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ON THE CONVERGENCE OF SERIES OF AUTOREGRESSIVE SEQUENCES

Necessary and sufficient conditions for the almost sure convergence of a series of autoregressive sequences are studied. In particular, the convergence of a series of zero-mean Gaussian Markov sequences are considered.

1. Introduction

Let $(\xi_k) = (\xi_k, k \geq 1)$ be *an autoregressive sequence* of random variables. Hereinafter, this means that

$$\xi_0 = 0, \quad \xi_k = \alpha_k \xi_{k-1} + \beta_k \theta_k, \quad k \geq 1, \tag{1}$$

where (α_k) and (β_k) are nonrandom real sequences such that $\alpha_1 = 1$, and (θ_k) is a sequence of independent symmetric random variables such that $\mathsf{P}\{\theta_k = 0\} < 1$, $k \geq 1$. We recall that the random variable θ is called *symmetric* (*symmetrically distributed*) if θ and $(-\theta)$ are identically distributed.

In particular, if (θ_k) is *a standard Gaussian sequence*, i.e., (θ_k) is a sequence of independent N(0,1) - distributed Gaussian random variables, then (ξ_k) is *a zero-mean Gaussian Markov sequence* [4].

For a given (ξ_k), we now consider the random series

$$\sum_{k=1}^{\infty} \xi_k \tag{2}$$

and the sequence of its partial sums

$$S_n = \sum_{k=1}^{n} \xi_k, \quad n \geq 1. \tag{3}$$

In this paper, the necessary and sufficient conditions for the convergence *almost surely* (*a.s.*) of series (2) are studied.

In order to find the necessary conditions for the convergence a.s. of series (2), we consider the sequence of its partial sums (3) as a series with independent symmetric terms in the Banach space of convergent real sequences. Such an approach allows one to use the theory of random series in Banach spaces of real sequences [1, 2]. In order to find the sufficient conditions, we use the theory of infinite-summability matrices [1, 2, 3, 5].

As an application of the general results, we consider series (2) for zero-mean Gaussian Markov sequences (ξ_k) in more details.

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2. Preliminaries

Let $\mathbf{R}^\infty$ be the space of all real sequences, and let $\mathbf{c}$ be the space of all convergent real sequences. The space $\mathbf{c}$ is a separable Banach space if it is endowed with the norm $\|x\|_\infty = \sup_{k\geq 1}|x_k|$, $x = (x_k) \in \mathbf{c}$, [6].

The recurrence equation (1) implies that sequence (3) can be represented in the form

$$(4)\qquad \begin{pmatrix} S_1 \\ S_2 \\ \vdots \\ S_n \\ \vdots \end{pmatrix} = \begin{pmatrix} A(1,1) \\ A(2,1) \\ \vdots \\ A(n,1) \\ \vdots \end{pmatrix} \theta_1 + \begin{pmatrix} 0 \\ A(2,2) \\ \vdots \\ A(n,2) \\ \vdots \end{pmatrix} \theta_2 + ... + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ A(n,n) \\ \vdots \end{pmatrix} \theta_n + \dots,$$

where

$$(5)\qquad A(n,k) = \beta_k A^{'}(n,k), \quad n,k \geq 1;$$

$$A^{'}(n,k) = 1 + \sum_{l=1}^{n-k} \Big(\prod_{j=k+1}^{k+l} \alpha_j \Big) \quad \text{for} \quad 1 \leq k \leq n-1; \quad A^{'}(n,k) = 0 \quad \text{for} \quad k > n;$$

$$A^{'}(n,n) = 1 \quad \text{for} \quad n \geq 1,$$

and

$$(6)\qquad S_n = \sum_{k=1}^{n} A(n,k)\theta_k, \;\; n \geq 1.$$

It is suitable to rewrite series (4) as

$$(7)\qquad \overrightarrow{S} = \sum_{k=1}^{\infty} \theta_k \overrightarrow{A}_k,$$

where

$$\overrightarrow{S} = \begin{pmatrix} S_1 \\ S_2 \\ \vdots \\ S_n \\ \vdots \end{pmatrix}, \;\; \overrightarrow{A}_k = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ A(k,k) \\ A(k+1,k) \\ \vdots \end{pmatrix}, \;\; k \geq 1.$$

It is worth noting that series (7) converges in the coordinate-wise sense. Thus, the sequence of partial sums (S_n) is represented in the form of series (7) with independent symmetric random terms in $\mathbf{R}^\infty$. By Theorem 2.1.1 [1], the following result holds.

Lemma 2.1. *The random series* (2) *converges a.s. if and only if* $\overrightarrow{A}_k \in \mathbf{c}$, $k \geq 1$, *and the random series* (7) *converges a.s. in the norm of the space* $\mathbf{c}$.

3. Necessary conditions

For $k \geq 1$, consider the nonrandom series

$$(8)\qquad A^{'}(\infty,k) = 1 + \sum_{l=1}^{\infty} \prod_{j=k+1}^{k+l} \alpha_j,$$

$$A(\infty,k)=\beta_k A^{'}(\infty,k)=\beta_k\Big(1+\sum_{l=1}^{\infty}\prod_{j=k+1}^{k+l}\alpha_j\Big), \tag{9}$$

and note that $A(\infty,k)=\lim_{n\to\infty}A(n,k)$, if $\lim_{n\to\infty}A(n,k)$ exists, i.e., series (9) converges.

The next result follows from Lemma 2.1 and Theorem 2.8.1 [2].

Theorem 3.1. *If the random series* (2) *converges a.s., then the nonrandom series* (9) *converges for any* $k\geq 1$, *and the random series*

$$\sum_{k=1}^{\infty}A(\infty,k)\theta_k \tag{10}$$

converges a.s. Moreover, the equality

$$\sum_{k=1}^{\infty}\xi_k=\sum_{k=1}^{\infty}A(\infty,k)\theta_k \quad a.s. \tag{11}$$

holds true.

For standard Gaussian sequences (θ_k), the above result is specialized as follows.

Corollary 3.1. *Suppose that* (θ_k) *is a standard Gaussian sequence, i.e.,* (ξ_k) *is a zero-mean Gaussian Markov sequence. If the random series* (2) *converges a.s., then the non-random series* (9) *converges for any* $k\geq 1$,*, and*

$$\sum_{k=1}^{\infty}(A(\infty,k))^2<\infty. \tag{12}$$

Moreover, equality (11) *holds true.*

Proof. Corollary 3.1 follows from Theorem 3.1, since the Gaussian random series (10) converges a.s. if and only if condition (12) holds true. □

Introducing some more notations, we obtain the necessary conditions for the convergence a.s. of the random series (2) for zero-mean Gaussian Markov sequences (ξ_k) in "correlation" terms.

For a zero-mean Gaussian Markov sequence (ξ_k), we consider two sequences: *the sequence of variance*, (σ_k^2), and *the sequence of correlation coefficients*, $(r_{k,k+1})$, where $\sigma_k^2=\mathsf{E}{\xi_k}^2$, $k\geq 1$, and $r_{k,k+1}=(\mathsf{E}\xi_k\xi_{k+1})/\sigma_k\sigma_{k+1}$, if $\sigma_k\sigma_{k+1}>0$, and $r_{k,k+1}=0$, if $\sigma_k\sigma_{k+1}=0$, $k\geq 1$.

It is well known [4] that

$$\mathsf{E}\xi_j\xi_m=\sigma_j\sigma_m\prod_{i=j}^{m-1}r_{i,i+1} \tag{13}$$

for any $m\geq 1$ and $1\leq j<m$.

For $k\geq 1$, we consider the nonrandom series

$$C(k)=\sigma_k+\sum_{l=k+1}^{\infty}\sigma_l\prod_{i=k}^{l-1}r_{i,i+1},$$

and set

$$B(k)=(1-r_{k-1,k}^2)^{1/2}C(k), \quad k\geq 1, \tag{14}$$

where $r_{0,1} = 0$. Note that, by (13),

$$\sum_{l=k}^{\infty} \mathsf{E}\xi_k\xi_l = \sigma_k C(k), \quad k \geq 1.$$

Corollary 3.2. *Suppose that* (ξ_k) *is a zero-mean Gaussian Markov sequence such that* $\sigma_k^2 > 0$, $k \geq 2$. *If the random series* (2) *converges a.s., then the nonrandom series* (14) *converges for any* $k \geq 1$,, *and*

$$\sum_{k=1}^{\infty} (B(k))^2 < \infty. \tag{15}$$

Moreover, if (ξ_k) *is generated by the standard Gaussian sequence* (θ_k) (*recall* (1)), *then the random series*

$$\sum_{k=1}^{\infty} B(k)\theta_k \tag{16}$$

converges a.s., and the equality

$$\sum_{k=1}^{\infty} \xi_k = \sum_{k=1}^{\infty} B(k)\theta_k \quad a.s. \tag{17}$$

holds true.

Proof. Corollary 3.2 follows from Corollary 3.1 since

$$\alpha_k = \frac{\sigma_k}{\sigma_{k-1}} r_{k-1,k}, \quad k \geq 2,$$

$$\beta_1^2 = \sigma_1^2, \quad \beta_k^2 = \sigma_k^2(1 - r_{k-1,k}^2), \quad k \geq 2,$$

and

$$A(\infty, k) = B(k), \quad k \geq 1.$$

□

4. Sufficient conditions

This section deals with sufficient conditions the for convergence a.s. of series (2). The method used in this section is based on the theory of infinite-summability matrices [1, 2, 3, 5].

Consider an infinite-summability real matrix $\Lambda = (\lambda_{n,k})_{n,k=1}^{\infty}$. This means that $\lim_{n\to\infty} \lambda_{n,k} = 1$ for all $k \geq 1$. Consider also a real series $\sum_{k=1}^{\infty} x_k$. To this series and to the matrix Λ, we relate the sequence of series $\sum_{k=1}^{\infty} \lambda_{n,k}x_k, n \geq 1$. Assume that all these series converge. We denote their sums by $\Xi_n, n \geq 1$. Then, if the sequence (Ξ_n) converges, the series $\sum_{k=1}^{\infty} x_k$ is called Λ-*summable*, and the limit $\lim_{n\to\infty} \Xi_n$ is called the Λ-*sum* of the series $\sum_{k=1}^{\infty} x_k$.

Let Λ be a summability matrix. If

$$Varn(\Lambda) = \sup_{n\geq 1} \sup_{m\geq 2} \Big[\Big(\sum_{k=1}^{m-1} |\lambda_{n,k} - \lambda_{n,k+1}| \Big) + |\lambda_{n,m}| \Big] < \infty,$$

then the matrix Λ is called *the matrix of bounded variation.*

In order to obtain sufficient conditions we use one result which asserts the equivalence of the summation by matrices of bounded variation (see [2], Theorem 2.8.2, and [3]). Theorem 2.8.2 [2] says that *if the sequence* (X_k) *is a sequence of independent symmetric random variables, and the series* $\sum_{k=1}^{\infty} X_k$ *is* Λ'-*summable a.s. by some matrix of bounded variation* Λ', *then it is* Λ-*summable a.s. by all the matrices of bounded variation, and all*

Λ-sums are equal a.s. to one another. In particular, this theorem says that *if the series $\sum_{k=1}^{\infty} X_k$ is* convergent a.s., *then it is Λ-summable a.s. by all the matrices of bounded variation, and all Λ-sums are equal a.s. to $\sum_{k=1}^{\infty} X_k$.*

Theorem 4.1. *Suppose that $\alpha_k \geq 0$, $k \geq 2$. If the nonrandom series* (9) *converges for any $k \geq 1$, and the random series* (10) *converges a.s., then the random series* (2) *converges a.s., and equality* (11) *holds true.*

Proof. Assume that the series $\sum_{k=1}^{\infty} A(\infty,k)\theta_k$ converges a.s. and remark that the sequences (β_k) with $\beta_k \neq 0$, $k \geq 1$, can be considered without loss of generality.

Consider the matrix $\Lambda = (\lambda_{n,k})_{n,k=1}^{\infty}$, where

$$\lambda_{n,k} = \begin{cases} \frac{A(n,k)}{A(\infty,k)}, & 1 \leq k \leq n, \\ 0, & k > n. \end{cases} \tag{18}$$

Observe that all $\lambda_{n,k}$ are well-defined, since the series $A(\infty,k)$ converges, and the series $A(\infty,k) \neq 0$ for any $k \geq 1$. Since $\lim_{n\to\infty} A(n,k) = A(\infty,k)$ for any $k \geq 1$, we have

$$\lim_{n\to\infty} \lambda_{n,k} = \lim_{n\to\infty} \frac{A(n,k)}{A(\infty,k)} = 1, \quad k \geq 1.$$

Hence, the matrix Λ is a summability matrix, and, for $X_k = A(n,k)\theta_k$, $k \geq 1$,

$$\sum_{k=1}^{\infty} \lambda_{n,k} X_k = \sum_{k=1}^{n} \frac{A(n,k)}{A(\infty,k)} A(\infty,k)\theta_k = \sum_{k=1}^{n} A(n,k)\theta_k, \quad n \geq 1.$$

Thus, by (6) and (2),

$$\sum_{k=1}^{\infty} \lambda_{n,k} X_k = \sum_{k=1}^{n} \xi_k, \quad n \geq 1. \tag{19}$$

Since $\alpha_k \geq 0$, $k \geq 2$, $\lambda_{n,k} \geq 0$ for any $k, n \geq 1$, and, by (5) and (8),

$$\lambda_{n,k} - \lambda_{n,k+1} = \frac{A(n,k)}{A(\infty,k)} - \frac{A(n,k+1)}{A(\infty,k+1)} = \frac{A'(n,k)}{A'(\infty,k)} - \frac{A'(n,k+1)}{A'(\infty,k+1)}$$

$$= \frac{1+\alpha_{k+1}A'(n,k+1)}{1+\alpha_{k+1}A'(\infty,k+1)} - \frac{A'(n,k+1)}{A'(\infty,k+1)} = \frac{A'(\infty,k+1) - A'(n,k+1)}{(1+\alpha_{k+1}A'(\infty,k+1))A'(\infty,k+1)}$$

$$= \frac{(\alpha_{k+2}\alpha_{k+3}\ldots\alpha_{n+1})A'(\infty,n+1)}{(1+\alpha_{k+1}A'(\infty,k+1))A'(\infty,k+1)} \geq 0$$

for any $1 \leq k \leq n-1$.

Therefore,

$$Varn(\Lambda) = \sup_{n\geq 2} \Big[\Big(\sum_{k=1}^{n-1} |\lambda_{n,k} - \lambda_{n,k+1}| \Big) + |\lambda_{n,n}| \Big] =$$

$$= \sup_{n\geq 2} \Big[\Big(\sum_{k=1}^{n-1} \lambda_{n,k} - \lambda_{n,k+1} \Big) + \lambda_{n,n} \Big] = \sup_{n\geq 2} (\lambda_{n,1}) = \sup_{n\geq 2} \frac{A'(n,1)}{A'(\infty,1)} \leq 1.$$

Thus, the matrix Λ is a summability matrix of bounded variation.

Since the sequence $(A(\infty,k)\theta_k)$ is a sequence of independent symmetric random variables, and the series $\sum_{k=1}^{\infty} A(\infty,k)\theta_k$ converges a.s., then, by Theorem 2.8.2 [2], this sequence is Λ-summable a.s., and its Λ-sum are equal a.s. to $\sum_{k=1}^{\infty} A(\infty,k)\theta_k$.

Therefore, by (19), the limit

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} \lambda_{n,k} X_k = \lim_{n\to\infty} \sum_{k=1}^{n} \xi_k = \sum_{k=1}^{\infty} \xi_k$$

exists a.s., and equality (11) holds. □

Theorem 4.1 implies the following two corollaries for zero-mean Gaussian Markov sequences.

Corollary 4.1. *Suppose that* $\alpha_k \geq 0$, $k \geq 2$, *and* (θ_k) *is a standard Gaussian sequence, i.e.,* (ξ_k) *is a zero-mean Gaussian Markov sequence. If the nonrandom series* (9) *converges for any* $k \geq 1$, *and* (12) *holds, then the random series* (2) *converges a.s., and equality* (11) *holds true.*

Corollary 4.2. *Suppose that* (ξ_k) *is a zero-mean Gaussian Markov sequence such that* $\sigma_k^2 > 0$, $k \geq 2$, *and* $r_{k-1,k} \geq 0, k \geq 2$. *If the nonrandom series* (14) *converges for any* $k \geq 1$, *and* (15) *holds, then the random series* (2) *converges a.s. Moreover, if* (ξ_k) *is generated by the standard Gaussian sequence* (θ_k) *(see* (1)*), then the random series* (16) *converges a.s., and equality* (17) *holds true.*

In the next theorem, we consider sequences (α_k) with the elements of alternating signs.

Theorem 4.2. *Assume that* ε *and* M *are two positive numbers such that*

$$0 < \varepsilon \leq |A(\infty, k)| \leq M < \infty \tag{20}$$

for any $k \geq 1$. *Assume also that*

$$H = \sup_{n\geq 1} \sum_{k=1}^{n} \prod_{j=k+2}^{n+1} |\alpha_j| < \infty. \tag{21}$$

If the nonrandom series (9) *converges for any* $k \geq 1$, *and the random series* (10) *converges a.s., then the random series* (2) *converges a.s., and equality* (11) *holds true.*

Proof. Consider the matrix $\Lambda = (\lambda_{n,k})_{n,k=1}^{\infty}$ which is defined at (18). By the proof of Theorem 4.1 above, we have

$$\lambda_{n,k} - \lambda_{n,k+1} = \frac{(\alpha_{k+2}\alpha_{k+3}\ldots\alpha_{n+1})A'(\infty, n+1)}{A'(\infty, k)A'(\infty, k+1)}$$

for any $n \geq 2$ and $1 \leq k \leq n-1$. Hence, by (20) and (21),

$$Varn(\Lambda) = \sup_{n\geq 2} \Big[\Big(\sum_{k=1}^{n-1} |\lambda_{n,k} - \lambda_{n,k+1}|\Big) + |\lambda_{n,n}|\Big]$$

$$= \sup_{n\geq 2} \Big[\Big(\sum_{k=1}^{n-1} \frac{|\alpha_{k+2}\alpha_{k+3}\ldots\alpha_{n+1}||A'(\infty, n+1)|}{|A'(\infty, k)A'(\infty, k+1)|}\Big) + \frac{1}{|A'(\infty, n)|}\Big] \leq \frac{MH}{\varepsilon^2} + \frac{1}{\varepsilon} < \infty.$$

Much of the following is repeated from the proof of Theorem 4.1. □

Example 4.1. Suppose that $0 < q < 1$ and $\alpha_k = (-1)^k q^k, \quad k \geq 2$. Then, for the sequence (α_k), all conditions of Theorem 4.2 hold.

5. Series of autoregressive sequences with weighted coefficients

For the autoregressive sequences (ξ_k) (recall (1)) and a real sequence (c_k) such that $c_k \neq 0$, $k \geq 1$, consider the random series

$$\sum_{k=1}^{\infty} c_k \xi_k. \tag{22}$$

Denote $\zeta_k = c_k \xi_k$, $k \geq 1$. It is clear that (ζ_k) is an autoregressive sequences, and

$$\zeta_0 = 0, \quad \zeta_k = \widetilde{\alpha}_k \zeta_{k-1} + \widetilde{\beta}_k \theta_k, \quad k \geq 1,$$

where

$$\widetilde{\alpha}_1 = 1, \quad \widetilde{\alpha}_k = \frac{c_k}{c_{k-1}}\, \alpha_k,\ k \geq 2; \quad \widetilde{\beta}_k = c_k \beta_k, \quad k \geq 1,$$

and (θ_k) is a sequence of independent symmetric random variables (see (1)).

Put

$$\widetilde{A}_c(n,k) = \beta_k A_c^{'}(n,k), \quad n,k \geq 1,$$

where

$$A_c^{'}(n,k) = c_k + \sum_{l=1}^{n-k} \Big(c_{k+l} \prod_{j=k+1}^{k+l} \alpha_j \Big) \quad \text{for} \quad 1 \leq k \leq n-1; \quad A_c^{'}(n,k) = 0 \quad \text{for} \quad k > n;$$

$$A_c^{'}(n,n) = 1 \quad \text{for} \quad n \geq 1,$$

and denote, for $k \geq 1$,

$$\widetilde{A}_c(\infty,k) = \lim_{n\to\infty} \widetilde{A}_c(n,k) = \beta_k \Big(c_k + \sum_{l=1}^{\infty} \Big(c_{k+l} \prod_{j=k+1}^{k+l} \alpha_j \Big)\Big) \tag{23}$$

if this limit exists.

The results of Section 4 and 5 yield the necessary and sufficient conditions for the convergence a.s. of series (22).

Theorem 5.1. *If the random series* (22) *converges a.s., then the nonrandom series* (23) *converges for any* $k \geq 1$, *and the random series*

$$\sum_{k=1}^{\infty} \widetilde{A}_c(\infty,k)\theta_k \tag{24}$$

converges a.s. Moreover, the following equality

$$\sum_{k=1}^{\infty} c_k \xi_k = \sum_{k=1}^{\infty} \widetilde{A}_c(\infty,k)\theta_k \quad a.s. \tag{25}$$

holds true.

Theorem 5.2. *Suppose that* $\widetilde{\alpha}_k \geq 0$, $k \geq 2$. *If the nonrandom series* (23) *converges for any* $k \geq 1$, *and the random series* (24) *converges a.s., then the random series* (22) *converges a.s., and equality* (25) *holds true.*

For zero-mean Gaussian Markov sequences (ξ_k), the above results are specialized as follows.

Corollary 5.1. *Suppose that* (θ_k) *is a standard Gaussian sequence, i.e.,* (ξ_k) *is a zero-mean Gaussian Markov sequence. If the random series* (22) *converges a.s., then the nonrandom series* (23) *converges for any* $k \geq 1$, *and*

$$\sum_{k=1}^{\infty} (\widetilde{A}_c(\infty,k))^2 < \infty. \tag{26}$$

Moreover, equality (25) *holds true.*

Corollary 5.2. *Suppose that* (θ_k) *is a standard Gaussian sequence, i.e.,* (ξ_k) *is a zero-mean Gaussian Markov sequence. Suppose also that* $\widetilde{\alpha}_k \geq 0$, $k \geq 2$. *If the nonrandom series* (23) *converges for any* $k \geq 1$, *and* (26) *holds, then the random series* (22) *converges a.s., and equality* (25) *holds true.*

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