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ON SIMULTANEOUS HITTING OF MEMBRANES BY TWO SKEW BROWNIAN MOTIONS

We consider two depending Wiener processes which have membranes at zero with different permeability coefficients. Starting from different points, the processes almost surely do not meet at any fixed point except that where membranes are situated. The necessary and sufficient conditions for the meeting of the processes are found. It is shown that the probability of meeting is equal to zero or one.

Introduction

Let $(w_1(t)), (w_2(t))_{t\geq 0}$ be a two-dimensional Wiener process with the correlation matrix

$$B = \begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix} t,$$

where $\alpha \in (-1,1)$ is some constant.

Consider the equations

$$x_1(t) = x_1(0) + w_1(t) + \varkappa_1 L^0_{x_1}(t), \tag{1}$$

$$x_2(t) = x_2(0) + w_2(t) + \varkappa_2 L^0_{x_2}(t), \tag{2}$$

where $\{\varkappa_1, \varkappa_2\} \in [-1,1]$ are constants,

$$L^0_{x_i}(t) = \lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon}\int_0^t 1\!\!1_{[-\varepsilon,\varepsilon]}(x_i(s))ds,\ i=1,2,$$

is a local time of the process $(x_i(t))_{t\geq 0}$ at zero. As is known (cf. [1]), each of Eqs. (1) and (2) has a unique solution which is a skew Brownian motion. Here, $\varkappa_1$ and $\varkappa_2$ can be treated as the coefficients of permeability. If $\varkappa_1 = 1$, the part of the process $(x_1(t))_{t\geq 0}$ on the positive semiaxis is a Wiener process with reflection at 0; if $\varkappa_1 = -1$, then the part of the process $(x_1(t))_{t\geq 0}$ on the negative semiaxis is a Wiener process with reflection at 0; if $\varkappa_1 \in (-1,1)$, then there is a semipermeable membrane at 0.

The aim of the paper is to calculate the probability of the simultaneous hitting of the membranes by the processes $(x_1(t))_{t\geq 0}$ and $(x_2(t))_{t\geq 0}$. This probability turns out to be determined by the sign of an expression involving $\varkappa_1, \varkappa_2, \alpha$. In addition, it is equal to zero or one.

The case of $\alpha = 1$ is studied in [2], [3]. In [2], it is proved that if $\{\varkappa_1, \varkappa_2\} \in [-1,1]\backslash\{0\}$, then the processes $(x_1(t))_{t\geq 0}$ and $(x_2(t))_{t\geq 0}$ meet in a finite time with probability 1. In [3], it is obtained that if $x_1(0) = x_2(0) = 0$, $0 < \varkappa_1 < \varkappa_2 < 1$, and $\varkappa_1 > \varkappa_2/(1+2\varkappa_2)$, then, for each $t_0 > 0$, there exists $t > t_0$ such that $x_1(t) = x_2(t)$. The problem of the simultaneous hitting of the sphere by two Brownian motions with normal reflection on the sphere is treated in [4] (two-dimensional case) and [5].

If there are no membranes, i.e. $\varkappa_1 = \varkappa_2 = 0, \alpha \neq 1$, then the process $x(t) = (x_1(t), x_2(t))\,, t \geq 0$, is a two-dimensional Wiener process. It reaches any fixed point

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$x_0 \in \mathbb{R}^2$, $x_0 \neq x(0)$, with probability 0. In particular, this implies that the process $(x(t))_{t\geq 0}$ almost surely does not hit any fixed point except the points, at which at least one membrane is situated.

There is one more problem of stochastic analysis, where the study of the simultaneous membrane visitation arises naturally. Assume that we are attending to construct a flow generated by a stochastic differential equation with a semipermeable membrane located on a hyperplane [7]. Note that there is no general results on the existence and uniqueness of a strong solution to such equations in a multidimensional space. In order to construct the flow on some probability space, it is sufficient to construct a sequence of consistent (weak) n-point motions [8]. One-point motion can be constructed by N. Portenko's methods [7]. There are no general results on the weak uniqueness for two-point motion, when both points start from a membrane. However, if the simultaneous visitation of the membrane has a probability 0, then there is a hope to construct n-point motion using the localization at a neighborhood of the membrane. Unfortunately, the results of the article show that the synchronous hitting of the membrane is quite natural.

1. Transformation of the processes

The pair of the processes $(x_1(t), x_2(t))_{t\geq 0}$ can be thought off as a new process in the Euclidean space $\mathbb{R}^2$ with membranes on the straight lines $S_1 = \{x_2 = 0\}$ and $S_2 = \{x_1 = 0\}$. The membranes act in the normal direction $\nu_1 = (0, 1)$ and $\nu_2 = (1, 0)$ to S_1 and S_2, respectively.

Let us make a coordinate transformation defined by the linear operator

$$A = B^{-1/2} = \frac{1}{c}\begin{pmatrix} a & b \\ b & a \end{pmatrix},$$

where

$$\begin{aligned} a &= \sqrt{1-\alpha} + \sqrt{1+\alpha}, \\ b &= \sqrt{1-\alpha} - \sqrt{1+\alpha}, \\ c &= 2\sqrt{1-\alpha^2}. \end{aligned}$$

As a result, we get a new process $(\tilde{x}_1(t), \tilde{x}_2(t))_{t\geq 0}$. From (1), (2), we see that its trajectories are solutions of the equations

$$\tilde{x}_1(t) = \tilde{x}_1(0) + \tilde{w}_1(t) + \varkappa_1 \frac{a}{c} L^0_{x_1}(t) + \varkappa_2 \frac{b}{c} L^0_{x_2}(t), \tag{3}$$

$$\tilde{x}_2(t) = \tilde{x}_2(0) + \tilde{w}_2(t) + \varkappa_1 \frac{b}{c} L^0_{x_1}(t) + \varkappa_2 \frac{a}{c} L^0_{x_2}(t), \tag{4}$$

where $\tilde{w}_1(t) = a/cw_1(t) + b/cw_2(t),\ \tilde{w}_2(t) = b/cw_1(t) + a/cw_2(t),\ t \geq 0$. It is easily seen that the correlation matrix of the vector $(\tilde{w}_1(t), \tilde{w}_2(t))$ is as follows:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} t.$$

This yields that $(\tilde{w}_1(t))_{t\geq 0}$ and $(\tilde{w}_2(t))_{t\geq 0}$ are independent Wiener processes.

Denote, by S_1' and S_2', the images of S_1 and S_2 under the transformation defined by the matrix A. Then Eqs. (3), (4) can be rewritten in the form

$$\tilde{x}_1(t) = \tilde{x}_1(0) + \tilde{w}_1(t) + \varkappa_1 \frac{a}{c} L^{S_1'}_{\tilde{x}}(t) + \varkappa_2 \frac{b}{c} L^{S_2'}_{\tilde{x}}(t), \tag{5}$$

$$\tilde{x}_2(t) = \tilde{x}_2(0) + \tilde{w}_2(t) + \varkappa_1 \frac{b}{c} L^{S_1'}_{\tilde{x}}(t) + \varkappa_2 \frac{a}{c} L^{S_2'}_{\tilde{x}}(t), \tag{6}$$

where $L_{\tilde{x}}^{S_i'}$ is a symmetric local time of the process $(\tilde{x}(t))_{t\geq 0}$ on the straight line S_i', that is,

$$L_{\tilde{x}}^{S_i'}(t) = \lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{A_\varepsilon^i}(\tilde{x}(s))ds, \ i = 1, 2, \tag{7}$$

$$A_\varepsilon^i = \{x \in \mathbb{R}^2 : \exists y \in S_i', s \in [-1, 1] \text{ such that } x = y + \varepsilon s \nu_i'\},$$

ν_i', $i = 1, 2$, is the image of ν_i under the transformation defined by the matrix A.

2. On the hitting of zero by the Wiener process on the plane with membranes on rays with a common endpoint

A Wiener process in $\mathbb{R}^2$ with membranes on rays $c_1, \ldots, c_n$ having a common endpoint was investigated in [6]. Let $(r, \varphi), r \geq 0, \varphi \in [0, 2\pi)$, be polar coordinates in $\mathbb{R}^2$, and let

$$c_k = \{(r, \varphi) : r \geq 0, \ \varphi = \varphi_k\},$$

where $0 \leq \varphi_1 < \cdots < \varphi_n < 2\pi$. Put $\varphi_{n+1} = \varphi_n$, $\xi_k = \varphi_{k+1} - \varphi_k, k = 1, 2, \ldots, n$, and $\xi_0 = \xi_n$.

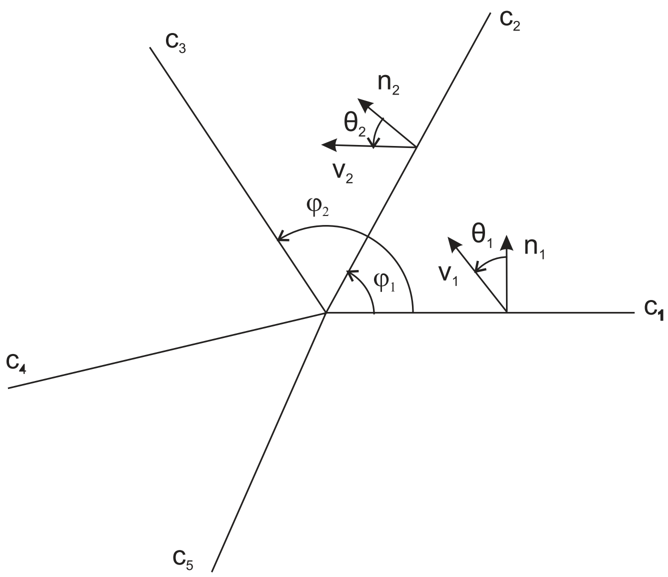

Figure 1

Denote, by $n_k, k = 1, \ldots, n$, the unit vector normal to c_k that points anticlockwise, and let v_k be a vector in $\mathbb{R}^2$ such that $(v_k, n_k) = 1$. The angle between n_k and v_k denoted by $\theta_k \in (-\frac{\pi}{2}, \frac{\pi}{2})$ is referred to as positive if and only if v_k points towards the origin. Let $\gamma_k, |\gamma_k| \leq 1, \ k = 1, \ldots, n$, be the membrane permeability coefficients. The case of $n = 5, \theta_1 > 0, \theta_2 > 0$ is shown in Fig. 1.

It was proved in [6] that there exists a unique strong solution to the equation

$$dx(t) = dw(t) + \sum_{k=1}^{n} \gamma_k v_k dL_x^{c_k}(t) \tag{8}$$

in $\mathbb{R}^2$ with the initial condition $x(0) = x^0,\ x^0 \in \mathbb{R}^2$, up to the time ζ, where $\zeta = +\infty$ or $x(\zeta-) = 0$.

Further on, we make use of the following Proposition on hitting 0 or ∞ by the process $(\tilde{x}(t))_{t\ge 0}$ (cf. [6]).

Proposition 1. *Let $\gamma_k \in [-1, 1],\ k = 1, \dots, n$, and let us consider the Markov chain with the state-space $\{1, \dots, n\}$ and the transition matrix*

$$\begin{pmatrix} 0 & \tilde{p}_1 & 0 & 0 & \dots & 0 & 0 & 0 & \tilde{q}_1 \\ \tilde{q}_2 & 0 & \tilde{p}_2 & 0 & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & \tilde{q}_{n-1} & 0 & \tilde{p}_{n-1} \\ \tilde{p}_n & 0 & 0 & 0 & \dots & 0 & 0 & \tilde{q}_n & 0 \end{pmatrix},$$

where

$$\tilde{p}_k = \frac{(1+\gamma_k)\xi_{k-1}}{(\xi_{k-1}+\xi_k) + \gamma_k(\xi_{k-1}-\xi_k)}, \tag{9}$$

$$\tilde{q}_k = \frac{(1-\gamma_k)\xi_k}{(\xi_{k-1}+\xi_k) + \gamma_k(\xi_{k-1}-\xi_k)}, \tag{10}$$

has a unique invariant distribution $(\pi_k)_{k=1}^n$.

If $\sum_{k=1}^n \gamma_k \pi_k \frac{\xi_{k-1}\xi_k}{(\xi_{k-1}+\xi_k)+\gamma_k(\xi_{k-1}-\xi_k)} \tan\theta_k > 0$, then the process $(\tilde{x}(t))_{t\ge 0}$ hits the origin almost surely;
if $\sum_{k=1}^n \gamma_k \pi_k \frac{\xi_{k-1}\xi_k}{(\xi_{k-1}+\xi_k)+\gamma_k(\xi_{k-1}-\xi_k)} \tan\theta_k \le 0$, then the process $(\tilde{x}(t))_{t\ge 0}$ does not hit the origin a.s.

3. The main result

Let us formulate our problem in terms of the previous Section. Let c_1, c_2, c_3, c_4 be the images of the rays $[0, \infty) \times \{0\}$, $\{0\} \times [0, \infty)$, $(-\infty, 0] \times \{0\}$, $\{0\} \times (-\infty, 0]$ under the linear transformation A. The images of $\nu_1 = (0, 1)$ and $\nu_2 = (1, 0)$ are the vectors $\nu_1' = (a/c, b/c)$ and $\nu_2' = (b/c, a/c)$. Denote, by ξ, the angle between them. Then

$$\cos\xi = \frac{(\nu_1', \nu_2')}{|\nu_1'| \cdot |\nu_2'|} = \frac{2ab}{a^2+b^2} = -\alpha.$$

The case of $\alpha < 0$ is shown in Fig. 2.

Put $\xi_1 = \xi_3 = \xi,\ \xi_2 = \xi_4 = \pi - \xi,\ \gamma_1 = -\gamma_3 = \varkappa_1,\ \gamma_2 = -\gamma_4 = -\varkappa_2,\ v_1 = (a/c, b/c)$, $v_2 = (-b/c, -a/c)$, $v_3 = (-a/c, -b/c)$, $v_4 = (b/c, a/c)$, $\theta_1 = \theta_3 = \xi - \pi/2$, $\theta_2 = \theta_4 = \pi/2 - \xi$. It is easy to check that $(v_i, n_i) = 1, i = 1, 2, 3, 4$, where n_i is the unit normal vector to c_i that points anticlockwise. Indeed,

$$(v_i, n_i) = \frac{\sqrt{a^2+b^2}}{c}\cos(\pi/2 - \xi) = \frac{2}{2\sqrt{1-\alpha^2}}\sqrt{1-\alpha^2} = 1, \qquad i = 1, 2, 3, 4.$$

Equations (5),(6) can be rewritten in the form

$$\tilde{x}(t) = \tilde{x}(0) + \tilde{w}(t) + \sum_{i=1}^4 \gamma_i v_i L_{\tilde{x}}^{c_i}(t),\ t \le \zeta, \tag{11}$$

where $\tilde{x}(t) = (\tilde{x}_1(t), \tilde{x}_2(t)), \tilde{w}(t) = (\tilde{w}_1(t), \tilde{w}_2(t))$, ζ is the first hitting time of 0 by the process $(\tilde{x}(t))_{t\ge 0}$. So (11) coincides with (8).

Now we calculate the expression from Proposition for $n = 4$:

$$S = \sum_{k=1}^4 \gamma_k \pi_k \frac{\xi_{k-1}\xi_k}{(\xi_{k-1}+\xi_k) + \gamma_k(\xi_{k-1}-\xi_k)} \tan\theta_k.$$

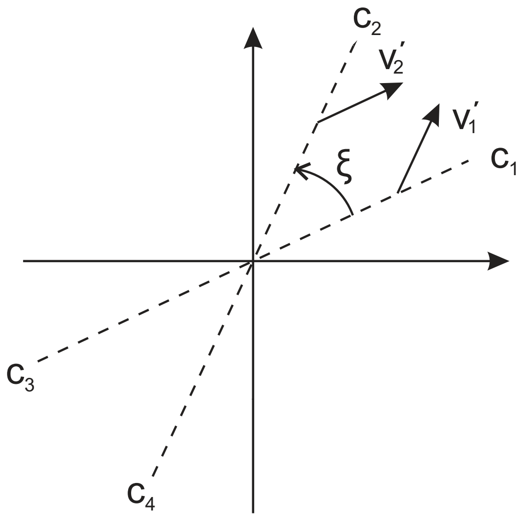

FIGURE 2

Let $\{\varkappa_1, \varkappa_2\} \in (-1,1)\setminus\{0\}$. The invariant distribution $(\pi_i)_{i=1}^4$ can be obtained directly. But we make use of formula (29) from [6]. We have

$$\pi_1 = \frac{\tilde q_2 \tilde q_3 \tilde q_4}{\tilde p_1 \tilde q_3 \tilde q_4 + \tilde p_1 \tilde p_2 \tilde q_4 + \tilde p_1 \tilde p_2 \tilde p_3 + \tilde q_2 \tilde q_3 \tilde q_4}.$$

Taking (9),(10) into account, we get

$$\begin{aligned}
\pi_1 &= \frac{(1-\gamma_2)\left((1+\gamma_1)(\pi-\xi)+(1-\gamma_1)\xi\right)}{D},\\
\pi_2 &= \frac{(1+\gamma_1)\left((1-\gamma_2)(\pi-\xi)+(1+\gamma_2)\xi\right)}{D},\\
\pi_3 &= \frac{(1+\gamma_2)\left((1-\gamma_1)(\pi-\xi)+(1+\gamma_1)\xi\right)}{D},\\
\pi_4 &= \frac{(1-\gamma_1)\left((1+\gamma_2)(\pi-\xi)+(1-\gamma_2)\xi\right)}{D},
\end{aligned}$$

where $D = 2\left[(1+\gamma_1\gamma_2)\xi + (1-\gamma_1\gamma_2)(\pi-\xi)\right] > 0$. Then

$$S = 2\frac{\xi(\pi-\xi)}{D}(-\varkappa_1\varkappa_2 \cot\xi).$$

The condition $\xi \in (0,\pi)$ yields $\cot\xi = -\frac{\alpha}{\sqrt{1-\alpha^2}}$. Consequently, $S > 0$ if and only if $\varkappa_1\varkappa_2\alpha > 0$.

It is obvious that the processes $(x_1(t))_{t\ge 0}$ and $(x_2(t))_{t\ge 0}$ meet in zero when and only when $(\tilde x(t))_{t\ge 0}$ hits zero.

If $\varkappa_1 = \varkappa_2 = 1$, then there exists a unique invariant distribution $\pi_1 = \pi_2 = 1/2,\ \pi_3 = \pi_4 = 0$. It is easy to see that now $S > 0$ if and only if $\alpha > 0$.

Finally, let $\varkappa_1 = 1,\ \varkappa_2 \in (-1,1)\setminus\{0\}$. Then the invariant distribution is of the form $\pi_1 = \tilde p_2/2,\ \pi_2 = 1/2,\ \pi_3 = \tilde q_2/2,\ \pi_4 = 0$. We get that $S > 0$ if and only if $\varkappa_2\alpha > 0$.

The other cases where the modulus of at least one permeability coefficient is equal to 1 can be treated analogously.

Now let $\varkappa_1 = 0$. Then the unique invariant distribution is as follows: $\pi_1 = \pi_3 = 0$, $\pi_2 = \pi_4 = 1/2$. It is easy to see that, in this case, $S = 0$. Analogously, $S = 0$ if $\varkappa_2 = 0$.

Thus, we have proved the following statement.

Theorem 1. *Let* $(x_1(0), x_2(0)) \in \mathbb{R}^2 \setminus \{(0,0)\}$, $\{\varkappa_1, \varkappa_2\} \subset [-1,1]$, $\alpha \in (-1,1)$. *Then*

$$1)\ \mathbb{P}\{\exists\ t_0 < \infty : x_1(t_0) = x_2(t_0) = 0\} = 1 \ \textit{if}\ \ \varkappa_1\varkappa_2\alpha > 0,$$
$$2)\ \mathbb{P}\{\exists\ t_0 < \infty : x_1(t_0) = x_2(t_0) = 0\} = 0 \ \textit{if}\ \ \varkappa_1\varkappa_2\alpha \le 0.$$

Remark 1. The conditions for the meeting of processes obtained in Theorem for $\alpha \in (-1,1)$ are completely different from those for $\alpha = 1$ obtained in [3].

Remark 2. As was mentioned above, the process $(x(t))_{t\ge 0}$ almost surely does not hit any fixed point except the points, at which at least one membrane is situated. It follows from statement 2) of Theorem that the process almost surely does not hit any fixed point, at which exactly one membrane is situated.

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