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EXISTENCE OF GENERALIZED LOCAL TIMES FOR GAUSSIAN RANDOM FIELDS

We consider a Gaussian centered random field that has values in the Euclidean space. We investigate the existence of local time for the random field as a generalized functional, an element of the Sobolev space constructed for our random field. We give the sufficient condition for such an existence in terms of the field covariation and apply it in a few examples: the Brownian motion with additional weight and the intersection local time of two Brownian motions.

Introduction

This article was motivated by the following problem. Let W_1, W_2 be two Brownian motions in the d-dimensional Euclidean space. Suppose that they are jointly Gaussian, and their covariation is as follows: $EW_1(s)W_2(t)^T = min(s,t)Q$ (Q is a $d \times d$ matrix). Consider the intersection local time on the time interval $[0,1]$ of these two processes as a limit of the approximations $L_\varepsilon = \int_0^1 \int_0^1 f_\varepsilon(W_1(s) - W_2(t))dsdt$, where f_ε is an approximation of the measure with unit mass at zero: $f_\varepsilon(x) = \frac{1}{\varepsilon^d} f(\frac{x}{\varepsilon}), f \in C_b^\infty(\mathbb{R}^d), f \geqslant 0, \int_{\mathbb{R}^d} f(x)dx = 1$ ($C_b^\infty(\mathbb{R}^d)$ is the space of bounded infinitely differentiable functions on $\mathbb{R}^d$). Find the conditions on d and Q such that these approximations converge in the Sobolev spaces $D_{2,\alpha}$ constructed over an abstract Wiener space associated with the processes (for the exact definition, see below).

To answer this question, we derive the sufficient condition in a more general statement. The main problem here is how to deal with approximations in order to find the index of the Sobolev space, where we have the convergence. In work [1], this was done for the self-intersection local time of the Brownian motion by finding the explicit form for the Itô—Wiener expansion kernels of approximations. But it is enough, in fact, to find the scalar product of those kernels in the space of square integrable functions. Moreover, it is enough to find the asymptotic behavior of this product, as the kernel index tends to infinity. The latter can be done using a certain integral representation of kernel scalar products. We are able to generalize this approach and implement it for an arbitrary Gaussian random field. In our generalization, we use the fact that the intersection local time of two Brownian motions may be thought as the local time of a Gaussian random field on the two-dimensional parameter space.

Let $(T, \mathfrak{B})$ be a measurable space with finite measure ν on it. Consider a centered Gaussian random field ξ on $(T, \mathfrak{B})$, where each $\xi(t)$ is a centered Gaussian random vector in $\mathbb{R}^d$. Denote $K_{ij}(s,t) = E\xi_i(s)\xi_j(t)$. We suppose that K_{ij} are measurable functions, and $\det K(t,t) > 0$ ν-a.s. Additionally, we suppose that ξ is separable. In other words, it is determined by its values in a countable set of points. Then there exists an abstract Wiener space (B, H, μ) (see [6]) such that ξ can be constructed on the probability space $(B, \mathfrak{F}, \mu)$ ($\mathfrak{F}$ is the σ-algebra of Borel sets in B) as a linear functional on

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it: $\xi^k(t)(\omega) = l_t^k(\omega), \omega \in B, l_t^k \in H, k = 1, \dots, d, t \in T$. Here, B is a separable Banach space, and μ is the centered Gaussian probability measure on it. Using the structure of an abstract Wiener space, we can introduce Sobolev spaces as in [5]. The space of square integrable functions can be represented as a direct sum of eigenspaces of the Ornstein—Uhlenbeck operator L: $L_2(B, \mu) = \oplus_{n=0}^{\infty} M_n$, where $Lx = -nx, x \in M_n$. Consequently, we can represent each square integrable function $f \in L_2(B, \mu)$ as a sum of eigenvectors $f = \sum_{n=0}^{\infty} f_n, f_n \in M_n$. Such a decomposition is called the Itô—Wiener decomposition. Now we have $L_2(\Omega) = L_2(B, \mu)$, so every square integrable ξ-measurable random variable has this decomposition. Consider $P(B, \mu) \subset L_2(B, \mu)$, being a subspace of the space of square integrable functions which consists of the functions that have only finite non-zero members in the associated Itô—Wiener decomposition. Introduce the norm indexed by α on the subspace $P(B, \mu)$:

$$\|f\|_{2,\alpha}^2 = \left\|(I - L)^{\frac{\alpha}{2}} f\right\|_2^2 = \sum_{n=0}^{N} (1+n)^{\alpha} \|f_n\|_2,$$

where N is the maximum of indices corresponding to non-zero decomposition members, and $\|\cdot\|_2$ is norm in $L_2(B, \mu)$. We denote the completion of $P(B, \mu)$ equipped with this norm as $D_{2,\alpha}$ and call it the Sobolev space with index $2, \alpha$.

We want to have the convergence of approximations for the local time in these spaces. We consider the local time at $x = 0$ of ξ with regard to some finite measure ν on $(T, \mathfrak{B})$ and approximations having form $L_\varepsilon = \int_T f_\varepsilon(\xi(t))\nu(dt)$, where f_ε is the same as above.

We have to note that this approach to the local time generalization is not unique. There exists an approach which makes use of the white noise theory (see, e.g., [2]). Another approach introduces generalized homogeneous functionals for the Brownian motion [3].

In the first part, we give the sufficient condition for the existence of a square integrable local time (as a limit of approximations). In the second one, we get the integral representation for the scalar products of Itô—Wiener expansion members. In the third part, we study the asymptotic behavior of this representation and use it to prove the convergence of approximations in the appropriate Sobolev space. In examples, we show how our conditions for the local time existence can be used in some cases including the problems on the intersection and the self-intersection local time for the Brownian motion.

Existence of square integrable local time

At first, we give the sufficient condition for the existence in $L_2(\Omega)$. The same condition for the processes mentioned in work [4]. By $B(s, t)$, we denote a symmetric matrix $2d \times 2d$ such that

$$B_{ij}(s,t) = K_{ij}(s,s), i = 1..d, j = 1..d$$

$$B_{ij}(s,t) = K_{i-d,j}(s,t), i = d+1..2d, j = 1..d$$

$$B_{ij}(s,t) = K_{i-d,j-d}(t,t), i = d+1..2d, j = d+1..2d.$$

In other words, $B(s, t)$ is the covariance matrix of a vector $(\xi(s), \xi(t))$.

Theorem 1. *Suppose that*

$$\int_T \int_T \frac{1}{\sqrt{\det B(s,t)}} \nu(ds)\nu(dt) < +\infty,$$

then the limit of L_ε exists in $L_2(\Omega)$.

Proof. It is sufficient to prove that $EL_{\varepsilon_1} L_{\varepsilon_2} \to C < +\infty, \varepsilon_1, \varepsilon_2 \to 0+$. Note that the condition of Theorem 1 gives: $\det B(s, t) > 0$ a.e. with regard to the measure $\nu \times \nu$.

Using the Fubini theorem and performing the change of variables $\varepsilon_1 u = x, \varepsilon_2 w = y$, we get

$$\begin{aligned} EL_{\varepsilon_1} L_{\varepsilon_2} &= \int_T \int_T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f_{\varepsilon_1}(x) f_{\varepsilon_2}(y) (2\pi)^{-d} (\det B(s,t))^{-\frac{1}{2}} \\ &\qquad \times \exp\left(-\frac{1}{2} \begin{pmatrix} x \\ y \end{pmatrix}^T B^{-1}(s,t) \begin{pmatrix} x \\ y \end{pmatrix}\right) dxdy\nu(ds)\nu(dt) \\ &= \int_T \int_T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(u) f(w) (2\pi)^{-d} (\det B(s,t))^{-\frac{1}{2}} \\ &\qquad \times \exp(-\frac{1}{2} \begin{pmatrix} \varepsilon_1 u \\ \varepsilon_2 w \end{pmatrix}^T B^{-1}(s,t) \begin{pmatrix} \varepsilon_1 u \\ \varepsilon_2 w \end{pmatrix}) dudw\nu(ds)\nu(dt). \end{aligned}$$

The function inside all integrals is monotonously converging to

$$f(u)f(w)(2\pi)^{-d}(\det B(s,t))^{-\frac{1}{2}}$$

when $\varepsilon_1, \varepsilon_2 \to 0+$. So, by applying the theorem on monotonous convergence, we conclude that $EL_{\varepsilon_1} L_{\varepsilon_2} \to \int_T \int_T (2\pi)^{-d} (\det B(s,t))^{-\frac{1}{2}} \nu(ds)\nu(dt)$, and Theorem 1 is proved.

Covariation formula

As we stated in Introduction, we can write an Itô—Wiener expansion for the following expression, because it is a bounded and therefore square integrable random variable:

$$\int_T f(\xi(t))\nu(dt) = \sum_{n=0}^{\infty} a_n(f), f \in C_b^\infty(\mathbb{R}^d), a_n(f) \in M_n$$

Note that this is the expression appeared in our local time approximations. We want to obtain a representation for $Ea_n(f)a_n(g), f, g \in C_b^\infty(\mathbb{R}^d)$.

Theorem 2. *For any functions $f, g \in C_b^\infty(\mathbb{R}^d)$,*

$$Ea_n(f)a_n(g) = \\ = \int_T \int_T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x)g(y) e^{i(x,\gamma_1)+i(y,\gamma_2)} q_n(s,t,\gamma_1,\gamma_2) dxdyd\gamma_1 d\gamma_2 \nu(ds)\nu(dt),$$

where

$$q_n(s,t,x,y) = (2\pi)^{-2d} \frac{(-1)^n}{n!} (K(s,t)x,y)^n e^{-\frac{1}{2}(K(s,s)x,x)} e^{-\frac{1}{2}(K(t,t)y,y)}$$

Proof. Consider the Ornstein—Uhlenbeck semigroup $\{T_\lambda, \lambda \geq 0\}$ on our abstract Wiener space. By definition (see [5]), it acts on any function $h \in L_2(B,\mu)$ in the following way: $T_\lambda h(x) = \int_B h(e^{-\lambda}x + \sqrt{1-e^{-2\lambda}}y)\mu(dy)$. It also has property that its action on the same function with the Itô—Wiener decomposion $h = \sum_{n=0}^\infty h_n, h_n \in M_n$ may be written as $T_\lambda h = \sum_{n=0}^\infty e^{-n\lambda} h_n$. Using this property, we get

$$E(\int_T f(\xi(t))\nu(dt) \cdot T_\lambda \int_T g(\xi(t))\nu(dt)) = \sum_{n=0}^{\infty} e^{-n\lambda} Ea_n(f)a_n(g).$$

Here, we used the fact that $Ea_n(f)a_m(g) = 0, m \neq n$. On the other hand, by the definition of the Ornstein—Uhlenbeck semigroup and because $\xi(t)(e^{-\lambda}x+\sqrt{1-e^{-2\lambda}}y) = e^{-\lambda}\xi(t)(x) + \sqrt{1-e^{-2\lambda}}\xi(t)(y), x, y \in B$, we have

$$
\begin{aligned}
&E(\int_T f(\xi(t))\nu(dt) \cdot T_\lambda \int_T g(\xi(t))\nu(dt)) = \\
&= E(\int_T f(\xi(t))\nu(dt) \cdot E(\int_T g(e^{-\lambda}\xi(t) + \sqrt{1-e^{-2\lambda}}\tilde{\xi}(t))\nu(dt)/\xi)) = \\
&= \int_T \int_T E(f(\xi(s))g(e^{-\lambda}\xi(t) + \sqrt{1-e^{-2\lambda}}\tilde{\xi}(t)))\nu(ds)\nu(dt),
\end{aligned}
$$

where $\tilde{\xi}$ is an independent copy of ξ. We can swap integrals and expectations, because the expression inside all integrals is bounded. Denote $B_\beta(s,t) = \begin{pmatrix} K(s,s) & \beta K(t,s) \\ \beta K(s,t) & K(t,t) \end{pmatrix}$. The distribution of $(\xi(s), e^{-\lambda}\xi(t) + \sqrt{1-e^{-2\lambda}}\tilde{\xi}(t))$ is centered Gaussian with the covariation matrix $B_{e^{-\lambda}}(s,t)$. By Lemma 1 and our assumptions on the covariation: if $\lambda > 0$, then $\det B_{e^{-\lambda}}(s,t) > 0$ for all $(s,t) \in T^2$ except a set of the zero measure $\nu \times \nu$. This yields

$$
\begin{aligned}
&E(\int_T f(\xi(t))\nu(dt) \cdot T_\lambda \int_T g(\xi(t))\nu(dt)) = \\
&= \int_T \int_T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x)g(y) \frac{1}{(2\pi)^d \sqrt{\det B_{e^{-\lambda}}(s,t)}} \cdot \\
&\cdot \exp(-\frac{1}{2} \begin{pmatrix} x \\ y \end{pmatrix}^T B^{-1}_{e^{-\lambda}}(s,t) \begin{pmatrix} x \\ y \end{pmatrix}) dxdy\nu(ds)\nu(dt) = \\
&= \int_T \int_T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x)g(y)e^{i(x,\gamma_1)+i(y,\gamma_2)} q^\lambda(s,t,\gamma_1,\gamma_2) d\gamma_1 d\gamma_2 dxdy\nu(ds)\nu(dt),
\end{aligned}
$$

where

$$
\begin{aligned}
q^\lambda(s,t,x,y) &= (2\pi)^{-2d} \exp(-\frac{1}{2} \begin{pmatrix} x \\ y \end{pmatrix}^T B_{e^{-\lambda}}(s,t) \begin{pmatrix} x \\ y \end{pmatrix}) = \\
&= (2\pi)^{-2d} \exp(-\frac{1}{2}((K(s,s)x,x) + (K(t,t)y,y) + 2e^{-\lambda}(K(s,t)x,y)))
\end{aligned}
$$

Obviously:

$$
q^\lambda(s,t,x,y) = \sum_{n=0}^{\infty} e^{-n\lambda} q_n(s,t,x,y), s,t \in T, x,y \in \mathbb{R}^d.
$$

We substitute q in the integral with this sum. By the Lebesgue dominated convergence theorem, we can exchange this sum with the integral by λ_1, λ_2. If we can show that it can be exchanged with other integrals for all $\lambda > 0$, we get two different representations for the starting expression in the form of a power series of $e^{-\lambda}$. Then we match members of these series and get the desired integral representation. Fix $(s,t) \in T^2$ such that $\det K(s,s) > 0$, $\det K(t,t) > 0$, and $\lambda > 0$. Denote:

$$
\begin{aligned}
p(s,t,x,y) &= \frac{1}{(2\pi)^d \sqrt{\det B_0(s,t)}} \exp(-\frac{1}{2} \begin{pmatrix} x \\ y \end{pmatrix}^T B_0^{-1}(s,t) \begin{pmatrix} x \\ y \end{pmatrix}) \\
&= \frac{1}{(2\pi)^d \sqrt{\det K(s,s) \det K(t,t)}} \exp(-\frac{1}{2}((K^{-1}(s,s)x,x) + (K^{-1}(t,t)y,y))), \\
r_n(s,t,x,y) &= \frac{1}{p(s,t,x,y)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(x,\gamma_1)+i(y,\gamma_2)} q_n(s,t,\gamma_1,\gamma_2) d\gamma_1 d\gamma_2,
\end{aligned}
$$

$$R_n(s,t,x,y) = \sum_{k=0}^{n} e^{-k\lambda} r_k(s,t,\cdot,\cdot),$$

$$\begin{aligned} R(s,t,x,y) &= \sum_{k=0}^{\infty} e^{-k\lambda} r_k(s,t,\cdot,\cdot) \\ &= \frac{1}{p(s,t,x,y)} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i(x,\gamma_1)+i(y,\gamma_2)} q^{\lambda}(s,t,\gamma_1,\gamma_2) d\gamma_1 d\gamma_2 \\ &= \frac{\sqrt{\det B_0(s,t)}}{\sqrt{\det B_{e^{-\lambda}}(s,t)}} \exp\left(-\frac{1}{2}\begin{pmatrix} x \\ y\end{pmatrix}^T (B^{-1}_{e^{-\lambda}} - B_0^{-1})(s,t)\begin{pmatrix} x \\ y\end{pmatrix}\right). \end{aligned}$$

We want to show that the integrals can be exchanged with the sum. Now it can be formulated as

$$\lim_{n\to\infty} \int_T\int_T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x)g(y)R_n(s,t,x,y)p(s,t,x,y)dxdy\nu(ds)\nu(dt) =$$
$$= \int_T\int_T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x)g(y)R(s,t,x,y)p(s,t,x,y)dxdy\nu(ds)\nu(dt).$$

Consider the space of functions on $\mathbb{R}^{2d}$ which are square integrable by a Gaussian measure with density $p(s,t,x,y)$ and denote it as $L_2(\mathbb{R}^{2d}, p(s,t,x,y)dxdy)$ with the corresponding scalar product $<\cdot,\cdot>$. It is easy to check that $< r_n(s,t,\cdot,\cdot), r_m(s,t,\cdot,\cdot) >= 0$ if $m \neq n$ (because r_n is a linear combination of multidimensional Hermitian polynomials of degree $2n$). Therefore,

$$\begin{aligned} &\sup_n < R_n(s,t,\cdot,\cdot), R_n(s,t,\cdot,\cdot) >= \\ &= \sup_n \sum_{k=0}^{n} e^{-2k\lambda} < r_k(s,t,\cdot,\cdot), r_k(s,t,\cdot,\cdot) >\le \\ &\le < R(s,t,\cdot,\cdot), R(s,t,\cdot,\cdot) > . \end{aligned}$$

Using Lemmas 2 and 3 (see below), we obtain

$$\begin{aligned} &< R(s,t,\cdot,\cdot), R(s,t,\cdot,\cdot) >= \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{1}{(2\pi)^d} \frac{\sqrt{\det B_0(s,t)(s,t)}}{\det B_{e^{-\lambda}}(s,t)} \exp(-\frac{1}{2}\begin{pmatrix} x \\ y\end{pmatrix}^T (2B^{-1}_{e^{-\lambda}} - B_0^{-1})(s,t)\begin{pmatrix} x \\ y\end{pmatrix})dxdy = \\ &= \sqrt{\det B_0^{-1} B_{e^{-\lambda}} B^{-1}_{-e^{-\lambda}}(s,t)} \frac{\sqrt{\det B_0(s,t)(s,t)}}{\det B_{e^{-\lambda}}(s,t)} = \frac{\det B_0(s,t)}{\sqrt{\det B_{e^{-\lambda}}(s,t) \det B_{-e^{-\lambda}}(s,t)}} \le \\ &\le (1-e^{-\lambda})^{-2d}. \end{aligned}$$

Applying the inequalities given above, we get

$$\begin{aligned} &\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(x)g(y)R_n(s,t,x,y)| p(s,t,x,y)dxdy \le \\ &\le \sqrt{< f(x)g(y), f(x)g(y) >< R_n(s,t,\cdot,\cdot), R_n(s,t,\cdot,\cdot) >} \le \\ &\le \sup_{x,y\in\mathbb{R}^d} |f(x)g(y)|(1-e^{-\lambda})^{-d}. \end{aligned}$$

We can see that the bound is independent of (s,t). So, by the Lebesgue dominated convergence theorem, we proved the equality and Theorem 2.

Now we prove a few technical results. Let m, n be positive integers, and let A be an $(m+n)\times(m+n)$ symmetric non-negative definite real matrix of the form

$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix}$, where A_{11}, A_{12}, A_{22} are matrices with sizes $m \times m, m \times n, n \times n$, respectively, and A_{11}, A_{22} are positive definite. Define $A_\alpha = \begin{pmatrix} A_{11} & \alpha A_{12} \\ \alpha A_{12}^T & A_{22} \end{pmatrix}, \alpha \in \mathbb{R}$.

Lemma 1. *If $|\alpha| < 1$, then A_α is positive definite.*

Proof. Take the $n+m$-vector $\begin{pmatrix} u \\ v \end{pmatrix}$ as a concatenation of the m-vector u and the n-vector v. Then we have to prove that $(A_\alpha \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}) = (\begin{pmatrix} A_{11}u + \alpha A_{12}v \\ \alpha A_{12}^T u + A_{22}v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}) = (A_{11}u, u) + 2\alpha(A_{12}v, u) + (A_{22}v, v)$ is always positive. Let $u_1 = \alpha u, v_1 = v$, and let us use that $A = A_1$ is non-negative definite. We have

$$0 \leqslant (A_1 \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}) = \alpha^2(A_{11}u, u) + 2\alpha(A_{12}v, u) + (A_{22}v, v) =$$

$$= (\alpha^2 - 1)(A_{11}u, u) + (A_\alpha \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}) \leqslant (A_\alpha \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}).$$

So we have that A_α is non-negative definite. Moreover, if

$$(A_\alpha \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}) = 0,$$

then the equality holds in the inequality above, and we get $(A_{11}u, u) = 0$. But A_{11} is positive definite, and so $u = 0$. Similarly using that A_{22} is positive definite, we get $v = 0$. Hence, A_α is positive definite.

Lemma 2. *If $|\alpha| < 1$, then $2A_\alpha^{-1} - A_0^{-1} = A_0^{-1}A_{-\alpha}A_\alpha^{-1}$, and the matrix on the right-hand side is positive definite.*

Proof. Note that A_α^{-1} and A_0^{-1} exist by the previous lemma. Consider the obvious relation $2A_0 - A_\alpha = A_{-\alpha}$. Multiplying it by A_α^{-1} from the right and by A_0^{-1} from the left, we get the first part of the assertion. From Lemma 1, we also have that $A_0^{-1}, A_\alpha^{-1}, and A_{-\alpha}$ are positive definite. Then $A_0^{-1}A_{-\alpha}A_\alpha^{-1}$ is positive definite as a product of positive definite symmetric matrices.

Lemma 3. *If $|\alpha| < 1$, then there exists a constant $C(\alpha) > 0$ such that, for any matrix A, $\det A_0 \leqslant C(\alpha) \det A_\alpha$ with $C(\alpha) = (1 - |\alpha|)^{-(m+n)}$.*

Proof. We will prove that if B and C are symmetric $l \times l$ non-negative definite matrices, then $\det(B + C) \geqslant \det B$. There exists the basis in $\mathbb{R}^l$ such that C is a diagonal matrix with diagonal elements $\{c_i \geqslant 0, i = 1, \ldots, l\}$. Convert our matrices to this basis. By C_i, we denote the matrix with only one non-zero element c_i at the place (i, i) such that $C = \sum_{i=1}^l C_i$. By B_i, we denote the matrix obtained from B by removing the row i and the column i. Then we have $\det B_i \geqslant 0$, because B is non-negative definite, and $\det(B + C_1) = \det B + c_1 \det B_1 \geqslant \det B$. The matrix $B + C_1$ is again non-negative definite and, by the same argument, $\det(B + C_1 + C_2) \geqslant \det(B + C_1) \geqslant \det B$. Proceeding by induction, we get $\det(B + C) = \det(B + \sum_{i=1}^l C_i) \geqslant \det B$.

Suppose $\alpha \geqslant 0$. From the statement above, we obtain

$$\det A_\alpha = \det((1 - \alpha)A_0 + \alpha A) \geqslant \det((1 - \alpha)A_0) = (1 - \alpha)^{m+n} \det A_0.$$

The case $\alpha < 0$ is similar:

$$\det A_\alpha = \det((1 + \alpha)A_0 - \alpha A_{-1}) \geqslant \det((1 + \alpha)A_0) = (1 + \alpha)^{m+n} \det A_0$$

(A_{-1} is non-negative definite by the same arguments as in Lemma 1).

Main result

We want to know the asymptotic behavior of $Ea_n(f_{\varepsilon_1})a_n(f_{\varepsilon_2})$ as $n \to \infty$, where f_ε approximates the delta-measure as defined in Introduction. Since

$$|Ea_n(f_{\varepsilon_1})a_n(f_{\varepsilon_2})|$$
$$= \left| \int_T\int_T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f_{\varepsilon_1}(x)f_{\varepsilon_2}(y)e^{i(x,\gamma_1)+i(y,\gamma_2)}q_n(s,t,\gamma_1,\gamma_2)dxdyd\gamma_1 d\gamma_2\nu(ds)\nu(dt)\right|$$
$$\le \int_T\int_T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f_{\varepsilon_1}(x)f_{\varepsilon_2}(y)|q_n(s,t,\gamma_1,\gamma_2)|dxdyd\gamma_1 d\gamma_2\nu(ds)\nu(dt)$$
$$= \int_T\int_T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |q_n(s,t,\gamma_1,\gamma_2)|d\gamma_1 d\gamma_2\nu(ds)\nu(dt),$$

we are interested in the asymptotic behavior of $\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|q_n(s,t,x,y)|dxdy$. Fix $(s,t)\in T^2$ such that $\det K(s,s)>0$ and $\det K(t,t)>0$. Denote

$$G(s,t) = K^{-1/2}(t,t)K(s,t)K^{-1/2}(s,s)$$
$$\|G(s,t)\| = \sup_{x\in\mathbb{R}^d, |x|=1}\|G(s,t)x\|$$

In other words, we have the correlation matrix and its operator norm.

Theorem 3. *There exists a constant $C>0$ such that, for any $(s,t)\in T^2$ satisfying $\det K(s,s)>0$ and $\det K(t,t)>0$, the following inequality holds for all integers $n\ge 0$:*

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|q_n(s,t,x,y)|dxdy \le C\left(\det K(s,s)\det K(t,t)\right)^{-1/2}\|G(s,t)\|^n(n+1)^{d/2-1}.$$

Proof. We have

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|q_n(s,t,x,y)|dxdy =$$
$$= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{1}{(2\pi)^{2d}}\frac{|(K(s,t)x,y)|^n}{n!}e^{-\frac12(K(s,s)x,x)}e^{-\frac12(K(t,t)y,y)}dxdy =$$
$$= \frac{1}{(2\pi)^{2d}}\left(\det K(s,s)\det K(t,t)\right)^{-1/2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|(G(s,t)x,y)|^n}{n!}e^{-\frac{\|x\|^2}{2}}e^{-\frac{\|y\|^2}{2}}dxdy.$$

Note that if $n=2k+1, k\ge 0$, then, by the Cauchy inequality,

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|(G(s,t)x,y)|^{2k+1}}{(2k+1)!}\frac{1}{(2\pi)^d}e^{-\frac{\|x\|^2}{2}}e^{-\frac{\|y\|^2}{2}}dxdy \le$$
$$\le \sqrt{\frac{2k+2}{2k+1}}\sqrt{\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|(G(s,t)x,y)|^{2k}}{(2k)!}\frac{1}{(2\pi)^d}e^{-\frac{\|x\|^2}{2}}e^{-\frac{\|y\|^2}{2}}dxdy}\times$$
$$\times\sqrt{\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|(G(s,t)x,y)|^{2k+2}}{(2k+2)!}\frac{1}{(2\pi)^d}e^{-\frac{\|x\|^2}{2}}e^{-\frac{\|y\|^2}{2}}dxdy}.$$

So, it is enough to consider the case $n=2k$. Using Lemma 4 (see below), we can establish a change of variables in the integral associated with the rotation around the origin in $\mathbb{R}^{2d}$ such that $(G(s,t)x,y)=\sum_{n=0}^d\lambda_n u_n v_n$, where u,v are the new variables, and $\lambda_i^2, i=1,\dots,n$, are the eigenvalues of $G(s,t)G^T(s,t)$. Note that λ_i can be chosen non-negative, and $\max_{i=1,\dots,n}\lambda_i=\sqrt{\|G(s,t)G^T(s,t)\|}=\|G(s,t)\|$. We note that

$$(\sum_{i=0}^d\lambda_i u_i v_i)^{2n} = (2n)!\sum_{\sum_{i=0}^d k_i=2n}\prod_{i=0}^d\frac{(\lambda_i u_i v_i)^{k_i}}{k_i!},$$

and all members of the sum with one of k_i being odd became zero after the integrating with a Gaussian measure. All other members are non-negative, so we can increase the sum by replacing all λ_i by $\|G(s,t)\|$. We get

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{(G(s,t)x,y)^{2n}}{(2n)!}e^{-\frac{\|x\|^2}{2}}e^{-\frac{\|y\|^2}{2}}dxdy =$$
$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{(\sum_{i=0}^d \lambda_i u_i v_i)^{2n}}{(2n)!}e^{-\frac{\|u\|^2}{2}}e^{-\frac{\|v\|^2}{2}}dudv \le$$
$$\le \|G(s,t)\|^{2n}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{(\sum_{i=0}^d u_i v_i)^{2n}}{(2n)!}e^{-\frac{\|u\|^2}{2}}e^{-\frac{\|v\|^2}{2}}dudv.$$

The last integral can be calculated precisely and satisfies the inequality

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{(\sum_{i=0}^d u_i v_i)^{2n}}{(2n)!}e^{-\frac{\|u\|^2}{2}}e^{-\frac{\|v\|^2}{2}}dudv \le \tilde{C}(2n+1)^{d/2-1}.$$

Hence we proved Theorem 3.

The following lemma was used in Theorem 3.

Lemma 4. *For any $d\times d$ matrix A, there exists the orthogonal $2d\times 2d$ matrix U $(U^TU = I)$ such that $(Ax,y) = \sum_{n=0}^d \lambda_n u_n v_n$, where $x,y,u,v \in \mathbb{R}^d$, $\begin{pmatrix} u \\ v \end{pmatrix} = U\begin{pmatrix} x \\ y \end{pmatrix}$ and $\lambda_i^2, i = 1,\dots,n$, are the eigenvalues of AA^T.*

Proof. Denote $z = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{2d}, x,y \in \mathbb{R}^d$, and $Q = \begin{pmatrix} 0 & A^T \\ A & 0 \end{pmatrix}$. Note that $(Qz,z) = 2(Ax,y)$, and Q is symmetric. All eigenvalues of Q are the set $\{\lambda_i, -\lambda_i, i = 1,\dots,n\}$, where λ_i are from the statement of Lemma 4. This is true, because if $\lambda^2 > 0$, then the equations $Qz = \lambda z$ and $AA^Ty = \lambda^2 y, \lambda x = A^Ty$ are equivalent. So we can find the orthogonal $2d\times 2d$ matrix U_1 such that $2(Ax,y) = \sum_{n=0}^d \lambda_n(\tilde{u}_n^2 - \tilde{v}_n^2)$ and $\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = U_1\begin{pmatrix} x \\ y \end{pmatrix}$. This easily yields the statement of Lemma 4.

We are ready to establish the convergence of approximations. Denote

$$J_n = \int_T\int_T \|G(s,t)\|^n\left(\det K(s,s)\det K(t,t)\right)^{-1/2}\nu(ds)\nu(dt)$$

Theorem 4. *1) If $J_n < +\infty$, then $\lim_{\varepsilon\to 0+} a_n(f_\varepsilon)$ exists in $L_2(\Omega)$. For odd n, the limit is equal to zero.*

2) If $\sum_{n=0}^\infty J_n(n+1)^{\alpha+\frac{d}{2}-1} < +\infty$, then $\lim_{\varepsilon\to 0+} L_\varepsilon$ exists in $D_{2,\alpha}$.

Proof. From Theorem 2, we know that

$$Ea_n(f_{\varepsilon_1})a_n(f_{\varepsilon_2})$$
$$= \int_T\int_T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f_{\varepsilon_1}(x)f_{\varepsilon_2}(y)e^{i(x,\gamma_1)+i(y,\gamma_2)}q_n(s,t,\gamma_1,\gamma_2)dxdyd\gamma_1 d\gamma_2\nu(ds)\nu(dt)$$
$$= \int_T\int_T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x)f(y)e^{i(\varepsilon_1 x,\gamma_1)+i(\varepsilon_2 y,\gamma_2)}q_n(s,t,\gamma_1,\gamma_2)dxdyd\gamma_1 d\gamma_2\nu(ds)\nu(dt).$$

From Theorem 3, we have

$$|Ea_n(f_{\varepsilon_1})a_n(f_{\varepsilon_2})| \le$$
$$\le \int_T\int_T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x)f(y)|q_n(s,t,\gamma_1,\gamma_2)|dxdyd\gamma_1 d\gamma_2\nu(ds)\nu(dt) \le .$$
$$\le CJ_n(n+1)^{d/2-1}$$

By the Lebesgue dominated convergence theorem

$$Ea_n(f_{\varepsilon_1})a_n(f_{\varepsilon_2}) \to \int_T\int_T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} q_n(s,t,x,x)dxdy\nu(ds)\nu(dt), \varepsilon_1, \varepsilon_2 \to 0+ .$$

We conclude that the limit of $a_n(f_\varepsilon)$ as $\varepsilon \to 0+$ exists in L_2. For odd n, the limit is obviously zero so we proved the first part of Theorem 4. The second part easily follows from the first part because each $a_n(f_\varepsilon)$ satisfies (as noted above) $\|a_n(f_\varepsilon)\|_2^2 \le CJ_n(n+1)^{d/2-1}$.

Examples

All conditions we imposed on ξ in Introduction are obviously satisfied when $T \subset \mathbb{R}^d$ and ξ is a continuous centered Gaussian random field. So, without a further notice, we will use our results for such a kind of random fields.

Example 1. Let $T = [0,1], v(dt) = dt, \xi(t) = t^r W(t)$, where $W(t)$ is the d-dimensional Brownian motion. In this case, we have

$$K_{ii}(s,t) = (st)^r \min(s,t), K_{ij}(s,t) = 0,$$
$$\det K(s,s) = s^{(2r+1)d},$$
$$\|G(s,t)\| = \frac{min(s,t)}{\sqrt{st}},$$
$$J_n = \int_0^1\int_0^1 (st)^{-\frac{n}{2}-\frac{(2r+1)d}{2}}(\min(s,t))^n dsdt.$$

It easy to see that the condition $J_n < +\infty$ is satisfied for $r < \frac{1}{d} - \frac{1}{2}$ for all $n \geqslant 0$ (and failed for all n in either case), and

$$J_n = \frac{4}{(n+2-(2r+1)d)(2-(2r+1)d)} = O(\frac{1}{n}).$$

Using Theorem 4, we conclude that the limit of L_ε exists in $D_{2,\alpha}$ for $\alpha < 1 - \frac{d}{2}$.

Example 2. Let $T = [0,1], v(dt) = dt, \xi(t) = \int_0^t s^r dW(s)$. The integral exists if $r > -\frac{1}{2}$. As in the previous example, we can easily compute J_n:

$$K_{ii}(s,t) = \frac{1}{2r+1}(\min(s,t))^{2r+1}, K_{ij}(s,t) = 0,$$
$$\det K(s,s) = (2r+1)^{-d}s^{(2r+1)d},$$
$$\|G(s,t)\| = \left(\frac{min(s,t)}{\sqrt{st}}\right)^{2r+1},$$
$$J_n = (2r+1)^{-d}\int_0^1\int_0^1 (st)^{-\frac{n(2r+1)}{2}-\frac{d(2r+1)}{2}}(\min(s,t))^{n(2r+1)}dsdt =$$
$$= (2r+1)^{-d}\frac{4}{(n(2r+1)+2-d(2r+1))(2-d(2r+1))} = O(\frac{1}{n}).$$

As in the previous example, $J_n < +\infty$ is satisfied for $r < \frac{1}{d} - \frac{1}{2}$, and L_ε converges in $D_{2,\alpha}$ for $\alpha < 1 - \frac{d}{2}$. This example and the previous one are the generalizations of the classical result about the existence of a local time for the Brownian motion. The additional multiplier $t^r, r \in (-\frac{1}{2}, -\frac{1}{2} + \frac{1}{d})$ prevents the explosion of Itô—Wiener expansion kernels. It is interesting to note that the explosion occurs near $t = 0$. So if we take $T = [\frac{1}{2}, 1]$ for this example and the previous one, then we may drop the restriction on r (and we may set $r = 0$ to get the usual Brownian motion). The convergence still takes place under same conditions on α.

Now we consider two special cases of the problem stated in Introduction for $Q = 0$ and $Q = I$. The second case can also be found in [1].

Example 3. Let $T = [0,1]^2, v(dt) = dt_1 dt_2, \xi(t_1,t_2) = W_1(t_1) - W_2(t_2)$, where $W_i(t), i = 1,2$, are two independent d-dimensional Brownian motions. We have

$$K_{ii}(s,t) = \min(s_1,t_1) + \min(s_2,t_2), K_{ij}(s,t) = 0,$$
$$\det K(s,s) = (s_1+s_2)^d,$$
$$\|G(s,t)\| = \frac{\min(s_1,t_1)+\min(s_2,t_2)}{\sqrt{(s_1+s_2)(t_1+t_2)}},$$
$$J_n = \int_0^1\int_0^1\int_0^1\int_0^1 \frac{(\min(s_1,t_1)+\min(s_2,t_2))^n}{((s_1+s_2)(t_1+t_2))^{\frac{n}{2}+\frac{d}{2}}} ds_1 ds_2 dt_1 dt_2.$$

To find the asymptotic behaviour for J_n, we divide the region of integration $T^2 = [0,1]^4$ into subregions $T_1 = \{0 \leqslant s_1 \leqslant t_1 \leqslant 1; 0 \leqslant s_2 \leqslant t_2 \leqslant 1\}$, $T_2 = \{0 \leqslant s_1 \leqslant t_1 \leqslant 1; 0 \leqslant t_2 \leqslant s_2 \leqslant 1\}$, $T_3 = \{0 \leqslant t_1 \leqslant s_1 \leqslant 1; 0 \leqslant s_2 \leqslant t_2 \leqslant 1\}$, $T_4 = \{0 \leqslant t_1 \leqslant s_1 \leqslant 1; 0 \leqslant t_2 \leqslant s_2 \leqslant 1\}$.

$$J_n^1 = \int_{T_1} \frac{(\min(s_1,t_1)+\min(s_2,t_2))^n}{((s_1+s_2)(t_1+t_2))^{\frac{n}{2}+\frac{d}{2}}} ds_1 ds_2 dt_1 dt_2$$
$$= \int_0^1\int_0^1\int_0^{t_2}\int_0^{t_1} (s_1+s_2)^{\frac{n-d}{2}} (t_1+t_2)^{\frac{-n-d}{2}} ds_1 ds_2 dt_1 dt_2$$
$$= \frac{1}{(1+\frac{n-d}{2})(2+\frac{n-d}{2})} \int_0^1\int_0^1 (t_1+t_2)^{\frac{-n-d}{2}} ((t_1+t_2)^{\frac{n-d}{2}+2} - t_1^{\frac{n-d}{2}+2} - t_1^{\frac{n-d}{2}+2}) dt_1 dt_2$$
$$\leqslant \tilde{C}\frac{1}{(n+1)^2}\int_0^1\int_0^1 (t_1+t_2)^{2-d} dt_1 dt_2 \leqslant \tilde{\tilde{C}}\frac{1}{(n+1)^2}\int_0^1 u^{3-d} du.$$

We can see that the sufficient condition for the integral to exist is $d < 4$ (the special cases $d = 2, n = 0; d = 3, n = 1$ behave themselves in a slightly different way: logarithms appear after the partial integration, but the corresponding integrals are still finite by same arguments).

$$J_n^2 = \int_{T_2} \frac{(\min(s_1,t_1)+\min(s_2,t_2))^n}{((s_1+s_2)(t_1+t_2))^{\frac{n}{2}+\frac{d}{2}}} ds_1 ds_2 dt_1 dt_2 =$$
$$= \int_0^1\int_0^1\int_{s_1}^1\int_{t_2}^1 (s_1+t_2)^n (s_1+s_2)^{\frac{-n-d}{2}} (t_1+t_2)^{\frac{-n-d}{2}} ds_2 dt_1 ds_1 dt_2 =$$
$$= \frac{1}{(1-\frac{n+d}{2})^2}\int_0^1\int_0^1 (s_1+t_2)^n((1+s_1)^{1-\frac{n+d}{2}} - (s_1+t_2)^{1-\frac{n+d}{2}})\times$$
$$\times ((1+t_2)^{1-\frac{n+d}{2}} - (s_1+t_2)^{1-\frac{n+d}{2}}) ds_1 dt_2 \leqslant$$
$$\leqslant C\frac{1}{(n+1)^2}\int_0^1\int_0^1 (s_1+t_2)^{2-d} ds_1 dt_2 \leqslant$$
$$\leqslant \tilde{C}\frac{1}{(n+1)^2}\int_0^1 u^{3-d} du.$$

We assumed above that $n + d > 2$. In all other cases, $d = 1, n = 0, 1; d = 2, n = 0$, we can check that the integral is finite. Again we have the sufficient condition $d < 4$. Two last subregions can be treated similarly, and we omit them. We conclude: $J_n < +\infty$ is satisfied for $d < 4$ and all $n \geqslant 0$. In this case, $J_n = O(\frac{1}{n^2})$, so the limit exists in $D_{2,\alpha}$ for $\alpha < 2 - \frac{d}{2}$. As a consequence, we have the existence of the intersection local time for two independent d-dimensional Brownian motions for $d = 1, 2, 3$.

Example 4. Let $T = [0,1]^2, v(dt) = dt_1dt_2, \xi(t_1,t_2) = W(t_1) - W(t_2)$. We have

$$K_{ii}(s,t) = \min(s_1,t_1) + \min(s_2,t_2) - \min(s_2,t_1) - \min(s_1,t_2), K_{ij}(s,t) = 0,$$

$$\det K(s,s) = (s_1 + s_2 - 2\min(s_1,s_2))^d,$$

$$\|G(s,t)\| = \frac{\min(s_1,t_1) + \min(s_2,t_2) - \min(s_2,t_1) - \min(s_1,t_2)}{\sqrt{(s_1+s_2-2\min(s_1,s_2))(t_1+t_2-2\min(t_1,t_2))}},$$

$$J_n = \int_0^1\int_0^1\int_0^1\int_0^1 \frac{(\min(s_1,t_1) + \min(s_2,t_2) - \min(s_2,t_1) - \min(s_1,t_2))^n}{((s_1+s_2-2\min(s_1,s_2))(t_1+t_2-2\min(t_1,t_2)))^{\frac{n}{2}+\frac{d}{2}}} ds_1ds_2dt_1dt_2.$$

As in the previous example, we have to split the region $[0,1]^4$ into a few subregions. We consider only three of them, others are similar: $T_1 = \{0 \leqslant s_1 \leqslant s_2 \leqslant t_1 \leqslant t_2 \leqslant 1\}$, $T_2 = \{0 \leqslant s_1 \leqslant t_1 \leqslant s_2 \leqslant t_2 \leqslant 1\}$, $T_3 = \{0 \leqslant s_1 \leqslant t_1 \leqslant t_2 \leqslant s_2 \leqslant 1\}$. In the first subregion, $K_{ii}(s,t) = 0$ and $J_n^1 = 0$. Consider the second subregion:

$$J_n^2 = \int_0^1\int_0^{s_2}\int_0^{t_1}\int_{s_2}^1 \frac{(s_2-t_1)^n}{(s_2-s_1)^{\frac{n+d}{2}}(t_2-t_1)^{\frac{n+d}{2}}} dt_2ds_1dt_1ds_2 = \frac{1}{(1-\frac{n+d}{2})^2}$$

$$\times \int_0^1\int_0^{s_2} (s_2-t_1)^n((1-t_1)^{1-\frac{n+d}{2}} - (s_2-t_1)^{1-\frac{n+d}{2}})(s_2^{1-\frac{n+d}{2}} - (s_2-t_1)^{1-\frac{n+d}{2}})dt_1ds_2$$

$$\leqslant C\frac{1}{(n+1)^2}\int_0^1\int_0^{s_2}(s_2-t_1)^{2-d}dt_1ds_2 \leqslant C\frac{1}{(n+1)^2}\int_0^1 u^{2-d}du.$$

We assumed that $n + d > 2$. In that case, the sufficient condition for the integral to exist is $d < 3$. In the cases $d = 1, n = 0,1$ and $d = 2, n = 0$, the integral is also finite.

$$J_n^3 = \int_0^1\int_0^{s_2}\int_{s_1}^{s_2}\int_{s_1}^{t_2} \frac{(t_2-t_1)^{\frac{n-d}{2}}}{(s_2-s_1)^{\frac{n+d}{2}}} dt_1dt_2ds_1ds_2 =$$

$$= \frac{1}{(1+\frac{n-d}{2})(2+\frac{n-d}{2})}\int_0^1\int_0^{s_2}(s_2-s_1)^{2-d}ds_1ds_2 \leqslant$$

$$\leqslant C\frac{1}{(n+1)^2}\int_0^1 u^{2-d}du.$$

The sufficient conditions for the integral to exist: $d < 3$ and $n > 2 - d$. Again we have to treat the cases $d = 1, n = 0,1$ and $d = 2, n = 0$ separately. But, in the last case, the integral is infinite:

$$J_0^3 = \int_0^1\int_0^{s_2}\int_{s_1}^{s_2}\int_{s_1}^{t_2} \frac{1}{(s_2-s_1)(t_2-t_1)} dt_1dt_2ds_1ds_2 =$$

$$= \int_0^1\int_0^{s_2}\int_{s_1}^{s_2}\left(\int_0^{t_2-s_1}\frac{1}{u}du\right)\frac{1}{(s_2-s_1)}dt_2ds_1ds_2 = +\infty.$$

Finally, $J_n < +\infty$ is satisfied for $d < 3$ and all $n \geqslant 0$ except $n = 0, d = 2$. The limit exists in $D_{2,\alpha}$, $\alpha < 2 - \frac{d}{2}$ for $d = 1$. If $d = 2$, then as our calculations suggest, it is possible to subtract the first term of the expansion and prove the convergence of the rest. It is known as the renormalization of the self-intersection local time for the Brownian motion in two dimensions (see [1]).

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