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RANDOM COVERS OF FINITE HOMOGENEOUS LATTICES

We develop and extend some results for the scheme of independent random elements distributed on a finite lattice. In particular, we introduce the concept of the cover of a homogeneous lattice L_n of rank n and derive the exact equations and estimations for the number of covers with a given number of blocks and for the total covers number of the lattice L_n. A theorem about the asymptotic normality of the blocks number in a random equiprobable cover of the lattice L_n is proved. The concept of the cover index of the lattice L_n, that extend the notion of the cover index of a finite set by its independent random subsets, is introduced. Applying the lattice moments method, the limit distribution as $n \to \infty$ for the cover index of a subspace lattice of the n-dimensional vector space over a finite field is determined.

1. Basic notions and preliminary results

In this paper, we use notions and results from [1, 2]. We refer also to [3, 4] for the terminology and detailed exposition of finite lattices theory.

Let $L = \{L_n : n = 0, 1, ...\}$ be a sequence of finite lattices. Denote the Moebius function, the maximal and minimal elements of the lattice L_n by $\mu_n, 1_n$, and 0_n respectively. We assume that the sequence L satisfies the following homogeneity conditions (see [4]):

(a) L_n is a graduate lattice with rank function r, where $r(1_n) = n$ for any $n = 0, 1, ...$;

(b) for any $X \in L_n$ such that $r(X) = n - k, k \in \overline{0, n}$, the interval $[X, 1_n]$ is isomorphic to the lattice L_k.

Let

$$w(n,k) = \sum_{a \in L_n : r(a)=k} \mu_n(0_n, a), \; W(n,k) = \sum_{a \in L_n : r(a)=k} 1, \tag{1}$$

where $k \in \overline{0, n}, n = 0, 1, \ldots$; $w(n, k) = W(n, k) = 0$ otherwise. The numbers $w(n, k)$ and $W(n, k)$ are called the k-th level numbers of the lattice L_n of the first kind and of the second kind respectively [3, 4].

By $\chi_n(z)$, we denote the characteristic polynomial of the lattice L_n,

$$\chi_n(z) = \sum_{k=0}^{n} w(n, n-k) z^k, \; n = 0, 1, \ldots . \tag{2}$$

In what follows, we assume that there exists a sequence of real numbers $a_i \geq 1$ ($i = 1, 2, \ldots$) such that

$$\chi_n(z) = \prod_{i=1}^{n} (z - a_i), \; n = 1, 2, \ldots , \; \chi_0(z) \equiv 1. \tag{3}$$

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Note that condition (3) holds for the characteristic polynomial of any finite supersolvable geometric lattice (see [4, 5]).

We consider the following most important examples of homogeneous lattices which satisfy condition (3). Other examples can be found in [3, 4, 5].

1. $L_n = B(n)$, where $B(n)$ is the set of all subsets of $N_n = \{1, 2, \dots, n\}$. The rank function, level numbers, and characteristic polynomial of the lattice $B(n)$ are equal, respectively, to $r(X) = \#X$ (where $\#X$ denotes the cardinality of a set $X \in B(n)$),

$$w(n,k) = (-1)^k \binom{n}{k},\ W(n,k) = \binom{n}{k},\ \chi_n(z) = (z-1)^n,\ k \in \overline{0,n},\ n = 0,1,\dots .$$

2. $L_n = L(n,q)$ is a subspace lattice of the n-dimensional vector space $V(n,q)$ over a field with q elements. In this case, the rank $r(X)$ of a subspace $X \in L_n$ is equal to the dimension of X,

$$w(n,k) = (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q,\ W(n,k) = \begin{bmatrix} n \\ k \end{bmatrix}_q,\ \chi_n(z) = \prod_{i=1}^{n} (z - q^{i-1}),$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^{-1})_n}{(q^{-1})_k (q^{-1})_{n-k}} q^{k(n-k)}$$

is the Gauss coefficient (the number of k-dimensional subspaces of the vector space $V(n,q), k \in \overline{0,n}$), $(q^{-1})_n = (1-q^{-1})(1-q^{-2})\dots(1-q^{-n}),\ n = 0,1,\dots .$

3. $L_n = \Re(n+1)$ is the lattice of all partitions of the set $N_{n+1} = \{1, 2, \dots, n+1\}$. The rank of a partition $\pi \in \Re(n+1)$ is equal to $r(\pi) = n + 1 - b(\pi)$, where $b(\pi)$ is the number of blocks in π. The level numbers and the characteristic polynomial of the lattice $\Re(n+1)$ are, respectively, equal to

$$w(n,k) = s(n+1, n+1-k),\ W(n,k) = S(n+1, n+1-k),\ \chi_n(z) = \prod_{i=1}^{n} (z-i),$$

where $k \in \overline{0,n}$, $n = 0,1,\dots$, $s(\cdot,\cdot)$ and $S(\cdot,\cdot)$ are Stirling numbers of the first kind and of the second kind, respectively.

Let's consider a sequence of random variables $\xi_0, \xi_1, \dots$, where ξ_n takes values in the set $\{0, 1, \dots, n\}$, $n = 0,1,\dots$. Let $p_{n,k} = \mathbf{P}\{\xi_n = k\}$, $B_{n,k} = \mathbf{E}W(\xi_n, \xi_n - k)$, $k \in \overline{0,n}$, $n = 0,1,\dots$. We call $B_{n,k}$ the k-th lattice moment of the random variable ξ_n [2].

The following statement was proved in [2].

Statement 1. *Let L be a sequence of finite homogeneous lattices with the level numbers (1) and the characteristic polynomials (2) that satisfies condition (3). Then*

1. We have the following mutually inverse relations:

$$B_{n,l} = \sum_{k=l}^{n} p_{n,k} W(k, k-l),\ p_{n,l} = \sum_{k=l}^{n} B_{n,k} w(k, k-l),\ l \in \overline{0,n},\ n = 0,1,\dots . \tag{4}$$

2. For the generating function of the random variable ξ_n, the following equalities hold:

$$p_n(z) = \sum_{l=0}^{n} p_{n,l} z^l = \sum_{k=0}^{n} B_{n,k} \chi_k(z),\ n = 0,1,\dots ,$$

$$\sum_{k=0}^{2\nu+1} B_{n,k} \chi_k(z) \le p_n(z) \le \sum_{k=0}^{2\nu} B_{n,k} \chi_k(z),\ z \in [0,1], \tag{5}$$

where $\nu = 0, 1, \dots$.

Assuming that $z = 0$ in (5), we obtain the following inequalities:

$$\sum_{k=0}^{2\nu+1} B_{n,k} w(k,k) \le p_{n,0} \le \sum_{k=0}^{2\nu} B_{n,k} w(k,k), \ \nu = 0, 1, \dots ,$$

and, for $\nu = 0$,

$$1 - B_{n,1} \le p_{n,0} \le 1. \tag{6}$$

2. Random covers of the lattice L_n

By $\Lambda_{n,T}$, we denote the collection of all sets $X = \{X_1, \dots, X_T\}$ such that $X_1 \dots X_T$ are different non-zero elements of the lattice L_n. Let's assign the equiprobable distribution to the $\Lambda_{n,T}$, assuming that

$$p(X) = \binom{\lambda_n - 1}{T}^{-1}, \ X = \{X_1, \dots, X_T\} \in \Lambda_{n,T}, \tag{7}$$

where $\lambda_n = \# L_n$. Put

$$\lambda_n(Y) = \#[0_n, Y], \ Y \in L_n, \tag{8}$$

$$\lambda^{(n-1)} = \max\{\lambda_n(Y) : Y \in L_n, r(Y) = n-1\}, \ n = 0, 1, \dots . \tag{9}$$

Definition 1. A set $X = \{X_1, \dots, X_T\} \in \Lambda_{n,T}$ will be called a T-block cover of the lattice L_n (and its elements be called blocks of the cover X), if $X_1 \vee \dots \vee X_T = 1_n$.

By $r_{n,T} = n - r(X_1 \vee \dots \vee X_T)$, we denote the random variable equal to the co-rank of the join of $X_1, \dots, X_T$, where $X = \{X_1, \dots, X_T\}$ is a random element distributed according to (7). Let's denote

$$p_{n,l}^{(T)} = \mathbf{P}\{r_{n,T} = l\}, \ B_{n,l}^{(T)} = \mathbf{E} W(r_{n,T}, r_{n,T} - l), \ l \in \overline{0,n}, n = 0, 1, \dots .$$

We denote the T-block cover number and the total block cover number of the lattice L_n by $D_{n,T}$ and $D_n = \sum_{T=1}^{\lambda_n - 1} D_{n,T}$, respectively.

For the case $L_n = B(n)$, exact formulas and estimations of the numbers $D_{n,T}$, D_n are obtained in [6, p. 269]. The following statement extends these results.

Statement 2. *The following relations hold:*

$$D_{n,T} = \binom{\lambda_n - 1}{T} \sum_{k=0}^{n} B_{n,k}^{(T)} w(k,k), \ D_n = \sum_{k=0}^{n} w(k,k) \sum_{T=1}^{\lambda_n - 1} \binom{\lambda_n - 1}{T} B_{n,k}^{(T)}, \tag{10}$$

$$\binom{\lambda_n - 1}{T} - B_{n,1}^{(T)} \binom{\lambda_n - 1}{T} \le D_{n,T} \le \binom{\lambda_n - 1}{T}, \tag{11}$$

$$2^{\lambda_n - 1} - 1 - \sum_{T=1}^{\lambda_n - 1} \binom{\lambda_n - 1}{T} B_{n,1}^{(T)} \le D_{n,T} \le 2^{\lambda_n - 1} - 1. \tag{12}$$

As this takes place, the following equation holds for any $l \in \overline{0,n}$, $T \in \overline{1, \lambda_n - 1}$*:*

$$B_{n,l}^{(T)} = \binom{\lambda_n - 1}{T}^{-1} \sum_{\substack{Y \in L_n: \\ r(Y) = n - l}} \binom{\lambda_n(Y) - 1}{T}, \ l \in \overline{0,n}, \ T \in \overline{1, \lambda_n - 1}. \tag{13}$$

Proof. The first equality in (10) is immediate from (4) and the equality

$$D_{n,T} = \binom{\lambda_n - 1}{T} p_{n,0}^{(T)};$$

the second equality in (10) follows from the first one. Inequalities (11) follow from (6) and the first equation in (10); inequalities (12) are obtained by the summation of (11) over $T \in \overline{1, \lambda_n - 1}$. Finally, the proof of (13) is similar to the proof of Theorem 1 [1] applying the Moebius inversion formula.

3. Asymptotic behavior of the block number distribution in a random equiprobable cover of the lattice L_n

Let ζ_n denote the random variable equal to the block number in a random equiprobable cover of the lattice L_n. Let's prove the following theorem generalizing a result from [7].

Theorem 1. *Suppose that*

$$\varlimsup_{n\to\infty} \lambda^{(n-1)}(\lambda_n)^{-1} < 1, \tag{14}$$

where $\lambda^{(n-1)}$ *is defined by (9). Then*

$$D_n = 2^{\lambda_n - 1}(1 + o(1)),\ n \to \infty, \tag{15}$$

$$\mathbf{P}\{\zeta_n = T\} = \frac{D_{n,T}}{D_n} = \frac{1}{\sqrt{\frac{\pi}{2}(\lambda_n - 1)}} \exp\left\{-\frac{(x_{n,T})^2}{2}\right\}(1 + o(1)), \tag{16}$$

provided n *and* T *tend to infinity in such a way that*

$$x_{n,T} \overset{\text{def}}{=} \frac{T - \frac{1}{2}(\lambda_n - 1)}{\frac{1}{2}\sqrt{\lambda_n - 1}} = o(\lambda_n^{\frac{1}{6}}). \tag{17}$$

Under condition (14), the remainder term on the right-hand side of (16) tends uniformly to zero for all T *such that* $x_{n,T}$ *lies in any fixed finite interval.*

Proof. First we show that, under assumption (14),

$$2^{-(\lambda_n - 1)} \sum_{\substack{Y \in L_n: \\ r(Y) = n-1}} (2^{\lambda_n(Y) - 1} - 1) = o(1),\ n \to \infty. \tag{18}$$

Applying the second equality from (1), we obtain

$$\lambda_n = \#L_n = \sum_{k=0}^{n} W(n,k) > W(n, n-1).$$

Whence and from (14), we have

$$2^{-(\lambda_n - 1)} \sum_{\substack{Y \in L_n: \\ r(Y) = n-1}} (2^{\lambda_n(Y) - 1} - 1) < \sum_{\substack{Y \in L_n: \\ r(Y) = n-1}} 2^{\lambda_n(Y)} \leq 2^{\lambda^{(n-1)} - \lambda_n} W(n, n-1) <$$

$$< \lambda_n 2^{\lambda^{(n-1)} - \lambda_n} = 2^{-\lambda_n\left(1 - \frac{\lambda^{(n-1)}}{\lambda_n} - \frac{\log \lambda_n}{\lambda_n}\right)} = o(1),\ n \to \infty. \tag{19}$$

So equality (18) is proved.

Now, taking into account (12) and (13), we obtain

$$2^{\lambda_n-1}-1-\sum_{T=1}^{\lambda_n-1}\binom{\lambda_n-1}{T}B_{n,1}^{(T)}=2^{\lambda_n-1}-\sum_{\substack{Y\in L_n:\\ r(Y)=n-1}}(2^{\lambda_n(Y)-1}-1)\le D_n\le 2^{\lambda_n-1}-1.$$

So, applying (18), we arrive at (15).

To prove equality (16), we set

$$\alpha(n,T)=\frac{1}{\sqrt{\frac{\pi}{2}(\lambda_n-1)}}\exp\left\{-\frac{(x_{n,T})^2}{2}\right\}.$$

Then

$$\left|\frac{D_{n,T}}{D_n}\alpha(n,T)^{-1}-1\right|\le$$

$$\le\left|\frac{D_{n,T}}{D_n}\alpha(n,T)^{-1}-\frac{1}{D_n}\binom{\lambda_n-1}{T}\alpha(n,T)^{-1}\right|+\left|\frac{1}{D_n}\binom{\lambda_n-1}{T}\alpha(n,T)^{-1}-1\right|. \tag{20}$$

From (11) and (13), we obtain that the augend on the right-hand side of (20) is not greater than

$$\alpha(n,T)^{-1}\frac{1}{D_n}\sum_{\substack{Y\in L_n:\\ r(Y)=n-1}}\binom{\lambda_n(Y)-1}{T}<\alpha(n,T)^{-1}\frac{1}{D_n}\lambda_n 2^{\lambda^{(n-1)}}=$$

$$\alpha(n,T)^{-1}\lambda_n 2^{\lambda^{(n-1)}-(\lambda_n-1)}(1+o(1)),\ n\to\infty$$

[see relations (19) and (15)]. Further, applying (17) and (19), we obtain

$$\alpha(n,T)^{-1}\lambda_n 2^{\lambda^{(n-1)}-(\lambda_n-1)}=O\left(\exp\left\{-\frac{(x_{n,T})^2}{2}\right\}(\lambda_n)^{\frac{3}{2}}2^{\lambda^{(n-1)}-\lambda_n}\right)=o(1),\ n,T\to\infty.$$

Due to these relations, the augend on the right-hand side of (20) tends to zero as $n,T\to\infty$, and this convergence is uniform for all T, for which $x_{n,T}$ lies in any fixed finite interval.

To estimate the addend on the right-hand side of (20), we employ equality (15) and the Moivre – Laplace local theorem. Thus, we obtain that

$$\left|\frac{1}{D_n}\binom{\lambda_n-1}{T}\alpha(n,T)^{-1}-1\right|=o(1),\ n,T\to\infty,$$

where $o(1)$ tends to zero uniformly for all T, for which $x_{n,T}$ lies in any fixed finite interval.

So equality (16) is completely proved, and so is the theorem.

Corollary. *Under condition (14), the sequence of random variables* $\{\zeta_n : n=0,1,\dots\}$ *is asymptotically normal with parameters* $\frac{1}{2}(\lambda_n-1),\frac{1}{2}\sqrt{\lambda_n-1}$*:*

$$\mathbf{P}\left\{\frac{\zeta_n-\frac{1}{2}(\lambda_n-1)}{\frac{1}{2}\sqrt{\lambda_n-1}}\le x\right\}\to\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{t^2}{2}}dt,\ n\to\infty.$$

Notice that condition (14) is fulfilled if L_n is one of the lattices $B(n), L(n,q), \Re(n+1)$ from Section 1. For $B(n)$, Theorem 1 and its Corollary were earlier obtained by Sachkov [7]. It is evident that, in this case, the limit on the left-hand side of (14) equals 1/2.

If $L_n = L(n, q),\ n = 0, 1, \dots$, then $\lambda_n = G_n, \lambda^{(n-1)} = G_{n-1}$, where $G_n = \sum_{k=0}^{n} \left[{n \atop k} \right]_q$ is the total subspace number of the vector space $V(n, q)$ (the Galois number) [6]. In this case, inequality (14) follows from the asymptotic formula [8]

$$G_n - 1 = \theta_{n(2)}(q) q^{\frac{n^2}{4}} (1 + O(q^{-\frac{n}{2}})),\ n \to \infty, \tag{21}$$

where $n(2)$ denotes the residue n modulo 2,

$$\theta_0(q) = \frac{1}{(q^{-1})_\infty} \sum_{n=-\infty}^{\infty} q^{-n^2},\ \theta_1(q) = \frac{1}{(q^{-1})_\infty} \sum_{n=-\infty}^{\infty} q^{-(n-\frac{1}{2})^2}, \tag{22}$$

$$(q^{-1})_\infty = \prod_{m=1}^{\infty} (1 - q^{-m}).$$

In the case of $L_n = \Re(n + 1),\ n = 0, 1, \dots$, (14) follows from the asymptotic formula for Bell numbers (see, for example, [6, p. 297]).

4. Probability distribution asymptotic behavior of the cover index of a finite homogeneous lattice

Let $\Xi = X_1, X_2, \dots$ be a sequence of independent random elements of the lattice L_n.

Definition 2. The cover index of the lattice L_n by elements of the sequence Ξ is the least $\theta = \theta(n, \Xi) \in N$ such that

$$X_1 \vee \dots \vee X_\theta = 1_n. \tag{23}$$

This definition extends the concept of the n-set cover index by its independent random subsets (see [9]).

Taking into account the equality $\{\theta(n, \Xi) \le T\} = \{r_{n,T} = 0\}$, where $r_{n,T} = n - r(X_1 \vee \dots \vee X_T)$ is the co-rank of the join of T first elements of the sequence Ξ, we can to apply the lattice moments method [2] for obtaining the limit distribution of the random sequence $\theta(n, \Xi)$.

For any natural n, T and $l \in \overline{0, n}$, we set

$$B_{n,l}^{(T)} = \sum_{\substack{Y \in L_n: \\ r(Y) = n - l}} \mathbf{P}\{X_1 \le Y\} \dots \mathbf{P}\{X_T \le Y\}. \tag{24}$$

Theorem 2. *Let n and $T = T(n)$ be such that, for any $l = 0, 1, \dots$,*

$$B_{n,l}^{(T)} \to B_l < \infty,\ n \to \infty. \tag{25}$$

Then, if the series

$$B(z) \stackrel{\text{def}}{=} \sum_{l=0}^{\infty} B_l \chi(z) = \sum_{l=0}^{\infty} B_l \left(\sum_{k=0}^{l} w(l, l - k) z^k \right) \tag{26}$$

is uniformly convergent in a neighborhood of zero, then

$$\lim_{n \to \infty} \mathbf{P}\{\theta(n, \Xi) \le T\} = \sum_{l=0}^{\infty} w(l, l) B_l. \tag{27}$$

Proof. It follows from Theorem 2 in [1] that numbers (24) are equal to the lattice moments of the random variable $r_{n,T}$. Thus, expression (27) follows directly from equalities (25) and $\{\theta(n, \Xi) \le T\} = \{r_{n,T} = 0\}$ and the statement of Theorem 2 in [2].

Theorem 2 yields the expression for the limit distribution law (as $n \to \infty$) of the subspace lattice cover index of the n-dimensional vector space over a field with q elements.

Theorem 3. *Let Ξ be a sequence of independent and equiprobable random subspaces in the vector space $V(n,q)$. Then*

$$\lim_{\substack{n\to\infty:\\ n\equiv i(\mod 2)}} \mathbf{P}\{\theta(n,\Xi)\le 2\} = \frac{1}{2}\left(\frac{1+C_i}{(-q^{-1/2})_\infty} + \frac{1-C_i}{(q^{-1/2})_\infty}\right), \tag{28}$$

where

$$i\in\{0,1\},\ C_0 = 4q^{-1/2},\ C_1 = 1/4q^{1/2},\ (x)_\infty = \prod_{m=0}^{\infty}(1-xq^{-m}). \tag{29}$$

In addition,

$$\lim_{n\to\infty} \mathbf{P}\{\theta(n,\Xi)\le 3\} = 1. \tag{30}$$

Proof. From (24), we get

$$B_{n,l}^{(T)} = \begin{bmatrix} n \\ l \end{bmatrix}_q \left(\frac{G_{n-l}}{G_n}\right)^T,\ n,T\in\{1,2,\dots\},\ l\in\overline{0,n}. \tag{31}$$

Hence, applying (21), it is easy to obtain that, for $T=3$,

$$B_0^{(3)} \overset{\text{def}}{=} \lim_{n\to\infty} B_{n,0}^{(3)} = 1, B_l^{(3)} = \lim_{n\to\infty} B_{n,l}^{(3)} = 0,\ l=1,2,\dots .$$

Note that series (26) corresponding to values $B_l = B_l^{(3)}$, $l=0,1,\dots$ is uniformly convergent over the complex plane. Hence, we obtain (30) from (27).

For $T=2$, applying (31) and (21), we get

$$B_{n,l}^{(2)} = \frac{q^{-l^2/2}}{(q^{-1})_l}\left(\frac{\theta_{(n-l)(2)}}{\theta_{n(2)}}\right)^2(1+o(1)),\ n\to\infty,\ l=0,1,\dots ,$$

Hence, taking into account the equality $\theta_1(q) = 2q^{-1/4}\theta_0(q)$ emerging from the Jacobi identity (see, for example, [10]), we obtain

$$B_{0,l} \overset{\text{def}}{=} \lim_{\substack{n\to\infty:\\ n\equiv 0(\mod 2)}} B_{n,l}^{(2)} = \frac{q^{-l^2/2}}{(q^{-1})_l},\ \text{if } l\equiv 0(\mod 2),$$

$$B_{0,l} = 4q^{-1/2}\frac{q^{-l^2/2}}{(q^{-1})_l},\ \text{if } l\equiv 1(\mod 2), \tag{32}$$

$$B_{1,l} \overset{\text{def}}{=} \lim_{\substack{n\to\infty:\\ n\equiv 1(\mod 2)}} B_{n,l}^{(2)} = \frac{q^{-l^2/2}}{(q^{-1})_l},\ \text{if } l\equiv 0(\mod 2),$$

$$B_{1,l} = 1/4q^{1/2}\frac{q^{-l^2/2}}{(q^{-1})_l},\ \text{if } l\equiv 0(\mod 2); \tag{33}$$

As far as series (26) corresponding to each sequence $\{B_l = B_{i,l} : l=0,1,\dots\}, i\in\{0,1\}$, is uniformly convergent over the whole complex plane, we obtain, by applying (27), the following equalities:

$$\lim_{\substack{n\to\infty:\\ n\equiv i(\mod 2)}} \mathbf{P}\{\theta(n,\Xi)\le 2\} = \sum_{l=0}^{\infty} w(l,l)B_{i,l} = \sum_{l=0}^{\infty}(-1)^l q^{\binom{l}{2}} B_{i,l},\ i\in\{0,1\}. \tag{34}$$

Substituting (32), (33), respectively, in (34) and applying the equalities

$$\sum_{\substack{k\geq 0:\\ k\equiv 0 (\mod 2)}} (-1)^k \frac{q^{-k/2}}{(q^{-1})_k} = \frac{1}{2}\left(\frac{1}{(-q^{-1/2})_\infty} + \frac{1}{(q^{-1/2})_\infty}\right),$$

$$\sum_{\substack{k\geq 0:\\ k\equiv 1 (\mod 2)}} (-1)^k \frac{q^{-k/2}}{(q^{-1})_k} = \frac{1}{2}\left(\frac{1}{(-q^{-1/2})_\infty} - \frac{1}{(q^{-1/2})_\infty}\right)$$

following from the identity $\sum_{k=0}^{\infty} \frac{t^k}{(q^{-1})_k} = \frac{1}{(t)_\infty}$, $|t| < 1$ (see [10]), we obtain equality (28) after simple transformations.

So, the theorem is proved.

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