weakly to the local time $L_W^{s,t}$ of the Wiener process at the point 0.

Functional (4.7) is closely related to the Doob decomposition for the sequence

$$\left| X_n \left(\frac{k}{n} \right) \right|, \qquad k = 0, 1, \dots.$$

Namely (e.g. [22], §IV.6), the latter sequence has representation in the form

$$\left| X_n \left(\frac{k}{n} \right) \right| = M_k^n + \phi_n^{0, \frac{k}{n}}, \quad k = 1, 2, \dots,$$

where $\{M_k^n, k \ge 1\}$ is a martingale w.r.t. a filtration $\{\mathcal{F}_k = \sigma(\xi_j, j \le k), k \ge 1\}$. This, together with the Donsker invariance principle, implies that the characteristics of the functionals ϕ_n converge uniformly to the function

$$f^{s,t}(x) = E|W(t-s) + x| - |x| \tag{4.8}$$

(the proof is quite straightforward and omitted; a more general statement will be proved in Proposition 2 below). But the Itô–Tanaka formula

$$|W(t) + x| = |x| + \int_0^t \operatorname{sign}(W(r) + x)\, dW(r) + L_{W+x}^{0,t}$$

provides that function (4.8) is the characteristic for the local time L_W, the latter being considered as a W-functional of the Wiener process W. Therefore, condition 2 of Theorem 1 holds true with $\phi = L_W$. All the other conditions of this theorem can be verified (almost) trivially. Thus, Theorem 1 gives an alternative way to prove the (well-known) fact that the normalized number of zeroes for Bernoulli's random walk converges weakly to the local time of the Wiener process.

In the previous consideration, it was crucial for X_n to be generated by a random walk and for the step ξ_k of the random walk to have the Bernoulli distribution. However, we will demonstrate below that *the same*, in essence, arguments allow one to prove the weak convergence for certain "canonic" additive functionals under very mild restrictions on the processes $\{X_n\}$.

Consider the sequence $\{Z_n\}$ of difference approximations of the one-dimensional diffusion process Z (Example 3 with $m = d = 1$, equalities (2.5),(2.6)). Suppose that $|a(x)| \le R, R^{-1} \le b^2(x) \le R, x \in \mathbb{R}$ for some $R > 1$. Under this condition, the standard estimates for the transition density $p_t(x,y)$ of the diffusion Z (e.g., [1], Appendix §6) yield $\sup_{x,y} p_t(x,y) \le \frac{C_T}{\sqrt{t}}, t \in (0,T]$ for every $T > 0$, and $p_t(x,y)$ is uniformly continuous on the set $[t_0,\infty) \times \mathbb{R} \times \mathbb{R}$ for every $t_0 > 0$. This implies that the function

$$f^{0,t}(x) = b^2(0) \int_0^t p_s(x,0)\, ds$$

is well defined, and

$$f_\varepsilon^{0,t}(x) \stackrel{\text{def}}{=} \frac{1}{2\varepsilon} \int_0^t \int_{-\varepsilon}^{\varepsilon} p_s(x,y) b(y)\, dy\, ds \to f^{0,t}(x)$$

uniformly by $x \in \mathbb{R}, t \le T$ for every T. The function f_ε is the characteristic for the W-functional

$$\phi_\varepsilon^{s,t} = \int_s^t \mathbf{1}_{|Z(r)|<\varepsilon} b^2(Z(r))\, dr.$$

Hence, Dynkin's theorem (Theorem 6.4 [1]) implies that there exists a mean square limit

$$\phi^{s,t} = \lim_{\varepsilon\to 0+} \frac{1}{2\varepsilon} \int_s^t \mathbf{1}_{|Z(r)|<\varepsilon} b^2(Z(r))\, dr, \quad (s,t) \in \mathbb{T}. \tag{4.9}$$

The functional ϕ is called *the local time* of the diffusion Z at the point 0.

We recall that there exists another standard way to define the local time L_Z of Z, based on the Itô–Tanaka formula

$$|Z(t)| - |Z(s)| = \int_s^t \operatorname{sign}(Z(r))\, dZ(r) + L_Z^{s,t}$$

(see [23], §IV.1). In the case under consideration, these two definitions are adjusted in the sense that $\phi^{s,t} = L_Z^{s,t}$.

Put

$$\phi_n^{s,t}(Z_n) \equiv \sum_{k\in[sn,tn)} \left| Z_n\left(\frac{k}{n}\right)\right| \cdot \left[2\mathbf{1}_{Z_n(\frac{k-1}{n})Z_n(\frac{k}{n})<0} + \mathbf{1}_{Z_n(\frac{k-1}{n})=0}\right], \tag{4.10}$$

and denote, by ψ_n, the random broken lines corresponding to ϕ_n.

Proposition 2. *Suppose the functions a, b be globally Lipschitz and satisfy the condition*

$$|a(x)| \le R, \quad R^{-1} \le b^2(x) \le R, \quad x \in \mathbb{R}$$

for some $R > 1$. Suppose also that the noise in the difference relation (2.5) that defines Z_n satisfies the moment condition

$$E|\xi_1|^{3+\delta} < +\infty$$

for some $\delta > 0$.

Then the processes ψ_n converge by distribution in $C(\mathbb{T},\mathbb{R})$ to the local time L_Z of the diffusion process Z at the point 0.

Remark 5. Functional (4.10) is a straightforward generalization of functional (4.7): if Z_n are the broken lines corresponding to Bernoulli's random walk and $Z_n(0) \in \frac{1}{\sqrt{n}}\mathbb{Z}$, then formula (4.10) defines exactly the same functional as (4.7).

Remark 6. Structure of functional (4.10) is quite simple and transparent: the term $\mathbf{1}_{Z_n(\frac{k}{n})=0}$ corresponds to the number of visits of Z_n to zero at the points of the partition $\frac{1}{n}\mathbb{Z}_+$, the term $\mathbf{1}_{Z_n(\frac{k-1}{n})Z_n(\frac{k}{n})<0}$ corresponds to the number of sign changes for Z_n at the neighbouring points of this partition. Thus, functional (4.10) can be treated as a properly normalized mixture of the number of visits of Z_n to zero and the number of sign changes for Z_n. It looks not very typical that the normalizing coefficients $\left|Z_n\left(\frac{k}{n}\right)\right|$ are non-constant and depend process Z_n. However, exactly the functional of the type (4.10) appears to be "canonic" and to converge to the local time of the limit process Z without any specific assumptions on the approximating sequence $\{Z_n\}$. For instance, in the case $a \equiv 0, b \equiv 1$ (i.e., where Z_n corresponds to a random walk), Proposition 2 does not require any structural conditions on the law of ξ_k. In particular, the local limit theorem is not involved in any form.

Proof of Proposition 2. The functionals ϕ_n can fail to satisfy condition 1 of Theorem 1, since the random variables $Z_n\left(\frac{k}{n}\right)$, in general, have no means to be bounded by a constant. Therefore, we cannot apply Theorem 1 to the functionals $\{\phi_n\}$ straightforwardly. We consider an auxiliary sequence of functionals

$$\theta_n^{s,t}(Z_n) \equiv \sum_{k\in[sn,tn)} \left|Z_n\left(\frac{k}{n}\right)\right| \cdot \left[2\mathbb{1}_{Z_n\left(\frac{k-1}{n}\right)Z_n\left(\frac{k}{n}\right)<0} + \mathbb{1}_{Z_n\left(\frac{k-1}{n}\right)=0}\right]\mathbb{1}_{|Z_n(\frac{k}{n})-Z_n(\frac{k-1}{n})|\le n^{\gamma-\frac{1}{2}}}$$

with a fixed $\gamma \in \left(\frac{1}{2+\delta}, \frac{1}{2}\right)$. The increments of Z_n have the form

$$Z_n\left(\frac{k}{n}\right) - Z_n\left(\frac{k-1}{n}\right) = a\left(Z_n\left(\frac{k-1}{n}\right)\right)\frac{1}{n} + b\left(Z_n\left(\frac{k-1}{n}\right)\right)\frac{\xi_k}{\sqrt{n}},$$

and the functions a, b are bounded. Therefore, there exists a constant $A > 0$ such that, for n large enough, the inequality

$$\left|Z_n\left(\frac{k}{n}\right) - Z_n\left(\frac{k-1}{n}\right)\right| > n^{\gamma-\frac{1}{2}}$$

yields the inequality $|\xi_k| > An^{\gamma}$. Then, keeping in mind that $\gamma(3+\delta) > \gamma(2+\delta) > 1$, we get

$$P(\exists\, s \le t \le T : \phi_n^{s,t} \ne \theta_n^{s,t}) \le P\left(\bigcup_{k\le Tn+1}\{|\xi_k| > An^{\gamma}\}\right) \le (Tn+2)A^{-3-\delta}n^{-\gamma(3+\delta)} \to 0, \tag{4.11}$$

$n \to \infty$. Therefore, the asymptotic behavior for the functionals ϕ_n as $n \to \infty$ is the same as that for the functionals θ_n.

If either $Z_n\left(\frac{k-1}{n}\right) = 0$ or $Z_n\left(\frac{k-1}{n}\right)Z_n\left(\frac{k}{n}\right) < 0$, then

$$\left|Z_n\left(\frac{k}{n}\right)\right| \le \left|Z_n\left(\frac{k}{n}\right) - Z_n\left(\frac{k-1}{n}\right)\right|.$$

Therefore, the functionals θ_n satisfy condition 1 of Theorem 1 with $\delta_n = 2n^{\gamma-\frac{1}{2}}$. Let us verify that these functionals satisfy also condition 2 of Theorem 1.

Denote the characteristic of ϕ_n by f_n and the characteristic of θ_n by g_n. One can see that there exists a constant $B > 0$ such that

$$\left|Z_n\left(\frac{k}{n}\right) - Z_n\left(\frac{k-1}{n}\right)\right| \cdot \mathbb{1}_{|Z_n(\frac{k}{n})-Z_n(\frac{k-1}{n})|\le n^{\gamma-\frac{1}{2}}} \le B|\xi_k|\mathbb{1}_{|\xi_k|>An^{\gamma}}.$$

Then, analogously to (4.11) for $s \le t \le T, x \in \mathbb{R}$, we have

$$|f_n^{s,t}(x) - g_n^{s,t}(x)| \le d_n(T) = 2B\sum_{k=0}^{nT+2} E|\xi_k|\mathbb{1}_{|\xi_k|>An^{\gamma}} \le$$

$$\le 2BA^{-2-\delta}(nT+2)n^{-\gamma(2+\delta)} \to 0, \quad n \to \infty. \tag{4.12}$$

The following relation can be verified straightforwardly:

$$|y| - |x| = (y-x)\operatorname{sign} x + |y|\Big[2\mathbb{1}_{xy<0} + \mathbb{1}_{x=0}\Big], \quad x, y \in \mathbb{R}$$

(the function sign$(\cdot)$, by convention, takes zero value at the point 0). Therefore, for $s,t\in \frac{1}{n}\mathbb{Z}_+$,

$$|Z_n(t)|-|Z_n(s)|=\phi_n^{0,t}(Z_n)+$$

$$+\sum_{k=ns}^{nt-1}\left[a\left(Z_n\left(\frac{k}{n}\right)\right)\frac{1}{n}+b\left(Z_n\left(\frac{k}{n}\right)\right)\frac{\xi_{k+1}}{\sqrt{n}}\right]\text{sign}\left(Z_n\left(\frac{k}{n}\right)\right). \tag{4.13}$$

Since, for every $k\geq 1$, the variable ξ_{k+1} is centered and is with $Z_n\left(\frac{k}{n}\right)$, formula (4.13) yields the following representation for the characteristic of the functional ϕ_n:

$$f_n^{s,t}(x)=E\left[\left|Z_n\left(\frac{[nt]}{t}\right)\right|\Big|Z_n(s)=x\right]-|x|-$$

$$-\frac{1}{n}E\left[\sum_{k=ns}^{[nt]-1}a\left(Z_n\left(\frac{k}{n}\right)\right)\text{sign}\left(Z_n\left(\frac{k}{n}\right)\right)\Big|Z_n(s)=x\right],\quad s\in\frac{1}{n}\mathbb{Z}_+, t\geq s.$$

We are going to prove that, for every $T>0$,

$$g_n^{s,t}(x)\to f^{s,t}(x)\stackrel{\text{def}}{=}E_x|Z(t)|-|x|-E_x\int_s^t a(Z(r))\text{sign}\,(Z(r))\,dr \tag{4.14}$$

uniformly by $x\in\mathbb{R}, t\leq T, s\in\frac{1}{n}\mathbb{Z}\cap[0,t]$. This will provide condition 2 for the sequence $\{\theta_n\}$ since, by the Itô–Tanaka formula, the left-hand side of (4.14) is exactly the characteristic of the local time $\phi=L_Z$. The functionals defined by (4.9),(4.10) are time homogeneous, thus it is enough to prove the uniform convergence $f_n^{0,t}\to f^{0,t}, t\leq T$.

We remark that the distribution P_x of Z conditioned by $Z(0)=x$ can be interpreted as the distribution of the solution to SDE (2.6) with the initial condition $Z(0)=x$. Similarly, the conditional distribution $P[\cdot|Z_n(0)=x]$ can be interpreted as the distribution of the solution to the difference relation (2.5) with $Z_n(0)=x$. Thus, in the sequel, we write $Z(x,\cdot)$, $Z_n(x,\cdot)$ for the corresponding solutions and write the usual expectation instead of conditional ones.

By using a difference analogue of the Gronwall lemma, one can verify that

$$E(Z_n(x,t)-Z_n(x,0))^2\leq Ct \tag{4.15}$$

for every $n\in\mathbb{N}, x\in\mathbb{R}, t\leq T$ with some constant C. We do not give a detailed exposition here, since the estimates are quite standard (e.g., see the proof of Lemma 3.1 [20]).

Let $x_n\to x\in\mathbb{R}$ be an arbitrary sequence. It follows from (4.15) and the explicit formula for Z_n that

$$E(Z_n(x_n,t_1)-Z_n(x_n,t_2))^2E(Z_n(x_n,t_3)-Z_n(x_n,t_2))^2\leq C_1(t_3-t_1)^2,$$

$0\leq t_1<t_2<t_3\leq T$, with some new constant C_1. Thus, the sequence of distributions of $Z_n(x_n,\cdot)$ in $\mathbb{D}([0,T])$ is weakly compact ([24], Theorem 15.6). It follows from Proposition 5.1 [16] that every weak limit point of the sequence $\{Z_n(x_n,\cdot)\}$ gives a weak solution to SDE (2.6). Since a,b are Lipschitz, (2.6) possesses a unique weak solution (and even a unique strong one). Hence, $Z_n(x_n,\cdot)$ converge weakly to $Z(x,\cdot)$ in $\mathbb{D}([0,T])$. Trajectories of $Z(x,\cdot)$ are continuous, and, therefore, the weak convergence of $Z_n(x_n,\cdot)$ to $Z(x,\cdot)$ implies that, for every sequence $\{t_n\}\subset[0,T], t_n\to t$, the random variables $Z_n(x_n,t_n)$ converge in distribution to the variable $Z(x,t)$. This and (4.15) yield

$$E|Z_n(x_n,t_n)|-|x_n|\to E|Z(x,t)|-|x|.$$

Now, let $\{r_n\}\subset[0,T], r_n\to r$ be an arbitrary sequence. The diffusion process Z possesses a transition probability density, and therefore $P(Z(x,r)=0)=0$. The function $\hat{a}(z)\stackrel{\text{def}}{=}a(z)\text{sign}\,(z)$ is continuous everywhere except one point $z=0$. Hence, this function is continuous almost surely w.r.t. a distribution of $Z(x,r)$. In addition, this function is bounded. Hence $E\hat{a}(Z_n(x_n,r_n)) \to E\hat{a}(Z(x,r))$. Therefore,

$$\lim_{\varepsilon\to 0+} \limsup_{n\to+\infty} \sup_{r,s\in[0,T],|s-r|\le\varepsilon} |E\hat{a}(Z_n(x_n,s)) - E\hat{a}(Z(x,r))| = 0,$$

and thus

$$\frac{1}{n}E\sum_{k=0}^{[nt]-1} \hat{a}\left(Z_n\left(x_n,\frac{k}{n}\right)\right) - \frac{1}{n}E\sum_{k=0}^{[nt]-1} \hat{a}\left(Z\left(x,\frac{k}{n}\right)\right) \to 0$$

uniformly by $t \le T$. Since $\hat{a}$ is bounded, the above-mentioned properties of the transition probability density $p_t(x,y)$ yield

$$\frac{1}{n}E\sum_{k=0}^{[nt]-1} \hat{a}\left(Z\left(x,\frac{k}{n}\right)\right) = \frac{1}{n}\sum_{k=0}^{[nt]-1}\int_{\mathbb{R}} p_{\frac{k}{n}}(x,y)\hat{a}(y)\,dy \to$$

$$\to \int_0^t\int_{\mathbb{R}} p_r(x,y)\hat{a}(y)\,dy\,dr = E\int_s^t a(Z(x,r))\mathrm{sign}\,(Z(x,r))\,dr$$

uniformly by $t \le T$. Thus, for every sequence $t_n \to t \in [0,T]$ and every bounded sequence $\{x_n\} \subset \mathbb{R}$,

$$f_n^{0,t_n}(x_n) - f^{0,t_n}(x_n) \to 0.$$

This, together with (4.12), yields

$$g_n^{0,t_n}(x_n) - f^{0,t_n}(x_n) \to 0. \tag{4.16}$$

At last, consider a sequence $x_n \to \infty$. It follows from the definition of functional $\theta_n^{0,\cdot}$ that the values of this functional do not exceed δ_n up to the first time moment when the process Z_n visits point 0. The estimates given above provide that the expectation of the random variable $\theta_n^{s,t}$ is estimated by some constant C_2 for $0 \le s \le t \le T$. Thus, by additivity of the functional g_n and the Markov property of X_n, the value $g_n^{0,t}(x_n)$ is estimated by $(C_2+\delta_n)P_{n,T}(x_n)$, where $P_{n,T}(x_n)$ denotes the probability for the process $Z_n(x_n,\cdot)$ to visit the level 0 before the time moment T. Straightforward considerations using the Gronwall lemma provide that $P_{n,T}(x_n) \to 0$ and thus $g_n^{0,T}(x_n) \to 0$. In the same way, one can show that $f^{0,T}(x_n) \to 0$ (also, one can deduce the latter relation from the estimates on the transition probability density for Z). Thus, (4.16) holds for every sequence $\{x_n\}$, and therefore (4.14) holds true. This implies that condition 2 of Theorem 1 holds for the functionals $\{\theta_n\}$.

The explicit formula for the characteristic f and the properties of the transition probability density for Z yield condition 3 of Theorem 1. We also know already that $\{Z_n\}$ provides the Markov approximation for the diffusion process Z (Example 3). Thus, Theorem 1 can be applied to the functionals $\{\theta_n\}$. From this theorem and (4.11), we get the required statement. The proposition is proved.

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