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ON THE LARGE-DEVIATION PRINCIPLE FOR THE WINDING ANGLE OF A BROWNIAN TRAJECTORY AROUND THE ORIGIN

In this article we analyse the possibility of obtaining the large-deviation principle for the winding angle of a Brownian motion trajectory around the origin. We prove the weak large-deviation principle and show that the full large-deviation principle cannot hold with any rate function.

1. Introduction

The study of the winding angle of a planar Brownian motion has a long history. F. Spitzer in 1958 proved [1] that $\frac{2\Phi(t)}{\ln t} \xrightarrow[t\to\infty]{d} \xi$. Here $\Phi(t)$ is the angle that the 2-dimensional Brownian motion started from non-zero point wound around the origin up to time t, ξ is a random variable with the standard Cauchy distribution, that is, a random variable with the distribution density $p(x) = \frac{1}{\pi(1+x^2)}$. More subtle asymptotics describing the behaviour of the winding angle were obtained in works of Zhan Shi [2], J. Bertoin and W. Werner [3]. For example, one of the results of [2] is that

$$\varliminf_{t\to\infty} \frac{\ln\ln\ln t}{\ln t} \sup_{0\le u\le t} |\Phi(u)| = \frac{\pi}{4} \text{ a.s.}$$

The asymptotical behaviour of mutual winding angles of several two-dimensional Brownian motions is studied in [4] in connection with the behaviour of solar flames. This problem was solved in the article [5]. In this article the following result was obtained.

Theorem 1.1 ([5]). *Let $w_1, \dots .w_n$ be independent two-dimentional standard Brownian motions starting from pairwise distinct points of a plane. Then for the winding angles $\Phi_{ij}(t)$ of the Brownian motion w_i around the Brownian motion w_j the following asymptotical relaton holds:*

$$\left(\frac{2}{\ln t}\Phi_{ij}(t), 1\le i<j\le n\right) \xrightarrow[t\to\infty]{d} \left(C_{ij}, 1\le i<j\le n\right).$$

Here $C_{ij}, 1 \le i < j \le n$, are independent random variables with the standard Cauchy distribution.

All the cited results deal with the asymptotics of winding angles as $t \to \infty$. In this article we study the asymptotical distribution of the winding angle process as $t \to 0$. We consider the possibility of obtaining the large-deviation principle for the winding angle of the Brownian motion. Let us remind the formulation of the large-deviation principle (LDP).

Definition 1.1. Let X be a metric space, $(\xi_\varepsilon)_{\varepsilon>0}$ be a family of random elements in X, $I\colon X \to [0,\infty]$ be some lower semicontinuous function. For any subset $A \subseteq X$ we denote

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$I(A) = \inf_{x \in A} I(x)$. We say that the large-deviation principle (LDP) with rate function I holds for the family $(\xi_\varepsilon)_{\varepsilon>0}$ if for any Borel set $A \subseteq X$ the following inequalities hold:

$$-I(A^\circ) \le \varliminf_{\varepsilon\to 0} \varepsilon \ln P(\xi_\varepsilon \in A) \le \varlimsup_{\varepsilon\to 0} \varepsilon \ln P(\xi_\varepsilon \in A) \le -I(\overline{A}).$$

Here we denote by A° the interior of a set A and by $\overline{A}$ the closure of A.

Definition 1.2. Let X, $(\xi_\varepsilon)_{\varepsilon>0}$, I be as in Definition 1.1. We say that the weak large-deviation principle (weak LDP) with rate function I holds for the family $(\xi_\varepsilon)_{\varepsilon>0}$ if for any open set $G \subseteq X$ and compact set $K \subseteq X$ the following inequalities hold:

$$-I(G) \le \varliminf_{\varepsilon\to 0} \varepsilon \ln P(\xi_\varepsilon \in G),$$

$$\varlimsup_{\varepsilon\to 0} \varepsilon \ln P(\xi_\varepsilon \in K) \le -I(K).$$

In the article we consider the asymptotics of the same expressions in the following situation. Let $X = C([0,1])$ with the uniform norm. Let us now define random elements Φ_ε with values in X.

To any continuous function $f\colon [0,1] \to \mathbb{R}^2, 0 \le t \le 1$ with $f(0) = \begin{pmatrix}1\\0\end{pmatrix}$, $f(t) \ne \begin{pmatrix}0\\0\end{pmatrix}$ for all $t \in [0,1]$, we can put in correspondence a function $\Phi(f) \in C([0,1])$, that is a continuous version of the winding angle of f around zero. So, we introduced a mapping

$$\Phi\colon \left\{ f \in C([0,1], \mathbb{R}^2) \mid f(0) = \begin{pmatrix}1\\0\end{pmatrix}, \forall t \in [0,1]\, f(t) \ne \begin{pmatrix}0\\0\end{pmatrix} \right\} \to C([0,1]).$$

Let w be a two-dimensional Wiener process starting from the point $\begin{pmatrix}1\\0\end{pmatrix}$. Denote by w_ε the process of the form $w_\varepsilon(t) = w(\varepsilon t)$, $t \in [0,1]$ for $\varepsilon > 0$. Now we can consider the family of the random elements $\Phi_\varepsilon = \Phi(w_\varepsilon)$ with values in $C([0,1])$. Note that these random elements are defined with probability 1, as for any ε the probability that w_ε hits the origin is 0.

In this article we consider the following question: can we find such a function J that for the family of random elements (Φ_ε) the weak LDP or LDP with rate function J holds? In Section 2 we show that the weak LDP holds for (Φ_ε). In Section 3 we show that the estimates of the LDP hold for the class of cylinder sets in $C([0,1])$. However, the full LDP for (Φ_ε) does not hold, as we show in Section 4. In Section 5 we apply the method used in the proof of mixed LDP [6] to obtaining the lower estimate on $\varliminf_{\varepsilon\to 0} \varepsilon \ln P(\Phi_\varepsilon \in G)$ and upper estimate on $\varlimsup_{\varepsilon\to 0} \varepsilon \ln P(\Phi_\varepsilon \in F)$ for open sets G and closed sets F. The use of this method is possible due to the representation of the two-dimensional Brownian moton in a skew-product form [7]. That is, a two-dimensional Brownian motion $w(t)$ can be represented in the form $w(t) = R(t)e^{i\theta(t)}$, where $R(t) = \|w(t)\|$ is a Bessel process, $\theta(t)$ is a Brownian motion with changed argument: $\theta(t) = \beta(U_t)$, where $U_t = \int\limits_0^t \frac{ds}{R_s^2}$, β is a one-dimensional Brownian motion. Here the processes R_t and β_t are independent.

2. Weak LDP for the winding anglle

Denote by $I(x)$ the rate function for two-dimensional Brownian motion starting from the point $\begin{pmatrix}1\\0\end{pmatrix}$, that is,

$$I(x) = \begin{cases} \frac{1}{2}\int\limits_0^1 \|x'(s)\|^2 ds, \; x(0) = \begin{pmatrix}1\\0\end{pmatrix}, \\ \infty, \; x(0) \neq \begin{pmatrix}1\\0\end{pmatrix}. \end{cases}$$

We adopt the agreement that $\int\limits_0^1 \|x'(s)\|^2 ds = \infty$, if x is not absolutely continuous. In this section we prove the following theorem.

Theorem 2.1. *For the random elements $\Phi_\varepsilon \in C([0,1])$, the weak LDP with the rate function $J(\phi) = I(\overline{\Phi^{-1}(\phi)})$ holds.*

Remark 2.1. Here and in what follows, we denote by $\overline{\Phi^{-1}(A)}$ the closure in $C([0,1],\mathbb{R}^2)$ of the set $\Phi^{-1}(A) = \{x \in C([0,1],\mathbb{R}^2) : x(0) = \begin{pmatrix}1\\0\end{pmatrix}, \forall t \in [0,1] \; \|x(t)\| > 0, \Phi(x) \in A\}$. We write $\Phi^{-1}(\phi)$ for $\Phi^{-1}(\{\phi\})$.

Note that while investigating the question of whether the LDP is valid for the family (Φ_ε), it would be natural to try to show that the LDP does hold with the help of contraction principle [8]. Indeed, the random elements Φ_ε are obtained from the random elements w_ε with the help of the mapping Φ. But this mapping is not continuous on $C([0,1],\mathbb{R}^2)$. Nevertheless, for some non-continuous mappings the LDP can be obtained [8], [9]. For example, in the article [10] the LDP for the stopped Wiener process was proved. More precisely, the random elements $w(\varepsilon t \wedge \tau)$ are considered. Here w is a d-dimensional Wiener process, $\tau = \inf\{t : w(t) \in B\} \wedge 1$, $B \subset \mathbb{R}^d$ is some closed set. These random elements are obtained from the random elements w_ε with the help of the mapping Ψ, where for $f \in C([0,1],\mathbb{R}^d)$

$$\tau(f) = \inf\{t \colon f(t) \in B\} \wedge 1; \; \Psi(f)(t) = f(t \wedge \tau(f)), t \in [0,1].$$

The proof of the upper estimate in [10] is based on the relation

$$I(\Psi^{-1}(F)) = I(\overline{\Psi^{-1}(F)})$$

for closed sets F. But in our case the analogous equality $I(\overline{\Phi^{-1}(F)}) = J(F)$ is valid not for all closed sets F. However, it holds for compact sets $F \subseteq C([0,1])$, and this fact allows to obtain a weak LDP for (Φ_ε).

First we show that the function J is lower semicontinuous. We need the following lemma.

Lemma 2.1. *For any compact set $K \subseteq C([0,1])$ we have*

$$\overline{\Phi^{-1}(K)} = \{r(t)e^{i\phi(t)}, 0 \le t \le 1 \mid \phi \in K, \phi(0) = 0, r \in C([0,1]), r(0) = 1\}.$$

That is, the closure of $\Phi^{-1}(K)$ contains only functions of the form $r(t)e^{i\phi(t)}$ with some $\phi \in K$.

Remark 2.2. This property does not hold for non-compact sets. For example, if

$$A = \{\phi \in C([0,1]) \colon \phi(0) = 0, \; \phi(1) \ge 1\},$$

then it can be easily seen that the closure of $\Phi^{-1}(A)$ contains the function $z(t) = 1 - t$, $0 \le t \le 1$, which does not have the form $r(t)e^{i\phi(t)}$ for any $\phi \in A$.

Proof. Let $x \in \overline{\Phi^{-1}(K)}$. Then there exists a sequence $x_n \to x_0, x_n \in \Phi^{-1}(K)$. Let $\phi_n = \Phi(x_n)$. For any n, $\phi_n \in K$. As K is compact, there exists a convergent subsequence $\{\phi_{n_k}\}$ with $\phi_{n_k} \to \phi_0$ for some $\phi_0 \in K$. Let $r_n(t) = \|x_n(t)\|$, $r_0(t) = \|x_0(t)\|$. We have $r_n \to r_0$ in $C([0,1])$. Thus, $x_{n_k}(t) = r_{n_k}(t)e^{i\phi_{n_k}(t)} \xrightarrow[k\to\infty]{} r_0(t)e^{i\phi_0(t)}$ for any $t \in [0,1]$. As the limit is unique, we get

$$\forall\, t \in [0,1]\, x(t) = r_0(t)e^{i\phi_0(t)}.$$

This proves the inclusion

$$\overline{\Phi^{-1}(K)} \subseteq \{r(t)e^{i\phi(t)}, 0 \le t \le 1 \mid \phi \in K, \phi(0) = 0, r \in C([0,1]), r(0) = 1\}.$$

The inclusion

$$\{r(t)e^{i\phi(t)}, 0 \le t \le 1 \mid \phi \in K, \phi(0) = 0, r \in C([0,1]), r(0) = 1\} \subseteq \overline{\Phi^{-1}(K)}$$

is obvious. □

Lemma 2.2. *The function* $J(\phi) = I(\overline{\Phi^{-1}(\phi)})$ *is lower semicontinuous on* $C([0,1])$.

Proof. We show that for any C the set $\{\phi \in C([0,1])\colon J(\phi) \le C\}$ is closed.

Let $\phi_n \in C([0,1])$ be such that $\phi_n \to \phi_0$, and $J(\phi_n) \le C$ for all $n \ge 1$. We prove that $J(\phi_0) \le C$ as well. Choose $x_n \in \overline{\Phi^{-1}(\phi_n)}$ with $I(x_n) \le J(\phi_n) + \frac{1}{n}$. As $I(x_n) \le C + 1$ for every n and the level sets of I are compact, we obtain that all x_n belong to the same compact $K = \{x\colon I(x) \le C + 1\}$. Thus, there exists a subsequence $\{x_{n_k}\}$ with $x_{n_k} \to x_0 \in K$. We have $I(x_0) \le \varliminf_{k\to\infty} I(x_{n_k}) \le C$.

For each $n \ge 1$ we have $x_n \in \overline{\Phi^{-1}(\phi_n)}$. Thus, by Lemma 2.1 applied to compact sets $\{\phi_n\}$ we obtain $x_n(t) = \|x_n(t)\|e^{i\phi_n(t)}$ for all n. As $x_{n_k} \to x_0$ and $\phi_{n_k} \to \phi_0$, we get

$$\|x_{n_k}(t)\|e^{i\phi_{n_k}(t)} \to \|x_0(t)\|e^{i\phi_0(t)}(k \to \infty).$$

On the other hand,

$$\|x_{n_k}(t)\|e^{i\phi_{n_k}(t)} = x_{n_k}(t) \to x_0(t)(k \to \infty).$$

As the limit is unique, we get $x_0(t) = \|x_0(t)\|e^{i\phi_0(t)}$ for any $t \in [0,1]$. Thus, $x_0 \in \overline{\Phi^{-1}(\phi_0)}$, and $J(\phi_0) \le I(x_0) \le C$. □

Now we prove the upper estimate in the weak LDP for Φ_ε.

Proposition 2.1. *For any compact set* $K \subseteq C([0,1])$ *the following holds:*

$$\varlimsup_{\varepsilon\to 0} \varepsilon \ln P(\Phi_\varepsilon \in K) \le -J(K).$$

Proof. We have from the LDP for Brownian motion:

$$\varlimsup_{\varepsilon\to 0} \varepsilon \ln P(\Phi_\varepsilon \in K) = \varlimsup_{\varepsilon\to 0} \varepsilon \ln P(w_\varepsilon \in \Phi^{-1}(K)) \le -I(\overline{\Phi^{-1}(K)}).$$

By Lemma 2.1, we have

$$\overline{\Phi^{-1}(K)} = \bigcup_{\phi\in K} \overline{\Phi^{-1}(\phi)}.$$

Thus, $I(\overline{\Phi^{-1}(K)}) = \inf_{\phi\in K} I(\overline{\Phi^{-1}(\phi)}) = \inf_{\phi\in K} J(\phi) = J(K)$. So, we get

$$\varlimsup_{\varepsilon\to 0} \varepsilon \ln P(\Phi_\varepsilon \in K) \le -I(\overline{\Phi^{-1}(K)}) = -J(K).$$

□

We proceed to the proof of the lower estimate in LDP. We need the following lemma.

Lemma 2.3. *For any open set* $G \subseteq C([0,1])$ *we have*

$$J(G) = I(\Phi^{-1}(G)).$$

Proof. It is clear that $J(G) = \inf_{\phi\in G} I(\overline{\Phi^{-1}(\phi)}) \leq \inf_{\phi\in G} I(\Phi^{-1}(\phi)) = I(\Phi^{-1}(G))$. Let us prove the opposite inequality, that is,

$$I(\Phi^{-1}(G)) \leq J(G) = I\Big(\bigcup_{\phi\in G} \overline{\Phi^{-1}(\phi)}\Big).$$

Take any $x_0 \in \overline{\Phi^{-1}(\phi_0)}$ for some $\phi_0 \in G$. We need to prove that $I(\Phi^{-1}(G)) \leq I(x_0)$.

First, consider the case when x_0 does not pass through the origin:

$$\forall t \in [0,1]\ \|x_0(t)\| \neq 0.$$

We show that in this case $x_0 \in \Phi^{-1}(G)$, and so the desired inequality holds. Indeed, by Lemma 2.1 applied to the compact set $\{\phi_0\}$, x_0 has the form $x_0(t) = \|x_0(t)\|e^{i\phi_0(t)}$. As x_0 does not pass through the origin, we get $\Phi(x_0) = \phi_0$. Thus, $x_0 \in \Phi^{-1}(\phi) \subseteq \Phi^{-1}(G)$.

Now, consider the case $x_0(t_0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for some $t_0 \in [0,1]$. Denote

$$\tau = \inf\{t \in [0,1] \colon \|x_0(t)\| = 0\}, \tau_\delta = \inf\{t \in [0,1] \colon \|x_0(t)\| = \delta\}, \delta \in (0,1).$$

Fix $\varepsilon > 0$ with $B_\varepsilon(\phi_0) \subseteq G$. Choose any function $\psi \in B_{\varepsilon/2}(\phi_0)$ with the property

$$\int_0^1 \psi'(s)^2 ds < +\infty.$$

Define for all $\delta > 0$ functions $x_\delta \in C([0,1], \mathbb{R}^2)$ in such a way:

$$x_\delta(t) = \begin{cases} x_0(t), 0 \leq t \leq \tau_\delta, \\ x_0(\tau_\delta), \tau_\delta \leq t \leq \tau, \\ \delta e^{i(\psi(t)-\psi(\tau)+\phi_0(\tau_\delta))}, t \geq \tau. \end{cases}$$

It is easily seen that $x_\delta \in \Phi^{-1}(G)$ for all δ small enough, and

$$I(x_\delta) = \frac{1}{2}\int_0^{\tau_\delta} \|x_0'(s)\|^2 ds + \frac{\delta^2}{2}\int_\tau^1 \psi'(s)^2 ds.$$

Thus, as $\int_0^1 \psi'(s)^2 ds < +\infty$, we get $\lim_{\delta\to 0} \delta^2 \int_\tau^1 \psi'(s)^2 ds = 0$. We also have

$$I(x_0) \geq \frac{1}{2}\int_0^{\tau_\delta} \|x_0'(s)\|^2 ds$$

for all $\delta > 0$. Therefore, $I(\Phi^{-1}(G)) \leq \overline{\lim_{\delta\to 0}} I(x_\delta) \leq I(x_0)$. □

Now we are ready to prove the lower estimate.

Proposition 2.2. *For any open set $G \subseteq C([0,1])$ the following holds:*

$$\underline{\lim_{\varepsilon\to 0}}\, \varepsilon \ln P(\Phi_\varepsilon \in G) \geq -J(G).$$

Proof. Denote for any $a \in \mathbb{R}^2, x \in C([0,1], \mathbb{R}^2)$

$$(T_a x)(t) = x(t) + a, t \in [0,1].$$

Then $T_a x \in C([0,1], \mathbb{R}^2)$. Set for $A \subseteq C([0,1], \mathbb{R}^2)$

$$T_a(A) = \{T_a x \mid x \in A\},\ T(A) = \bigcup_{a\in\mathbb{R}^2} T_a(A).$$

For any open $G \subseteq C([0,1])$ the set $T(\Phi^{-1}(G))$ is open in $C([0,1],\mathbb{R}^2)$, and

$$I(\Phi^{-1}(G)) = I(T(\Phi^{-1}(G))).$$

(Note that $\Phi^{-1}(G)$ is not open in $C([0,1],\mathbb{R}^2)$, as $x(0) = \begin{pmatrix}1\\0\end{pmatrix}$ for any $x \in \Phi^{-1}(G)$). Thus, we have by the LDP for Wiener process and Lemma 2.3:

$$\begin{aligned}\underline{\lim}_{\varepsilon\to 0}\, \varepsilon \ln P(\Phi_\varepsilon \in G) &= \underline{\lim}_{\varepsilon\to 0}\, \varepsilon \ln P(w_\varepsilon \in \Phi^{-1}(G)) = \\ &= \underline{\lim}_{\varepsilon\to 0}\, \varepsilon \ln P(w_\varepsilon \in T(\Phi^{-1}(G))) \geq -I(T(\Phi^{-1}(G))) = -I(\Phi^{-1}(G)) = -J(G).\end{aligned}$$

□

From Propositions 2.1 and 2.2 we obtain Theorem 2.1.

3. LDP for cylinder sets

In this section we prove the upper estimate of the LDP for cylinder sets in $C([0,1])$. As we will see in Section 4, the full LDP does not hold for (Φ_ε).

Theorem 3.1. *Let $B \subseteq \mathbb{R}^m$ be a closed set, $0 < t_1 < \ldots < t_m \leq 1$,*

$$A = \{\phi \in C([0,1])\colon (\phi(t_1),\ldots,\phi(t_m)) \in B\}.$$

Then the following estimation holds:

$$\overline{\lim_{\varepsilon\to 0}}\, \varepsilon \ln P(\Phi_\varepsilon \in A) \leq -J(A).$$

Remark 3.1. The lower estimate of the LDP for all open sets $G \subset C([0,1])$ was obtained in Section 2.

For the proof we need several lemmas.

Lemma 3.1. *Let $A \subseteq C([0,1])$ be closed, $x_0 \in \overline{\Phi^{-1}(A)}$. If $\|x_0(t)\| > 0$ for any $t \in [0,1]$ (that is, if x_0 does not pass through the origin), then $x_0 \in \Phi^{-1}(A)$.*

Proof. Choose $x_n \in \Phi^{-1}(A)$ with $x_n \to x_0$. As Φ is continuous at x_0, we have

$$\Phi(x_n) \to \Phi(x_0).$$

As A is closed, we get $\Phi(x_0) \in A$, and thus $x_0 \in \Phi^{-1}(A)$. □

Lemma 3.2. *Let $A \subseteq C([0,1])$, $x_0 \in \overline{\Phi^{-1}(A)}$. Let $\tau = \inf\{t \in [0,1]\colon \|x_0(t)\| = 0\} \wedge 1$, $y_0(t) = x_0(t \wedge \tau)$. Then $y_0 \in \overline{\Phi^{-1}(A)}$.*

Proof. It is sufficient to consider only the case when x_0 passes through the origin. Choose $x_n \in \Phi^{-1}(A)$ with $x_n \to x_0$. Set $\tau_\delta = \inf\{t : \|x_0(t)\| = \delta\}$ for $0 < \delta < 1$. Let

$$y^n_\delta(t) = \begin{cases} x_n(t), t \leq \tau_\delta, \\ \frac{\|x_n(\tau_\delta)\|}{\|x_n(t)\|} x_n(t), t \geq \tau_\delta. \end{cases}$$

Then $\Phi(y^n_\delta) = \Phi(x_n) \in A$, $y^n_{1/n} \to y_0\ (n \to \infty)$ in $C([0,1],\mathbb{R}^2)$. Thus, $y_0 \in \overline{\Phi^{-1}(A)}$. □

Lemma 3.3. *Let $t_1 < t_2$ be real numbers, $\phi\colon [t_1,t_2] \to \mathbb{R}$ be a continuous function with $\int_{t_1}^{t_2} \phi'(s)^2 ds < +\infty$, $h\colon [t_1,t_2] \to \mathbb{R}$ be a positive continuous function, $\{\alpha_n\}_{n=1}^\infty$ and $\{\beta_n\}_{n=1}^\infty$ be two sequences of real numbers with $\alpha_n \to 0$, $\beta_n \to 0$. Then there exists a sequence of functions $\psi_n \in C([t_1,t_2])$ with $\int_{t_1}^{t_2} \psi_n'(s)^2 ds < +\infty$ that satisfies the following conditions:*

- *$\psi_n(t_1) = \phi(t_1) + \alpha_n$ for every n;*

- $\psi_n(t_2) = \phi(t_2) + \beta_n$ *for every* n*;*
- $\int\limits_{t_1}^{t_2} h(s)\psi_n'(s)^2 ds \to \int\limits_{t_1}^{t_2} h(s)\phi'(s)^2 ds.$

Proof. Set $l_n(t) = \alpha_n + \frac{\beta_n - \alpha_n}{t_2 - t_1}(t - t_1)$, $\psi_n(t) = \phi(t) + l_n(t)$. We have

$$\int\limits_{t_1}^{t_2} h(s)\psi_n'(s)^2 ds - \int\limits_{t_1}^{t_2} h(s)\phi'(s)^2 ds = \int\limits_{t_1}^{t_2} h(s)l_n'(s)^2 ds + 2\int\limits_{t_1}^{t_2} h(s)\phi'(s)l_n'(s) ds =$$

$$= \left(\frac{\beta_n - \alpha_n}{t_2 - t_1}\right)^2 \int\limits_{t_1}^{t_2} h(s) ds + 2\left(\frac{\beta_n - \alpha_n}{t_2 - t_1}\right) \int\limits_{t_1}^{t_2} h(s)\phi'(s) ds \to 0\,(n \to \infty).$$

□

Lemma 3.4. *Let* $B \subseteq \mathbb{R}^m$ *be a closed set,* $0 < t_1 < \ldots < t_m \le 1$,

$$A = \{\phi \in C([0,1])\colon \phi(0) = 0, (\phi(t_1), \ldots, \phi(t_m)) \in B\}.$$

Then

$$I(\Phi^{-1}(A)) = I(\overline{\Phi^{-1}(A)}) = J(A).$$

Proof. As $I(\overline{\Phi^{-1}(A)}) \le J(A) \le I(\Phi^{-1}(A))$, we need to prove only

$$I(\Phi^{-1}(A)) \le I(\overline{\Phi^{-1}(A)}).$$

Take any $x_0 \in \overline{\Phi^{-1}(A)}$. We will show that $I(\Phi^{-1}(A)) \le I(x_0)$.

Without loss of generality, we consider $t_m = 1$ everywhere in the proof. First consider the case when x_0 does not pass through the origin. By Lemma 3.1, we get $x_0 \in \Phi^{-1}(A)$, and thus $I(\Phi^{-1}(A)) \le I(x_0)$.

Now, we assume that x_0 passes through the origin. Denote

$$\tau = \inf\{t \in [0,1] : \|x(t)\| = 0\}.$$

By Lemma 3.2, we may consider $x_0(t) = \begin{pmatrix}0\\0\end{pmatrix}$ for $t \ge \tau$. Set $t_0 = 0$. Let $k, 1 \le k \le m$ be such that $\tau \in (t_{k-1}, t_k]$. We have then

$$x_0(t_0) = \begin{pmatrix}1\\0\end{pmatrix} \ne \begin{pmatrix}0\\0\end{pmatrix}, x_0(t_1) \ne \begin{pmatrix}0\\0\end{pmatrix}, \ldots, x_0(t_{k-1}) \ne \begin{pmatrix}0\\0\end{pmatrix}, x_0(t_k) = \begin{pmatrix}0\\0\end{pmatrix}.$$

Choose a sequence $x_n \to x_0$ with $x_n \in \Phi^{-1}(A)$ for each n. Denote $\phi_n = \Phi(x_n)$. Let $\phi(t)$ be a winding angle of $x_0(t)$ defined on $[0, \tau)$. We have $\phi_n(t_i) \to \phi(t_i)$, $n \to \infty$ for $i = 1, \ldots, k-1$.

Fix any $\alpha > 1$. Choose functions $\psi_{n,i}\colon [t_{i-1}, t_i] \to \mathbb{R}$ for $i = 1, \ldots, k-1$ with the properties

- $\psi_{n,i}(t_{i-1}) = \phi_n(t_{i-1})$;
- $\psi_{n,i}(t_i) = \phi_n(t_i)$;
- $\int\limits_{t_{i-1}}^{t_i} \|x_0(s)\|^2 \psi_{n,i}'(s)^2 ds \to \int\limits_{t_{i-1}}^{t_i} \|x_0(s)\|^2 \phi'(s)^2 ds\,(n \to \infty)$.

Such functions exist by Lemma 3.3.

We put

$$\psi_n(t) = \begin{cases} \psi_{n,i}(t), t_{i-1} \le t \le t_i, i = 1, \ldots, k-1, \\ \phi_n(t_{k-1}), t_{k-1} \le t \le t_{k-1} + \frac{t_k - t_{k-1}}{\alpha}, \\ \phi_n(t_i), i = k, k+1, \ldots, m, \\ \text{linear on each closed interval } [t_{k-1} + \frac{t_k - t_{k-1}}{\alpha}, t_k], [t_k, t_{k+1}], \ldots, [t_{m-1}, t_m]. \end{cases}$$

As ψ_n is piecewise linear on $[t_{k-1}+\frac{t_k-t_{k-1}}{\alpha},1]$, we can choose $\delta_n>0$ with

$$\delta_n^2 \int\limits_{t_{k-1}+\frac{t_k-t_{k-1}}{\alpha}}^{1} \psi_n'(s)^2 ds < \frac{1}{2^n}$$

and

$$\tau_n = \inf\{t\colon \|x_0(t)\| = \delta_n\} > t_{k-1}.$$

Let

$$\rho_n(t) = \begin{cases} \|x_0(t)\|, 0 \le t \le t_{k-1}, \\ \|x_0(t_{k-1}+\alpha(t-t_{k-1}))\|, 0 \le t \le t_{k-1}+\frac{\tau_n-t_{k-1}}{\alpha}, \\ \delta_n, t \ge t_{k-1}+\frac{\tau_n-t_{k-1}}{\alpha}. \end{cases}$$

Set $y_n(t)=\rho_n(t)e^{i\psi_n(t)}$, $t\in[0,1]$. We get

$$\begin{aligned} 2I(y_n) = \sum_{i=1}^{k-1}\left(\int\limits_{t_{i-1}}^{t_i}\left(\frac{d}{ds}\|x_0(s)\|\right)^2 ds + \int\limits_{t_{i-1}}^{t_i}\|x_0(s)\|^2\psi_{n,i}'(s)^2 ds\right) + \\ + \alpha^2\int\limits_{t_{k-1}}^{\tau_n}\left(\frac{d}{ds}\|x_0(s)\|\right)^2 ds + \delta_n^2\int\limits_{t_{k-1}+\frac{t_k-t_{k-1}}{\alpha}}^{1}\psi_n'(s)^2 ds \le \\ \le \sum_{i=1}^{k-1}\left(\int\limits_{t_{i-1}}^{t_i}\left(\frac{d}{ds}\|x_0(s)\|\right)^2 ds + \int\limits_{t_{i-1}}^{t_i}\|x_0(s)\|^2\psi_{n,i}'(s)^2 ds\right) + \\ + \alpha^2\int\limits_{t_{k-1}}^{1}\|x_0'(s)\|^2 ds + \frac{1}{2^n} \xrightarrow[n\to\infty]{} \\ \xrightarrow[n\to\infty]{} \sum_{i=1}^{k-1}\left(\int\limits_{t_{i-1}}^{t_i}\left(\frac{d}{ds}\|x_0(s)\|\right)^2 ds + \int\limits_{t_{i-1}}^{t_i}\|x_0(s)\|^2\phi'(s)^2 ds\right) + \alpha^2\int\limits_{t_{k-1}}^{1}\|x_0'(s)\|^2 ds \le \\ \le \alpha^2\int\limits_0^1\|x_0'(s)\|^2 ds = 2\alpha^2 I(x_0). \end{aligned}$$

We obtain therefore

$$\varliminf_{n\to\infty} I(y_n) \le \alpha^2 I(x_0).$$

As $\Phi(y_n)\in A$ for each n, we get $I(\Phi^{-1}(A)) \le \varliminf_{n\to\infty} I(y_n)$, and thus

$$I(\Phi^{-1}(A)) \le \alpha^2 I(x_0).$$

As $\alpha>1$ is arbitrary, we get

$$I(\Phi^{-1}(A)) \le I(x_0).$$

□

Now we prove Theorem 3.1.

Proof. From the LDP for Brownian motion we have

$$\varlimsup_{\varepsilon\to 0}\varepsilon\ln P(\Phi_\varepsilon\in A) \le -I(\overline{\Phi^{-1}(A)}).$$

By Lemma 3.4 we have $J(A) = I(\overline{\Phi^{-1}(A)})$. Thus,

$$\overline{\lim_{\varepsilon\to 0}}\, \varepsilon \ln P(\Phi_\varepsilon \in A) \le -J(A).$$

□

4. The absence of the large-deviation principle for the family (Φ_ε)

Let us show that the LDP for the family $(\Phi_\varepsilon)_{\varepsilon>0}$ cannot hold. First we prove that the LDP with the rate function $J(\phi) = \inf_{x\in\overline{\Phi^{-1}(\phi)}} I(x)$, where

$$I(x) = \begin{cases} \frac{1}{2}\int\limits_0^1 \|x'(s)\|^2 ds,\ x(0) = \begin{pmatrix}1\\0\end{pmatrix}, \\ \infty,\ x(0) \ne \begin{pmatrix}1\\0\end{pmatrix}, \end{cases}$$

is not satisfied.

Proposition 4.1. *There exists such a closed set $A \subseteq C([0,1])$ that the following conditions hold:*

- $\overline{\lim_{\varepsilon\to 0}}\, \varepsilon \ln P(\Phi_\varepsilon \in A) \ge -\frac{1}{2}$;
- *for some $C > \frac{1}{2}$: $I(\Phi^{-1}(A)) \ge C$, and for any $\phi \in A$*

$$I(\overline{\Phi^{-1}(\phi)}) \ge C.$$

The proof of this propositon is based on the following lemma.

Lemma 4.1. *For any $\alpha > \frac{\pi}{2}$, with probability* 1 *the following relation holds:*

$$\lim_{\varepsilon\to 0} \varepsilon \ln P(\Phi_\varepsilon(1) \ge \alpha) = -\frac{1}{2}.$$

Proof. We fix some $\alpha > \frac{\pi}{2}$. We have to prove the following:

$$-\frac{1}{2} \le \underline{\lim_{\varepsilon\to 0}}\, \varepsilon \ln P(\Phi_\varepsilon(1) \ge \alpha) \le \overline{\lim_{\varepsilon\to 0}}\, \varepsilon \ln P(\Phi_\varepsilon(1) \ge \alpha) \le -\frac{1}{2}.$$

First we make the estimate from above. We have

$$\left\{x = \begin{pmatrix}x^{(1)}\\x^{(2)}\end{pmatrix} \in C([0,1],\mathbb{R}^2) : x(0) = \begin{pmatrix}1\\0\end{pmatrix}, \Phi(x)(1) \ge \alpha\right\} \subseteq$$
$$\subseteq \left\{x : \Phi(x)(1) \ge \frac{\pi}{2}\right\} \subseteq \{x : x^{(1)}(1) \le 0\}.$$

So,

$$\overline{\lim_{\varepsilon\to 0}}\, \varepsilon \ln P(\Phi_\varepsilon(1) \ge \alpha) \le \overline{\lim_{\varepsilon\to 0}}\, \varepsilon \ln P(w^{(1)}_\varepsilon(1) \le 0) = -\frac{1}{2}.$$

Here $w = \begin{pmatrix}w^{(1)}\\w^{(2)}\end{pmatrix}$ is a two-dimensional Wiener process starting from the point $\begin{pmatrix}1\\0\end{pmatrix}$, $w^{(1)}_\varepsilon(t) = w^{(1)}(\varepsilon t), t \in [0,1]$. Now we make the lower estimate. For any $\delta \in (0,1)$ we denote $\beta_\delta = \frac{2\alpha}{\delta}$ and consider the trajectory $\begin{pmatrix}x_\delta(t)\\y_\delta(t)\end{pmatrix} \in C([0,1],\mathbb{R}^2)$, defined by relations

$$x_\delta(t) + iy_\delta(t) = z_\delta(t), z_\delta(t) = \begin{cases} 1-t, 0 \le t \le 1-\delta; \\ \delta e^{i\beta_\delta(t-(1-\delta))}, 1-\delta \le t \le 1. \end{cases}$$

It can be easily seen that

$$I(z_\delta) = \frac{1}{2}(1 - \delta + \beta_\delta^2 \delta^3) = \frac{1}{2}(1 - \delta + 4\alpha^2\delta) \to \frac{1}{2}(\delta \to 0).$$

Let $G = \left\{ x \in C([0,1],\mathbb{R}^2)\colon x(0) = \begin{pmatrix}1\\0\end{pmatrix}, \forall t \in [0,t]\ \|x(t)\| > 0, \Phi(x)(1) > \alpha \right\}$. We have then $z_\delta \in G$, and

$$\varliminf_{\varepsilon\to 0} \varepsilon \ln P(\Phi_\varepsilon(1) \geq \alpha) \geq \varliminf_{\varepsilon\to 0} \varepsilon \ln P(w \in G) \geq -I(G) \geq -I(z_\delta).$$

Using $I(z_\delta) \to \frac{1}{2} (\delta \to 0)$, we get $\varliminf_{\varepsilon\to 0} \varepsilon \ln P(\Phi_\varepsilon(1) \geq \alpha) \geq -\frac{1}{2}$. □

Now we prove the Proposition 4.1.

Proof. We divide our proof into 3 parts. In the first part we construct the set A. In the second part we prove that the set A is closed. In the third part we find such $C > \frac{1}{2}$ that the second condition of the proposition is satisfied.

(1) Let $a \in (0, \frac{\pi}{2})$ be some positive number such that $\frac{\sin x}{x} > \frac{3}{4}$ for $0 < x < a$. We fix an increasing sequence $\alpha_k \to \infty\,(k \to \infty)$, such that $\alpha_k > \frac{\pi}{2}$ for any k, and a decreasing sequence $\varepsilon_k \to 0\,(k \to \infty)$. We also need a decreasing sequence $t_k \to 0\,(k \to \infty)$ with $0 < t_k < \frac{a^2}{2}$ for each k, which will be built later. Set $A = \bigcup_{k=1}^{\infty} A_k$, where A_k are defined as

$$A_k = \left\{ \phi \in C([0,1])\colon \phi(0) = 0, \phi(1) \geq \alpha_k, \sup_{t\in[t_k,t_{k-1}]} \frac{\phi(t)}{\sqrt{2t}} \geq 1 \right\}.$$

Now we specify the sequence t_k. We choose t_k inductively in the following way. Set $t_0 = \frac{a^2}{4}$. Having constructed t_{k-1} for some $k \geq 1$, choose $n = n(k) \geq k$ such that

$$\varepsilon_n \ln P(\Phi_{\varepsilon_n}(1) \geq \alpha_k) > -\frac{1}{2} - \frac{1}{2^k}.$$

This choice is possible due to Lemma 4.1. Now find $t_k, 0 < t_k < t_{k-1}$, in such a way that

$$P(\Phi_{\varepsilon_n} \in A_k) > \frac{1}{2} P(\Phi_{\varepsilon_n}(1) \geq \alpha_k).$$

This can be done, as

$$\lim_{u\to 0} P\left(\sup_{t\in[u,t_{k-1}]} \frac{\Phi_{\varepsilon_n}(t)}{\sqrt{2t}} \geq 1 \right) = 1,$$

which follows easily from the law of the iterated logarithm.

So, we provided an algorithm to construct sets A_k. Now we have

$$\begin{aligned} &\varepsilon_{n(k)} \ln P(\Phi_{\varepsilon_{n(k)}} \in A) \geq \varepsilon_{n(k)} \ln P\left(\Phi_{\varepsilon_{n(k)}} \in A_k\right) \geq \\ \geq &\varepsilon_{n(k)} \ln \left(\frac{1}{2} P(\Phi_{\varepsilon_{n(k)}}(1) \geq \alpha_k) \right) > -\varepsilon_{n(k)} \ln 2 - \frac{1}{2} - \frac{1}{2^k}. \end{aligned}$$

From here we get $\varlimsup_{k\to\infty} \varepsilon_{n(k)} \ln P(\Phi_{\varepsilon_{n(k)}} \in A) \geq -\frac{1}{2}$.

(2) We show that the set A is closed. Let the sequence $\{\phi_n\}_{n=1}^{\infty}$ be such that for any n: $\phi_n \in A$, and $\phi_n \to \phi (n \to \infty)$. Let us show that $\phi \in A$ as well. As $A = \bigcup_{k=1}^{\infty} A_k$, then for any n there exists a number $k(n)$ such that $\phi_n \in A_{k(n)}$. As $\phi_n(1) \to \phi(1) (n \to \infty)$, then the sequence $\{\phi_n(1)\}$ is bounded, and so the set $\{k(n)\}$ is bounded. Therefore, there exists k_0 such that $\phi_n \in A_{k_0}$ for infinitely many indices n. It can be easily seen that all sets A_k are closed, and thus $\phi \in A_{k_0} \subseteq A$.

(3) Now we check the second condition of the proposition. Let us estimate $I(\overline{\Phi^{-1}(\phi)})$ for any $\phi \in A$. Choose any $z \in \Phi^{-1}(\phi)$. Since $\sup\limits_{t\in[t_k,t_{k-1}]} \frac{\phi(t)}{\sqrt{2t}} \geq 1$ for some k, there exists $h \in [t_k, t_{k-1}]$ such that $\phi(h) \geq \sqrt{2h}$. Thus, the trajectory z has to cross the line l defined by the equation $y = x \tan \sqrt{2h}$ before the moment h, and the same property obviously holds for any $z \in \overline{\Phi^{-1}(\phi)}$. As the distance from the point $z(0) = \begin{pmatrix}1\\0\end{pmatrix}$ to the line l is equal to $\sin \sqrt{2h}$, and $h < \frac{a^2}{2}$, then

$$I(x) \geq \frac{1}{2}\int_0^h |x'(u)|^2 du \geq \frac{(\sin\sqrt{2h})^2}{2h} = \left(\frac{\sin\sqrt{2h}}{\sqrt{2h}}\right)^2 > \left(\frac{3}{4}\right)^2 = \frac{9}{16}.$$

Thus, for any $\phi \in A$ $I(\overline{\Phi^{-1}(\phi)}) \geq \frac{9}{16}$. The same considerations show that $I(\Phi^{-1}(A)) \geq \frac{9}{16}$. So, the second condition of the proposition is satisfied with $C = \frac{9}{16}$.

□

Now we show that the family of random elements (Φ_ε) can not satisfy LDP with any rate function $\tilde{I}$. For this we need several lemmas. We denote

$$I(x) = \frac{1}{2}\int_0^1 \|x'(u)\|^2 du.$$

Lemma 4.2. *For any $\phi \in C([0,1])$ the following equality holds:*

$$\bigcap_{\delta>0} \Phi^{-1}(B_\delta(\phi)) = \Phi^{-1}(\phi).$$

Proof. If $x \in \Phi^{-1}(B_\delta(\phi))$ for all $\delta > 0$, then $\Phi(x) \in B_\delta(\phi)$ for any $\delta > 0$. This means that $\Phi(x) = \phi$. □

Lemma 4.3. *For any $\phi \in C([0,1])$ such that $\phi(0) = 0$ the following holds:*

$$\bigcap_{\delta>0} \overline{\Phi^{-1}(B_\delta(\phi))} = \overline{\Phi^{-1}(\phi)}.$$

Proof. Let $x_0 \in \bigcap\limits_{\delta>0} \overline{\Phi^{-1}(B_\delta(\phi))}$. Then for any $\delta > 0$ there exists $x_\delta \in \Phi^{-1}(B_\delta(\phi))$ such that $\|x_\delta - x_0\| < \delta$. Therefore, $x_\delta \xrightarrow[\delta\to 0]{} x_0$.

Now we choose y_δ in such a way that $y_\delta \in \Phi^{-1}(\phi)$ and $y_\delta \to x_0$ $(\delta \to 0)$.

Let $x_\delta(t) = r_\delta(t)e^{i\phi_\delta(t)}$. Set $y_\delta(t) = r_\delta(t)e^{i\phi(t)}$. We show that $\|y_\delta - x_\delta\| \to 0$ $(\delta \to 0)$. For any $t \in [0,1]$:

$$\|y_\delta(t) - x_\delta(t)\| = r_\delta(t)|e^{i\phi(t)} - e^{i\phi_\delta(t)}| \leq r_\delta(t)|\phi(t) - \phi_\delta(t)|.$$

Thus, $\|y_\delta - x_\delta\| \leq |r_\delta| \cdot \|\phi - \phi_\delta\| \to 0$ $(\delta \to 0)$. Now we have $x_\delta \to x_0$, $\|y_\delta - x_\delta\| \to 0$. Therefore, $y_\delta \to x_0 (\delta \to 0)$. As $\Phi(y_\delta) = \phi$, then $y_\delta \in \Phi^{-1}(\phi)$. So, $x_0 \in \overline{\Phi^{-1}(\phi)}$. □

Lemma 4.4. *If $I(\overline{\Phi^{-1}(\phi)}) < +\infty$, then $I(\overline{\Phi^{-1}(B_\delta(\phi))}) \xrightarrow[\delta\to 0]{} I(\overline{\Phi^{-1}(\phi)})$.*

Proof. We show that for any sequence $\delta_n \to 0, \delta_n > 0$ the following holds:

$$I(\overline{\Phi^{-1}(B_{\delta_n}(\phi))} \xrightarrow[n\to\infty]{} I(\overline{\Phi^{-1}(\phi)}).$$

As $I(\overline{\Phi^{-1}(B_{\delta_n}(\phi))} \leq I(\overline{\Phi^{-1}(\phi)})$, then all we need to show is that for any $\varepsilon > 0$ the inequality

$$I(\overline{\Phi^{-1}(B_{\delta_n}(\phi))}) \leq I(\overline{\Phi^{-1}(\phi)}) - 2\varepsilon$$

can not hold for all n.

Suppose the opposite, that for some $\varepsilon > 0$ we have for all n:

$$I(\overline{\Phi^{-1}(B_{\delta_n}(\phi))}) \le I(\overline{\Phi^{-1}(\phi)}) - 2\varepsilon.$$

Then for any n we can find $x_n \in \overline{\Phi^{-1}(B_{\delta_n}(\phi))}$ such that $I(x_n) \le I(\overline{\Phi^{-1}(\phi)}) - \varepsilon$.

But $I(\overline{\Phi^{-1}(\phi)}) < +\infty$ by the condition of lemma. Thus, $I(x_n) \le I(\Phi^{-1}(\phi)) - \varepsilon < +\infty$ for all n.

The set $K = \{x : I(x) \le I(\Phi^{-1}(\phi)) - \varepsilon\}$ is compact. Therefore, all x_n are in one compact K. Thus, there exists a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of the sequence $\{x_n\}$. Let $x_{n_k} \to x_0 (k \to \infty)$. As x_0 is in the same compact K, then $I(x_0) \le I(\overline{\Phi^{-1}(\phi)}) - \varepsilon$. On the other hand, $x_0 \in \bigcap\limits_k \overline{\Phi^{-1}(B_{\delta_{n_k}}(\phi))} = \overline{\Phi^{-1}(\phi)}$.

So, $x_0 \in \overline{\Phi^{-1}(\phi)}$ and $I(x_0) \le I(\overline{\Phi^{-1}(\phi)}) - \varepsilon$. We got a contradiction. □

Lemma 4.5. *If $I(\overline{\Phi^{-1}(\phi)}) = +\infty$, then $I(\overline{\Phi^{-1}(B_\delta(\phi))}) \xrightarrow[\delta \to 0]{} +\infty$.*

Proof. It is clear that $I(\overline{\Phi^{-1}(B_\delta(\phi))})$ does not decrease as $\delta \to 0$. Therefore, there exists a finite or infinite limit $\lim\limits_{\delta \to 0} I(\overline{\Phi^{-1}(B_\delta(\phi))})$. Suppose that this limit is finite:

$$\lim_{\delta \to 0} I(\overline{\Phi^{-1}(B_\delta(\phi))}) = A < +\infty.$$

Then for any sufficiently small $\delta > 0$ there exists $x_\delta \in \overline{\Phi^{-1}(B_\delta(\phi))}$ with $I(x_\delta) \le A+1$. As the level sets of I are compact, we get, as in proof of Lemma 4.4, that $x_{\delta_n} \to x_0$ for some sequence $\{\delta_n\}_{n=1}^{\infty}$, $\delta_n \to 0\,(n \to \infty)$. Therefore, we have

- $I(x_0) \le A + 1$;
- $x_0 \in \bigcap\limits_n \overline{\Phi^{-1}(B_{\delta_n}(\phi))} = \overline{\Phi^{-1}(\phi)}$.

Thus, $I(\overline{\Phi^{-1}(\phi)}) \le I(x_0) \le A + 1 < +\infty$. This is a contradiction. □

Lemma 4.6. *For any $\phi \in C([0,1])$ the following convergence holds:*

$$I(\overline{\Phi^{-1}(B_\delta(\phi))}) \xrightarrow[\delta \to 0]{} I(\overline{\Phi^{-1}(\phi)}).$$

This lemma is a consequence of Lemmas 4.4 and 4.5.

Lemma 4.7. *If for the random elements (Φ_ε) the large-deviation principle with a rate function $\tilde{I}$ holds, then for any $\phi \in C([0,1])$, $\phi(0) = 0$, the following inequality holds:*

$$\tilde{I}(\phi) \ge I(\overline{\Phi^{-1}(\phi)}) = \inf_{x \in \overline{\Phi^{-1}(\phi)}} \frac{1}{2} \int_0^1 \|x'(u)\|^2 du.$$

Proof. With the help of the supposed LDP for (Φ_ε) and LDP for (w_ε) we have:

$$- \tilde{I}(B_\delta(\phi)) \le \overline{\lim_{\varepsilon \to 0}}\, \varepsilon \ln P(\Phi_\varepsilon \in B_\delta(\phi)) = \overline{\lim_{\varepsilon \to 0}}\, \varepsilon \ln P(\Phi(w_\varepsilon) \in B_\delta(\phi)) =$$
$$= \overline{\lim_{\varepsilon \to 0}}\, \varepsilon \ln P(w_\varepsilon \in \Phi^{-1}(B_\delta(\phi))) \le -I(\overline{\Phi^{-1}(B_\delta(\phi))}).$$

From here we get $I(\overline{\Phi^{-1}(B_\delta(\phi))}) \le \tilde{I}(B_\delta(\phi))$. But $\tilde{I}(B_\delta(\phi)) \le \tilde{I}(\phi)$. So, we get $I(\overline{\Phi^{-1}(B_\delta(\phi))}) \le \tilde{I}(\phi)$.

Tending $\delta \to 0$ and using Lemma 4.6, we get $I(\overline{\Phi^{-1}(\phi)}) \le \tilde{I}(\phi)$. □

Theorem 4.1. *The large-deviation principle with any rate function $\tilde{I}$ cannot hold for the family (Φ_ε).*

Proof. We consider the set $A = \bigcup_{k=1}^{\infty} A_k$ from Proposition 4.1. By Lemma 4.7, we get

$$\forall \phi \in A \;\; \tilde{I}(\phi) \geq I(\overline{\Phi^{-1}(\phi)}).$$

On the other hand, by Proposition 4.1, for any $\phi \in A$

$$I(\overline{\Phi^{-1}(\phi)}) \geq C.$$

Thus, $\tilde{I}(A) \geq C > \frac{1}{2}$. But this contradicts the inequality

$$\varlimsup_{\varepsilon \to 0} \varepsilon \ln P(\Phi_\varepsilon \in A) \geq -\frac{1}{2}.$$

□

5. Exponential estimates on winding angles

Despite of the abscence of the LDP for the family of random elements (Φ_ε), the exponential estimates on the behaviour of the probabilities $P(\Phi_\varepsilon \in A)$, while $\varepsilon \to 0$, still can be found. One of the methods to obtain such estimates is to apply the LDP for the Wiener process to probabilities $P(w_\varepsilon \in \Phi^{-1}(A))$. Here we use another approach based on the representation of the winding angle of the Wiener process w in the form $\Phi(t) = \beta\left(\int_0^t \frac{ds}{\|w(s)\|^2}\right)$. This approach is analogous to the mixed large-deviation principle from [6]. But in our case the estimates obtained in such a way coincide with the estimates obtained with the help of the first approach.

In this section we use the following notation:

- w is a two-dimensional Wiener process, $w(0) = \begin{pmatrix}1\\0\end{pmatrix}$;
- $w_\varepsilon(t) = w(\varepsilon t), t \in [0,1]$;
- β is an idependent from w one-dimensional Wiener process, $\beta(0) = 0$;
- $\beta_\varepsilon(t) = \beta(\varepsilon t), t \in [0, \infty)$;
- $\mathfrak{B} = \left\{x \in C([0,1], \mathbb{R}^2) \colon x(0) = \begin{pmatrix}1\\0\end{pmatrix}, \forall t \in [0,1] \; \|x(t)\| > 0\right\}$;
- $\mathfrak{D} = \{r \in C([0,1]) \colon r(0) = 1, \forall t \in [0,1] \, r(t) > 0\}$.

From the relation $\Phi_\varepsilon \overset{d}{=} \beta_\varepsilon\left(\int_0^\cdot \frac{ds}{\|w_\varepsilon(s)\|^2}\right)$ it follows that the study of the asympotical behaviour of the distributions of the random elements Φ_ε is equivalent to the study of random elements $\beta_\varepsilon\left(\int_0^\cdot \frac{ds}{\|w_\varepsilon(s)\|^2}\right)$.

We will need several technical lemmas.

Lemma 5.1. *Let $A \subseteq C([0,T])$ be a measurable set, $x_0 \in C([0,T], \mathbb{R}^2)$ be some function satisfying the conditions $x_0(0) = \begin{pmatrix}1\\0\end{pmatrix}$, $\|x_0(t)\| > 0$ for all $t \in [0,1]$. Then the following estimation takes place:*

$$-\frac{1}{2} \inf_{\phi \in A^\circ, \phi(0)=0} \int_0^T \|x_0(u)\|^2 \phi'(u)^2 du \leq \varliminf_{\varepsilon \to 0} \varepsilon \ln P\left(\beta_\varepsilon\left(\int_0^\cdot \frac{ds}{\|x_0(s)\|^2}\right) \in A\right) \leq$$

$$\leq \varlimsup_{\varepsilon \to 0} \varepsilon \ln P\left(\beta_\varepsilon\left(\int_0^\cdot \frac{ds}{\|x_0(s)\|^2}\right) \in A\right) \leq -\frac{1}{2} \inf_{\phi \in \overline{A}, \phi(0)=0} \int_0^T \|x_0(u)\|^2 \phi'(u)^2 du.$$

Proof. Let $h(t) = \int\limits_0^t \frac{ds}{\|x_0(s)\|^2}$, $t \in [0, T]$; $B = \{\phi \circ h^{-1} \mid \phi \in A\}$.

Then we have

$$P\left(\beta_\varepsilon\left(\int\limits_0^{\cdot} \frac{ds}{\|x_0(s)\|^2}\right) \in A\right) = P(\beta_\varepsilon(h(\cdot)) \in A) = P(\beta_\varepsilon|_{[0,h(T)]} \in B).$$

By the LDP for Wiener process, we get

$$-\frac{1}{2} \inf_{\psi \in B^\circ, \psi(0)=0} \int\limits_0^{h(T)} \psi'(u)^2 du \le \varliminf_{\varepsilon \to 0} \varepsilon \ln P(\beta_\varepsilon|_{[0,h(T)]} \in B) \le$$

$$\le \varlimsup_{\varepsilon \to 0} \varepsilon \ln P(\beta_\varepsilon|_{[0,h(T)]} \in B) \le -\frac{1}{2} \inf_{\psi \in \overline{B}, \psi(0)=0} \int\limits_0^{h(T)} \psi'(u)^2 du.$$

Now the use of the change of variables formula gives the needed estimation. ☐

Lemma 5.2. *Let $x_0 \in C([0, T_0], \mathbb{R}^2)$ be a function such that*

$$x_0(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \forall t \in [0, T_0] \ \|x_0(t)\| > 0.$$

Then for any $L > 0$, $\mu > 0$ there exists a neighborhood $U_\eta(x_0)$, $\eta = \eta(L) > 0$, such that

$$\varlimsup_{\varepsilon \to 0} \varepsilon \ln P\Bigg(\exists x \in U_\eta(x_0)\, \exists t \in [0, T_0]\colon \left|\beta_\varepsilon\left(\int\limits_0^t \frac{ds}{\|x(s)\|^2}\right) - \beta_\varepsilon\left(\int\limits_0^t \frac{ds}{\|x_0(s)\|^2}\right)\right| > \mu\Bigg) < -L.$$

Proof. We choose h in such a way that $\frac{\mu^2}{2h} > L$. Find a neighbourhood $U_\eta(x_0)$ such that the following condition holds:

$$\forall x \in U_\eta(x_0)\, \forall t \in [0, T_0] \left| \int\limits_0^t \frac{ds}{\|x(s)\|^2} - \int\limits_0^t \frac{ds}{\|x_0(s)\|^2} \right| < h.$$

Let $T = \sup\limits_{x \in U_\eta(x_0)} \int\limits_0^{T_0} \frac{ds}{\|x(s)\|^2}$. Then

$$P\left(\exists x \in U_\eta(x_0)\, \exists t \in [0, T_0]\colon \left|\beta_\varepsilon\left(\int\limits_0^t \frac{ds}{\|x(s)\|^2}\right) - \beta_\varepsilon\left(\int\limits_0^t \frac{ds}{\|x_0(s)\|^2}\right)\right| > \mu\right) \le$$

$$\le P\left(\exists s_1, s_2 \in [0, T]\colon |s_1 - s_2| \le h, |\beta_\varepsilon(s_1) - \beta_\varepsilon(s_2)| \ge \mu\right).$$

Put

$$F = \{\phi \in C([0, T_0]) \mid \phi(0) = 0, \exists\, s, t \in [0, T]:\ 0 < t - s \le h, |\phi(t) - \phi(s)| \ge \mu\}.$$

It can be easily seen that the set F is closed. But for any function $\psi \in F$ the following holds:

$$\int\limits_0^{T_0} |\psi'(u)|^2 du \ge \int\limits_s^t |\psi'(u)|^2 du \ge \frac{|\psi(t) - \psi(s)|^2}{t - s} \ge \frac{\mu^2}{h}.$$

Therefore, $I(F) \geq \frac{\mu^2}{2h}$. Thus,

$$\overline{\lim_{\varepsilon\to 0}} \varepsilon \ln P\left(\exists x \in U_\eta(x_0)\, \exists t \in [0, T_0]\colon \left|\beta_\varepsilon\left(\int_0^t \frac{ds}{\|x(s)\|^2}\right) - \beta_\varepsilon\left(\int_0^t \frac{ds}{\|x_0(s)\|^2}\right)\right| > \mu\right) \leq$$

$$\leq \overline{\lim_{\varepsilon\to 0}} \varepsilon \ln P(\beta_\varepsilon \in F) \leq -\frac{\mu^2}{2h} < -L.$$

□

Let us now obtain the lower estimate on the probabilities for the random elements Φ_ε to lie in an open set G.

Theorem 5.1. *Let $G \subseteq C[0,1]$ be an open set. Then*

$$\underline{\lim_{\varepsilon\to 0}} \varepsilon \ln P(\Phi_\varepsilon \in G) \geq -\frac{1}{2} \inf_{x\in\mathfrak{B}, \phi\in G, \phi(0)=0} \int_0^1 (\|x'(u)\|^2 + \|x(u)\|^2 \phi'(u)^2) du.$$

Proof. We will use the relation that follows from the mentioned in the end of Section 1 representation of the two-dimensional Brownian motion in a skew-product form:

$$\Phi_\varepsilon \overset{d}{=} \beta_\varepsilon\left(\int_0^\cdot \frac{ds}{\|w_\varepsilon(s)\|^2}\right).$$

Consider any function $x_0 \in \mathfrak{B}$. Choose any $\phi_0 \in G$ and any open ball $U_\delta(\phi_0) \subseteq G$. Fix $L > 0$. Choose a neighbourhood $U_\eta(x_0)$ in such a way that

$$\overline{\lim_{\varepsilon\to 0}} \varepsilon \ln P\left(\sup_{\substack{t\in[0,1]\\ x\in U_\eta(x_0)}} \left|\beta_\varepsilon\left(\int_0^t \frac{ds}{\|x(s)\|^2}\right) - \beta_\varepsilon\left(\int_0^t \frac{ds}{\|x_0(s)\|^2}\right)\right| > \frac{\delta}{2}\right) < -L. \tag{1}$$

This can be done by Lemma 5.2. We have

$$\begin{aligned}
P\left(\beta_\varepsilon\left(\int_0^\cdot \frac{ds}{\|w_\varepsilon(s)\|^2}\right) \in G\right) &\geq \\
\geq P\Bigg(\beta_\varepsilon\left(\int_0^\cdot \frac{ds}{\|x_0(s)\|^2}\right) &\in U_{\frac{\delta}{2}}(\phi_0), \\
\sup_{\substack{t\in[0,1]\\ x\in U_\eta(x_0)}} \left|\beta_\varepsilon\left(\int_0^t \frac{ds}{\|x(s)\|^2}\right) - \beta_\varepsilon\left(\int_0^t \frac{ds}{\|x_0(s)\|^2}\right)\right| &< \frac{\delta}{2}, w_\varepsilon \in U_\eta(x_0)\Bigg) \geq \\
\geq P\left(\beta_\varepsilon\left(\int_0^\cdot \frac{ds}{\|x_0(s)\|^2}\right) \in U_{\frac{\delta}{2}}(\phi_0), w_\varepsilon \in U_\eta(x_0)\right) &- \\
- P\left(\sup_{\substack{t\in[0,1]\\ x\in U_\eta(x_0)}} \left|\beta_\varepsilon\left(\int_0^t \frac{ds}{\|x(s)\|^2}\right) - \beta_\varepsilon\left(\int_0^t \frac{ds}{\|x_0(s)\|^2}\right)\right| > \frac{\delta}{2}\right)&.
\end{aligned} \tag{2}$$

As β_ε and w_ε are independent, we get

$$(3)\quad \varliminf_{\varepsilon\to 0} \varepsilon \ln P\left(\beta_\varepsilon\left(\int\limits_0^t \frac{ds}{\|x_0(s)\|^2}\right) \in U_{\frac{\delta}{2}}(\phi_0), w_\varepsilon \in U_\eta(x_0)\right) \geq$$

$$\geq \varliminf_{\varepsilon\to 0} \varepsilon \ln P\left(\beta_\varepsilon\left(\int\limits_0^t \frac{ds}{\|x_0(s)\|^2}\right) \in U_{\frac{\delta}{2}}(\phi_0)\right) + \varliminf_{\varepsilon\to 0} \varepsilon \ln P\left(w_\varepsilon \in U_\eta(x_0)\right).$$

By Lemma 5.1, we get

(4)

$$\varliminf_{\varepsilon\to 0} \varepsilon \ln P\left(\beta_\varepsilon\left(\int\limits_0^t \frac{ds}{\|x_0(s)\|^2}\right) \in U_{\frac{\delta}{2}}(\phi_0)\right) \geq -\frac{1}{2} \inf_{\phi\in U_{\frac{\delta}{2}}(\phi_0), \phi(0)=0} \int\limits_0^1 \|x_0(u)\|^2 \phi'(u)^2 du.$$

By the LDP for Brownian motion, we have

$$(5)\qquad \varliminf_{\varepsilon\to 0} \varepsilon \ln P\left(w_\varepsilon \in U_\eta(x_0)\right) \geq -I(U_\eta(x_0)) \geq -I(x_0) = -\frac{1}{2}\int\limits_0^1 \|x_0'(u)\|^2 du.$$

Define the function α_L by

$$\alpha_L(s) = \begin{cases} s,\ s > -L; \\ -\infty,\ s \leq -L. \end{cases}$$

From (1), (2), (3), (4) and (5) we get

$$\varliminf_{\varepsilon\to 0} \varepsilon \ln P\left(\beta_\varepsilon\left(\int_0^\cdot \frac{ds}{\|w_\varepsilon(s)\|^2}\right) \in G\right) \geq$$

$$\geq \alpha_L\left(-\frac{1}{2} \inf_{\phi\in U_{\frac{\delta}{2}}(\phi_0), \phi(0)=0} \int\limits_0^1 \|x_0(u)\|^2 \phi'(u)^2 du - \frac{1}{2}\int\limits_0^1 \|x_0'(u)\|^2 du\right).$$

As $x_0 \in \mathfrak{B}$ and $\phi_0 \in G$ are arbitrary, then

$$\varliminf_{\varepsilon\to 0} \varepsilon \ln P\left(\beta_\varepsilon\left(\int_0^\cdot \frac{ds}{\|w_\varepsilon(s)\|^2}\right) \in G\right) \geq$$

$$\geq \alpha_L\left(-\frac{1}{2} \inf_{x\in\mathfrak{B}, \phi\in G, \phi(0)=0}\left(\int\limits_0^1 \|x(u)\|^2 \phi'(u)^2 du + \int\limits_0^1 \|x'(u)\|^2 du\right)\right).$$

As L is arbitrary, then, taking the limit as $L \to \infty$, we get the needed estimate. □

Remark 5.1. In fact,

$$\frac{1}{2} \inf_{x\in\mathfrak{B}, \phi\in G, \phi(0)=0} \int\limits_0^1 (\|x'(u)\|^2 + \|x(u)\|^2 \phi'(u)^2) du = I(\Phi^{-1}(G)).$$

Indeed, denote $r(t) = \|x(t)\|$. It is easily seen that

$$\int\limits_0^1 (\|x'(u)\|^2 + \|x(u)\|^2 \phi'(u)^2) du \geq \int\limits_0^1 (r'(u)^2 + r(u)^2 \phi'(u)^2) du.$$

Therefore,

$$\inf_{x\in\mathfrak{B}, \phi\in G, \phi(0)=0} \int\limits_0^1 (\|x'(u)\|^2 + \|x(u)\|^2 \phi'(u)^2) du =$$

$$= \inf_{r\in\mathfrak{D},\phi\in G,\phi(0)=0} \int_0^1 (r'(u)^2 + r(u)^2\phi'(u)^2)du.$$

For $z(t) = r(t)e^{i\phi(t)}, r(0) = 1, \phi(0) = 0$ we have

$$\frac{1}{2}\int_0^1 (r'(u)^2 + r(u)^2\phi'(u)^2)du = I(z).$$

Thus,

$$\frac{1}{2}\inf_{x\in\mathfrak{B},\phi\in G,\phi(0)=0} \int_0^1 (\|x'(u)\|^2 + \|x(u)\|^2\phi'(u)^2)du = I(\Phi^{-1}(G)).$$

Now we obtain the upper estimate on the probabilities for the random elements Φ_ε to lie in a closed set $F \subseteq C([0,1])$. We will use the following notation:

- $\tau_\delta(x) = \inf\{t\colon x(t) \in B_\delta(0)\}$ for $x \in C([0,1],\mathbb{R}^2)$;
- $\mathfrak{F}_\delta = \Big\{(x,\phi)\colon x \in C([0,1],\mathbb{R}^2), x(0) = \begin{pmatrix}1\\0\end{pmatrix}, \phi \in C([0,\tau_\delta(x)]), \phi(0) = 0,$
  $\phi \in \overline{F|_{[0,\tau_\delta(x)]}}\Big\},$
  where by $\overline{F|_{[0,\tau_\delta(x)]}}$ we mean the closure in $C([0,\tau_\delta(x)])$ of
  $$F|_{[0,\tau_\delta(x)]} = \{\phi \in C([0,\tau_\delta(x)]) \mid \exists\psi \in F\colon \phi = \psi|_{[0,\tau_\delta(x)]}\};$$
- $\mathfrak{F}_{\mu,\delta} = \big\{(x,\phi)\colon x \in C([0,1],\mathbb{R}^2),\ x(0) = \begin{pmatrix}1\\0\end{pmatrix},$
  $\phi \in C([0,\tau_\delta(x)]), \phi(0) = 0, \phi \in \overline{\left(F|_{[0,\tau_\delta(x)]}\right)^\mu}\Big\},$
  where
  $$\left(F|_{[0,\tau_\delta(x)]}\right)^\mu = \left\{\phi \in C([0,\tau_\delta(x)]) \mid \exists\psi \in F|_{[0,\tau_\delta(x)]}\colon \sup_{s\in[0,\tau_\delta(x)]} |\phi(s) - \psi(s)| < \mu\right\};$$
- $\mathfrak{F}_{\mu,\delta,x_0} = \overline{\left(F|_{[0,\tau_\delta(x_0)]}\right)^\mu}.$

Theorem 5.2. *Let $F \subseteq C([0,1])$ be a closed set. Then for any $\delta > 0$:*

$$\overline{\lim_{\varepsilon\to 0}}\, \varepsilon \ln P\left(\beta_\varepsilon\left(\int_0^\cdot \frac{ds}{\|w_\varepsilon(s)\|^2}\right) \in F\right) \le \tag{6}$$

$$\le -\frac{1}{2}\inf_{(x,\phi)\in\mathfrak{F}_\delta} \int_0^{\tau_\delta(x)} (\|x'(u)\|^2 + \|x(u)\|^2\phi'(u)^2)du.$$

Proof. Fix some constant numbers $L,\mu,\chi,\delta > 0$. Choose $h > 0$ such that $\frac{\mu^2}{2h} > L$. Consider the compact

$$K_L = \left\{ x \in C([0,1],\mathbb{R}^2)\colon x(0) = \begin{pmatrix}1\\0\end{pmatrix},\ \frac{1}{2}\int_0^1 \|x'(u)\|^2 du \le L \right\}.$$

Let us build a covering of the set K_L by open sets. Take any point $x_0 \in K_L$. Let $\tau_\delta(x_0) = \inf\{t : x_0(t) \in B_\delta(0)\}$. We cover x_0 by a neighbourhood

$$U_\eta(x_0) = \big\{x \in C([0,1],\mathbb{R}^2)\colon \forall t \in [0,\tau_\delta(x_0)]\ \|x(t) - x_0(t)\| < \eta\big\}.$$

Here $\eta > 0$ is chosen in such a way that the following conditions hold:

$$\forall x \in U_\eta(x_0)\, \forall t \in [0, \tau_\delta(x_0)]\, x(t) \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}; \tag{7}$$

$$\forall x \in U_\eta(x_0)\, \forall t \in [0, \tau_\delta(x_0)] \left| \int\limits_0^t \frac{ds}{\|x(s)\|^2} - \int\limits_0^t \frac{ds}{\|x_0(s)\|^2} \right| < h; \tag{8}$$

$$I(\overline{U_\eta(x_0)}) \geq I(x_0|_{[0,\tau_\delta(x_0)]}) - \chi. \tag{9}$$

Choosing for any $x_0 \in K_L$ the neighbourhood $U_\eta(x_0)$ that covers x_0, we get an open covering of the compact K_L. Now choose its finite subcovering.

For any neighbourhood $U_\eta(x_0)$ from our finite covering we estimate the probability $P\left(w_\varepsilon \in U_\eta(x_0), \beta_\varepsilon\left(\int_0^\cdot \frac{ds}{\|w_\varepsilon(s)\|^2}\right) \in F|_{[0,\tau_\delta(x_0)]}\right)$. We have

$$P\left(w_\varepsilon \in U_\eta(x_0), \beta_\varepsilon\left(\int_0^\cdot \frac{ds}{\|w_\varepsilon(s)\|^2}\right) \in F|_{[0,\tau_\delta(x_0)]}\right) \leq$$
$$\leq P\left(w_\varepsilon \in U_\eta(x_0), \beta_\varepsilon\left(\int_0^\cdot \frac{ds}{\|x_0(s)\|^2}\right) \in (F|_{[0,\tau_\delta(x_0)]})^\mu\right) +$$
$$+ P\left(\exists x \in U_\eta(x_0)\, \exists t \in [0, \tau_\delta(x_0)]\colon \left|\beta_\varepsilon\left(\int\limits_0^t \frac{ds}{\|x(s)\|^2}\right) - \beta_\varepsilon\left(\int\limits_0^t \frac{ds}{\|x_0(s)\|^2}\right)\right| > \mu\right).$$

We estimate the first summand in our sum. We have

$$\overline{\lim_{\varepsilon\to 0}}\, \varepsilon \ln P\left(w_\varepsilon \in U_\eta(x_0), \beta_\varepsilon\left(\int_0^\cdot \frac{ds}{\|x_0(s)\|^2}\right) \in \left(F|_{[0,\tau_\delta(x_0)]}\right)^\mu\right) \leq$$
$$\leq \overline{\lim_{\varepsilon\to 0}}\, \varepsilon \ln P(w_\varepsilon \in U_\eta(x_0)) + \overline{\lim_{\varepsilon\to 0}}\, \varepsilon \ln P\left(\beta_\varepsilon\left(\int_0^\cdot \frac{ds}{\|x_0(s)\|^2}\right) \in \left(F|_{[0,\tau_\delta(x_0)]}\right)^\mu\right).$$

By the LDP for Brownian motion, with the help of (9) we get:

$$\overline{\lim_{\varepsilon\to 0}}\, \varepsilon \ln P(w_\varepsilon \in U_\eta(x_0)) \leq -I(\overline{U_\eta(x_0)}) \leq -I(x_0|_{[0,\tau_\delta(x_0)]}) + \chi.$$

By Lemma 5.1, we have

$$\overline{\lim_{\varepsilon\to 0}}\, \varepsilon \ln P\left(\beta_\varepsilon\left(\int_0^\cdot \frac{ds}{\|x_0(s)\|^2}\right) \in \left(F|_{[0,\tau_\delta(x_0)]}\right)^\mu\right) \leq$$
$$\leq -\frac{1}{2} \inf_{\phi \in \mathfrak{F}_{\mu,\delta,x_0}, \phi(0)=0} \int\limits_0^{\tau_\delta(x_0)} \|x_0(u)\|^2 \phi'(u)^2 du.$$

Now estimate the second summand. By Lemma 5.2, we have

$$\overline{\lim_{\varepsilon\to 0}}\, \varepsilon \ln P\Bigg(\exists x \in U_\eta(x_0)\, \exists t \in [0, \tau_\delta(x_0)]\colon$$
$$\left|\beta_\varepsilon\left(\int\limits_0^t \frac{ds}{\|x(s)\|^2}\right) - \beta_\varepsilon\left(\int\limits_0^t \frac{ds}{\|x_0(s)\|^2}\right)\right| > \mu\Bigg) < -L.$$

We finally get

$$\overline{\lim_{\varepsilon\to 0}}\, \varepsilon \ln P\left(w_\varepsilon \in U_\eta(x_0), \beta_\varepsilon\left(\int_0^\cdot \frac{ds}{\|w_\varepsilon(s)\|^2}\right) \in F\right) \leq$$

$$\leq \left(-I(\overline{U_\eta(x_0)}) - \frac{1}{2} \inf_{\phi \in \mathfrak{F}_{\mu,\delta,x_0}, \phi(0)=0} \int_0^{\tau_\delta(x_0)} \|x_0(u)\|^2 \phi'(u)^2 du \right) \vee (-L) \leq$$

$$\leq \left(-\frac{1}{2} \int_0^{\tau_\delta(x_0)} \|x_0'(u)\|^2 du - \frac{1}{2} \inf_{\phi \in \mathfrak{F}_{\mu,\delta,x_0}, \phi(0)=0} \int_0^{\tau_\delta(x_0)} \|x_0(u)\|^2 \phi'(u)^2 du + \chi \right) \vee (-L).$$

Putting together such estimates for all neighbourhoods from our finite covering, we obtain

$$\overline{\lim_{\varepsilon \to 0}} \varepsilon \ln P \left(\beta_\varepsilon \left(\int_0^{\cdot} \frac{ds}{\|w_\varepsilon(s)\|^2} \right) \in F \right) \leq$$

$$\leq \left(-\frac{1}{2} \inf_{(x,\phi) \in \mathfrak{F}_{\mu,\delta}} \int_0^{\tau_\delta(x)} (\|x'(u)\|^2 + \|x(u)\|^2 \phi'(u)^2) du + \chi \right) \vee (-L).$$

We sequentially take the limits as $L \to \infty, \chi \to 0$ and get

$$\overline{\lim_{\varepsilon \to 0}} \varepsilon \ln P \left(\beta_\varepsilon \left(\int_0^{\cdot} \frac{ds}{\|w_\varepsilon(s)\|^2} \right) \in F \right) \leq$$

$$\leq -\frac{1}{2} \inf_{(x,\phi) \in \mathfrak{F}_{\mu,\delta}} \int_0^{\tau_\delta(x)} (\|x'(u)\|^2 + \|x(u)\|^2 \phi'(u)^2) du.$$

Taking the limit as $\mu \to 0$, due to the function $j(\phi) = \int_0^{\tau_\delta(x)} \|x(u)\|^2 \phi'(u)^2 du$ being lower semicontinuous and its level sets being compact, we get

$$\overline{\lim_{\varepsilon \to 0}} \varepsilon \ln P \left(\beta_\varepsilon \left(\int_0^{\cdot} \frac{ds}{\|w_\varepsilon(s)\|^2} \right) \in F \right) \leq$$

$$\leq -\frac{1}{2} \inf_{(x,\phi) \in \mathfrak{F}_{\delta}} \int_0^{\tau_\delta(x)} (\|x'(u)\|^2 + \|x(u)\|^2 \phi'(u)^2) du.$$

□

It remains to take the limit in (6) as $\delta \to 0$. This is what we do now.

Lemma 5.3. *Let $F \subseteq C([0,1])$ be a closed set. If $0 < t_1 < t_2 \leq 1, \phi \in \overline{F|_{[0,t_2]}}$, then $\phi|_{[0,t_1]} \in \overline{F|_{[0,t_1]}}$.*

Proof. As $\phi \in \overline{F|_{[0,t_2]}}$, then there exists a sequence $\phi_n \to \phi$, $\phi_n \in F|_{[0,t_2]}$. It is clear that the restriction to $[0,t_1]$ conserves this convergence:

$$\phi_n|_{[0,t_1]} \to \phi|_{[0,t_1]}.$$

But $\phi_n|_{[0,t_1]} \in F|_{[0,t_1]}$, and thus $\phi|_{[0,t_1]} \in \overline{F|_{[0,t_1]}}$. □

Lemma 5.4. *Under the conditions of Theorem 5.2, there exists the limit*

$$\lim_{\delta \to 0} \inf_{(x,\phi) \in \mathfrak{F}_\delta} \int_0^{\tau_\delta(x)} (\|x'(u)\|^2 + \|x(u)\|^2 \phi'(u)^2) du.$$

Proof. We say that the pair (x, ϕ), where $x \in C([0,1], \mathbb{R}^2)$, $\phi \in C([0, \tau_\delta(x)])$, is suitable for δ if

$$\phi \in \overline{F|_{[0,\tau_\delta(x)]}}.$$

By Lemma 5.3, we obtain that if a pair (x, ϕ) is suitable for δ_1, then for $\delta_2 > \delta_1$ the pair $(x, \phi|_{[0,\tau_{\delta_2}(x)]})$ is also suitable. So,

$$\inf_{(x,\phi)\in\mathfrak{F}_\delta} \int\limits_0^{\tau_\delta(x)} (\|x'(u)\|^2 + \|x(u)\|^2\phi'(u)^2)du$$

does not increase on δ. □

Theorem 5.3. *Under the conditions of Theorem 5.2, the following relation holds:*

$$\lim_{\delta\to 0} \inf_{(x,\phi)\in\mathfrak{F}_\delta} \int\limits_0^{\tau_\delta(x)} (\|x'(u)\|^2 + \|x(u)\|^2\phi'(u)^2)du = \inf_{y\in\overline{\Phi^{-1}(F)}} \int\limits_0^1 \|y'(u)\|^2 du.$$

To prove this theorem we will need the following lemma.

Lemma 5.5. *If $y(t) = r(t)e^{i\phi(t)}$, $y \in \overline{\Phi^{-1}(F)}$, then for $\tau_\delta = \inf\{t : |r(t)| \le \delta\}$ the following inclusion holds: $\phi|_{[0,\tau_\delta]} \in \overline{F|_{[0,\tau_\delta]}}$ for any $\delta > 0$.*

Proof. If $y \in \overline{\Phi^{-1}(F)}$, then there exists a sequence $\{y_n\} \subseteq \Phi^{-1}(F)$ such that $y_n \to y$. But as $y_n \to y$, then $y_n|_{[0,\tau_\delta]} \to y|_{[0,\tau_\delta]}$ as well. As $y|_{[0,\tau_\delta]}$ does not pass through zero, then the mapping Φ is continuous at $y|_{[0,\tau_\delta]}$. Therefore, we obtain that

$$\Phi(y_n|_{[0,\tau_\delta]}) \to \Phi(y|_{[0,\tau_\delta]}).$$

But $\Phi(y_n|_{[0,\tau_\delta]}) \in F|_{[0,\tau_\delta]}$ for any n. On the other hand, $\Phi(y|_{[0,\tau_\delta]}) = \phi|_{[0,\tau_\delta]}$. So, we get $\phi|_{[0,\tau_\delta]} \in \overline{F|_{[0,\tau_\delta]}}$. □

Now we return to the proof of Theorem 5.3. With the help of Lemma 5.5 we get:

$$\lim_{\delta\to 0} \inf_{(x,\phi)\in\mathfrak{F}_\delta} \int\limits_0^{\tau_\delta(x)} (\|x'(u)\|^2 + \|x(u)\|^2\phi'(u)^2)\, du \le \inf_{y\in\overline{\Phi^{-1}(F)}} \int\limits_0^1 \|y'(u)\|^2 du. \tag{10}$$

Let us show that the opposite inequality also holds, that is,

$$\lim_{\delta\to 0} \inf_{(x,\phi)\in\mathfrak{F}_\delta} \int\limits_0^{\tau_\delta(x)} (\|x'(u)\|^2 + \|x(u)\|^2\phi'(u)^2)du \ge \inf_{y\in\overline{\Phi^{-1}(F)}} \int\limits_0^1 \|y'(u)\|^2 du. \tag{11}$$

If

$$\lim_{\delta\to 0} \inf_{(x,\phi)\in\mathfrak{F}_\delta} \int\limits_0^{\tau_\delta(x)} (\|x'(u)\|^2 + \|x(u)\|^2\phi'(u)^2)du = \infty,$$

then we have nothing to prove. So, we suppose that

$$\lim_{\delta\to 0} \inf_{(x,\phi)\in\mathfrak{F}_\delta} \int\limits_0^{\tau_\delta(x)} (\|x'(u)\|^2 + \|x(u)\|^2\phi'(u)^2)du = 2\alpha < \infty.$$

In this case there exists a subsequence (x_n, δ_n, ϕ_n) such that $\delta_n \to 0$, $\phi_n \in \overline{F|_{[0,\tau_{\delta_n}(x_n)]}}$, $\phi_n(0) = 0$, and

$$\lim_{n\to\infty} \int_0^{\tau_{\delta_n}(x_n)} (\|x_n'(u)\|^2 + \|x_n(u)\|^2 \phi_n'(u)^2) du = 2\alpha.$$

We consider y_n defined in the following way:

$$y_n(t) = \begin{cases} \|x_n(t)\| e^{i\phi_n(t)}, & t \le \tau_{\delta_n}(x_n); \\ \delta_n e^{i\phi_n(\tau_{\delta_n}(x_n))}, & \tau_{\delta_n}(x_n) \le t \le 1. \end{cases}$$

It is clear that $\overline{\lim\limits_{n\to\infty}} I(y_n) \le \alpha$. Therefore, we can select a subsequence from $\{y_n\}$ that belongs to the compact $\{y \in C([0,1], \mathbb{R}^2) \colon I(y) \le \alpha + 1\}$. So, we can select even a convergent subsequence. Let us consider $\{y_n\}$ to be convergent itself.

Put $y = \lim\limits_{n\to\infty} y_n$. We will show that $y \in \overline{\Phi^{-1}(F)}$. To do this, we build a sequence from $\Phi^{-1}(F)$ that converges to y. As $\phi_n \in \overline{F|_{[0,\tau_{\delta_n}(x_n)]}}$, then for any $\mu > 0$ there exists $\psi_n \in F$ such that $\rho\left(\phi_n, \psi_n|_{[0,\tau_{\delta_n}(x_n)]}\right) < \mu$. Let us choose these ψ_n in such a way that

$$\sup_{t\in[0,\tau_{\delta_n}(x_n)]} \left| \|x_n(t)\| e^{i\phi_n(t)} - \|x_n(t)\| e^{i\psi_n(t)} \right| \to 0 \, (n \to \infty).$$

Define z_n in the following way:

$$z_n(t) = \begin{cases} \|x_n(t)\| e^{i\psi_n(t)}, & t \le \tau_{\delta_n}(x_n); \\ \delta_n e^{i\psi_n(t)}, & \tau_{\delta_n}(x_n) \le t \le 1. \end{cases}$$

It is clear that $z_n \to y \, (n \to \infty)$. But it is also clear that $z_n \in \Phi^{-1}(F)$ for any n. Therefore, $y \in \overline{\Phi^{-1}(F)}$. Further, $y_n \to y$, and so $I(y) \le \underline{\lim}\, I(y_n) \le \alpha$. This finishes the proof of the inequality (11). Theorem 5.3 is also proved.

So, from Theorems 5.2 and 5.3 we obtain for closed sets $F \subseteq C([0,1])$:

$$\overline{\lim_{\varepsilon\to 0}}\, \varepsilon \ln P(\Phi_\varepsilon \in F) \le \frac{1}{2} \inf_{y \in \overline{\Phi^{-1}(F)}} \int_0^1 \|y'(u)\|^2 du = I(\overline{\Phi^{-1}(F)}).$$

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