Theory of Stochastic Processes
Vol. 12 (28), no. 1–2, 2006, pp. 94–105

UDC 519.21

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ON STOCHASTIC DIFFERENTIAL INCLUSIONS WITH UNBOUNDED RIGHT SIDES

The paper deals with one-dimensional homogeneous stochastic differential inclusions without drift with a Borel measurable right side. Using a new method of explicit solutions, the necessary and sufficient conditions for the existence of weak solutions of the inclusions with locally unbounded right sides are given.

Introduction

The following driftless homogeneous stochastic differential inclusion (SDI) is investigated:

$$dX_t \in B(X_t)\, dW_t, \quad t \geq 0, \tag{1}$$

where $B : \mathbb{R} \to comp(\mathbb{R})$ is a multi-valued Borel measurable mapping, $comp(\mathbb{R})$ - the set of all non-empty compact subsets of $\mathbb{R}$ with the Hausdorff metric which is defined $\forall A, B \in comp(\mathbb{R})$ by $\rho(A, B) = \max(\beta(A, B), \beta(B, A))$, where $\beta(A, B) = \sup_{x\in A}(\inf_{y\in B} |x - y|)$ is the excess of A over B, and W is a one-dimensional Wiener process.

The theory of differential inclusions originated, basically, from two problems. The first one is the generalization of the equations without solutions to inclusions, which have solutions, and treatment of the solutions of the equations in the sense of the solutions of the corresponding inclusions. This theory was systematized in [13] by A.F. Filippov, who introduced the main properties, which the solutions of the inclusions should satisfy to. The main property is the preservation of the solution set for the equations having solutions in the classic context. The second problem arose in the models described by the differential equations with multi-valued right sides (cf. [3]). The solutions of the equations appeared to be well-defined in terms of the corresponding inclusions.

The investigation of multi-valued random processes (cf. [14]) motivated the creation of a technique for stochastic differential equations with multi-valued right sides, that led to the use of the ideas developed in the theory of differential inclusions in stochastic models (cf. [6]). Stochastic differential inclusions as a separate theory were introduced by P. Kree in [19]. N.U. Ahmed, E. Cepa, G. Da Prato, H. Frankovska, M. Kisielewicz, A.A. Levakov, M. Michta, J. Motyl, R. Petterson who considered the stochastic differential inclusions with different right sides and gave sufficient conditions for the existence of weak solutions of SDIs (cf. [1,5,7,17,18,20,23]). The known results contain the existence conditions for weak solutions of stochastic differential inclusions, whose right sides are required to be bounded at least locally.

This paper deals with the existence conditions for weak solutions of SDIs with locally unbounded right sides. The main result is the necessary and sufficient existence conditions for weak solutions of SDIs for every initial distribution.

2000 *AMS Mathematics Subject Classification.* Primary 34A60, 60G44, 60H10, 60J65, 60H99.

Key words and phrases. Stochastic differential equations, stochastic differential inclusions, measurable coefficients.

Preliminaries

Solutions of stochastic differential equations with unbounded coefficients.

The main idea of the paper is to express the solutions of stochastic differential inclusions in terms of the theory of stochastic differential equations (SDEs) and to use the developed technique of the theory. That is why, at first we will recall several important facts from the SDEs theory.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space with filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. As usually, assume, that the filtration $\mathbb{F}$ satisfies the natural conditions, e.c. it is right-continuous and $\mathcal{F}_0$ contains all $\mathbf{P}$ - zero subsets of $\mathcal{F}$. For a stochastic process $(X_t)_{t\geq 0}$ defined on $(\Omega, \mathcal{F}, \mathbf{P})$, we will write $(X, \mathbb{F})$, if X is $\mathbb{F}$-adapted.

SDI (1) is, actually, the generalization of SDE of the following type:

$$dX_t = b(X_t)\, dW_t, \quad t \geq 0, \tag{2}$$

where $b : \mathbb{R} \to \mathbb{R}$ - Borel measurable diffusion coefficient, W - one-dimensional Wiener process.

It is known that if the process X is a stochastic integral driven by a Wiener process (for example, it is a solution of Eq. (2), then it has continuous modification (if X is a solution of Eq. (2), then its continuous modification is also a solution of (2)). Taking that into consideration, we will deal with the trajectory space $C(\mathbb{R}_+, \mathbb{R})$ equipped with the Borel σ-algebra of cylindric sets $\mathcal{B}(C(\mathbb{R}_+, \mathbb{R}))$.

A stochastic process $(X, \mathbb{F})$, defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ and trajectories in $C(\mathbb{R}_+, \mathbb{R})$ is called a *weak solution of SDE* (2) with initial distribution X_0, if there exists a Wiener process $(\overline{W}, \mathbb{F})$ with $\overline{W}_0 = 0$ such that $\mathbf{P}$- a.s. for all $t \geq 0$

$$X_t = X_0 + \int_0^t b(X_s)\, d\overline{W}_s.$$

It should be emphasized that the initial distribution X_0 in the solution definition can be an arbitrary random value in the space, where the solution exists. The existence conditions of the paper will guarantee the existence of solutions *for every initial distribution*, that means: for every probabilistic measure $\bar{P}$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, there exists a solution X with initial distribution X_0 such that

$$\mathbf{P}(\{X_0 \in C\}) = \bar{P}(C), \quad \forall C \in \mathcal{B}(\mathbb{R}).$$

The idea of the proof of our main result is based on the Engelbert—Schmidt theorem (cf. [12], Theorem 4.17) which states that the weak solutions of Eq. (2) exist for every initial distribution if and only if the following holds:

$$M_b \subseteq N_b, \tag{3}$$

where the sets in (3) are defined for every measurable function $v : \mathbb{R} \to \mathbb{R}$ by

$$M_v = \left\{ x \in \mathbb{R} \,\middle|\, \int_{U(x)} v^{-2}(y)\, dy = \infty, \forall U(x) \text{ - open neighborhood of the point } x \right\} \tag{4}$$

$$N_v = \{x \in \mathbb{R} | v(x) = 0\} \tag{5}$$

The solutions of SDE (2) are always non-exploding (cf. [12], Proposition 4.11).

Random time change.

Now we will focus attention on the method of random time change. Assume that a filtrated probability space is given. Any increasing right-continuous family $A = (A_t)$ of $\mathbf{P}$-a.s. finite $\mathbb{F}$-stopping times is called $\mathbb{F}$-*time change*. Having defined the right-inverse process $T_t = \inf\{s \geq 0 : A_s > t\}$, one can give the main property of a random time change $A_{T_t} = t \quad \mathbf{P}$-a.s. $\forall t \geq 0$. Additionally, if the process A is strictly increasing, then the process T is continuous and $T_{A_t} = t \quad \mathbf{P}$-a.s. $\forall t \geq 0$.

Now we will give some generalization of the random time change that was introduced by H.J. Engelbert and W. Schmidt (cf. [10]). Let $A = (A_t)$ be a right-continuous increasing process with values in $[0, +\infty]$ and $A_\infty = \lim_{t\to\infty} A_t$. Having defined the inverse increasing process $T_t = \inf\{s \geq 0 : A_s > t\}$, we can pathwisely construct the Lebesgue integral $\int_0^t Z_s dA_s = \int_0^{t\wedge T_\infty} Z_s dA_s, \forall t \geq 0$, where Z is an arbitrary measurable process.

Lemma 1 ([10], Lemma 1.6). *For every non-negative measurable process Z, the following holds:*

$$\int_0^t Z_s dT_s = \int_0^{T_t} Z_{A_s} ds, \quad \forall t \geq 0.$$

Different kinds of random time changes proved to be helpful in the solution of various problems of stochastic analysis. Let us consider a time change which will be used in this article. Let us be given with an arbitrary Borel measurable function $v : \mathbb{R} \to \mathbb{R}$ and a Wiener process $(\tilde{W}, \tilde{\mathbb{F}})$ on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with arbitrary initial distribution. We can define the increasing process

$$T_t^v = \int_0^{t+} v^{-2}(\tilde{W}_s) ds, \forall t \geq 0 \tag{6}$$

and its right-inverse process

$$A_t^v = inf\{s \geq 0 : T_s^v > t\}. \tag{7}$$

Additionally, let $U(M_v)$ be the first entry time of $\tilde{W}$ in M_v,

$$U(M_v) = inf\{s \geq 0 : \tilde{W}_s \in M_v\}. \tag{8}$$

The following properties will be used in the proof of the main results of the paper.

Lemma 2 ([11],Lemma 1,[9],Theorem 3). *Under the definitions above,* $\mathbf{P}$*-a.s.*

$$\int_0^t v^{-2}(\tilde{W}_s) ds < +\infty, \ \forall t < U(M_v); \qquad \int_0^{U(M_v)+} v^{-2}(\tilde{W}_s) ds = +\infty;$$

$$A_\infty = U(M_v); \qquad and \qquad T_\infty^v = +\infty.$$

Right sides of stochastic differential inclusions.

In this section, we will closely investigate the right sides of SDI (1). The following notations of measure will be used in the whole paper: l is the Lebesgue measure on $\mathbb{R}$, and l_+ is a Lebesgue measure on $\mathbb{R}_+ = [0, \infty)$.

Let X and Y be arbitrary sets. The function $v : X \to Y$ is called the *selection* of a multi-valued mapping B of the set X into some set of subsets from Y, if $v(x) \in B(x), \forall x \in X$ (cf. [4]). The definition can be extended. Let (X, ν) be a space with measure ν, and let Y be an arbitrary set. Then the function $u : X \to Y$ is called *ν-a.e. selection* of the mapping B of the set X into some set of subsets from Y, if $u(x) \in B(x)$, for ν-almost all $x \in X$ (cf. [22]).

Selections of the multi-valued mapping $B(x)$ can be interpreted in two ways

1) *Explicit selections* of the mapping $B : \mathbb{R} \to comp(\mathbb{R})$ such functions $v : \mathbb{R} \to \mathbb{R}$ that $v(x) \in B(x)$.

2) *Composition selections* $B(X) : \mathbb{R}_+ \times \Omega \to comp(\mathbb{R})$ ($l_+ \times \mathbf{P}$*-a.e. composition selections*) - such functions $u : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ that $u(t, \omega) \in B(X(t, \omega))$.

The *measurability* of the considered multi-valued mappings is meant in the sense of the measurability of functions with values in the space $(comp(\mathbb{R}), \rho)$ with the Hausdorff metric. The corresponding σ-algebra of Borel subsets $\mathcal{B}(comp(\mathbb{R}))$ is generated by open sets of the form $[A]^\varepsilon = \{B \in comp(\mathbb{R}) | \rho(A, B) < \varepsilon\}$, $\forall A \in comp(\mathbb{R})$, $\varepsilon > 0$. Let us denote $op(\mathbb{R})$ and $cl(\mathbb{R})$ - all open and closed sets from $\mathbb{R}$ generated by the Euclidian metric.

Proposition 1 ([4], Theorem 3.2.). *The multi-valued mapping $B : \mathbb{R} \to comp(\mathbb{R})$ is Borel measurable if and only if*

$$B^-(U) = \{x \in \mathbb{R} | B(x) \cap U \neq \emptyset\} \in \mathcal{B}(\mathbb{R})$$

holds either for all $U \in op(\mathbb{R})$ or for all $U \in cl(\mathbb{R})$.

Let us define *the internal characteristic selection* of the mapping B

$$b_{int}(x) = \begin{cases} \beta(0, B(x)), & \beta(0, B(x)) \in B(x); \\ -\beta(0, B(x)), & \beta(0, B(x)) \notin B(x); \end{cases}$$

and *external characteristic selection* of the mapping B

$$b_{ext}(x) = \begin{cases} \beta(B(x), 0), & \beta(B(x), 0) \in B(x); \\ -\beta(B(x), 0), & \beta(B(x), 0) \notin B(x). \end{cases}$$

Proposition 2. *The internal and external characteristic selections of a Borel measurable multi-valued mapping are Borel measurable functions.*

Proof. Let $U(x, r) \subset \mathbb{R}$ denote an open ball around $x \in \mathbb{R}$ with radius r. For every $r > 0$,

$$\{x \in \mathbb{R} | \beta(0, B(x)) < r\} = \{x \in \mathbb{R} | B(x) \cap U(0, r) \neq \emptyset\} = B^-(U(0, r))$$

holds, and

$$\{x \in \mathbb{R} | \beta(B(x), 0) < r\} = \mathbb{R} \setminus \{x \in \mathbb{R} | B(x) \cap (\mathbb{R} \setminus U(0, r)) \neq \emptyset\} = \mathbb{R} \setminus (B^-(\mathbb{R} \setminus U(0, r)).$$

The functions $\beta(0, B(x))$ and $\beta(B(x), 0)$ are Borel measurable from Proposition 1 and the fact that the mapping B is Borel measurable. Hence, the statement of Proposition 2 holds due to the fact that the internal and external characteristic selections are the compositions of Borel measurable functions.

Remark 1. a) The functions b_{int} and b_{ext} are really selections of the mapping B due to $\forall x \in \mathbb{R} : b_{int}(x) \in B(x), b_{ext}(x) \in B(x)$.

b) For every explicit selection v of the mapping B, the following holds: $|b_{int}(x)| \leq |v(x)| \leq |b_{ext}(x)|, \forall x \in \mathbb{R}$.

Additionally, let us define *the optimal characteristic selection* of the mapping B:

$$b_{opt}(x) = \begin{cases} 0, & 0 \in B(x); \\ b_{ext}(x), & 0 \notin B(x); \end{cases}$$

Remark 2. a) The function b_{opt} is a Borel measurable selection of the mapping B, because $\forall x \in \mathbb{R} : b_{opt}(x) \in B(x)$ and it is a composition of Borel measurable functions.

b) $|b_{int}(x)| \leq |b_{opt}(x)| \leq |b_{ext}(x)|, \forall x \in \mathbb{R}$.

It is worthwhile to provide a few facts on the integrability of multi-valued mappings. The Lebesgue integral of multi-valued mappings was described in detail by R.J. Auman in [3]. Let B be a multi-valued mapping of $\mathbb{R}$ into $comp(\mathbb{R})$ and $\mathcal{I}_B$ be the set of all Lebesgue integrable selections of B. Then the integral of B over interval $[0, t] \subset \mathbb{R}, t > 0$ is the set of integrals of all integrable selections

$$\int_0^t B(s)ds = \{\int_0^t u(s)ds : u \in \mathcal{I}_B\}.$$

It is evident that, for the complete description of the integral, one can deal only with Borel measurable selections. A stochastic integral driven by the Wiener process of multi-valued mappings can be defined in the same way. Namely, let a multi-valued process $B : \mathbb{R}_+ \times \Omega \to comp(\mathbb{R})$ be defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with filtration $\mathbb{F}$, and let the process be measurable and $\mathbb{F}$-adapted. Then we can define the set $\mathcal{I}_B$ of all $(\mathcal{B}(\mathbb{R}_+) \times \mathcal{F})$-measurable, $\mathbb{F}$-adapted processes u such that $u(t, w)$ is $l_+ \times \mathbf{P}$-a.e. selection of the mapping $B(t, w)$, and $E(\int_0^{+\infty} |u(s)|^2 ds) < \infty$. The stochastic integral driven by a Wiener process over the interval $[0, t] \subset \mathbb{R}, t > 0$ of the multi-valued mapping B can be defined by

$$\int_0^t B(s)dW_s = \{\int_0^t u(s)dW_s : u \in \mathcal{I}_B\}.$$

Taking into consideration the fact that the right side of the stochastic differential inclusion (1) is the composition $B(X) : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ and that the selections can be interpreted in two ways, we have a right, for fixed measurable $\mathbb{F}$-adapted process X, to consider $\overline{\mathcal{I}}_B$ - the set of all explicit Borel measurable selections v of the mapping $B : \mathbb{R} \to comp(\mathbb{R})$ such that $E(\int_0^{+\infty} |v(X_s)|^2 ds) < \infty$ and introduce the set

$$\{\int_0^t v(X_s)dW_s : v \in \overline{\mathcal{I}}_B\}.$$

Using the excess β between sets from $comp\mathbb{R}$, we can define four sets which will be used in the statements of the theorems given below:

$$\underline{M}_B = \left\{ x \in \mathbb{R} \,\middle|\, \int_{U(x)} \beta(0, B(y))^{-2}\, dy = \infty, \forall U(x) \text{ - open neighborhood of point } x \right\},$$

$$\bar{M}_B = \left\{ x \in \mathbb{R} \,\middle|\, \int_{U(x)} \beta(B(y), 0)^{-2}\, dy = \infty, \forall U(x) \text{ - open neighborhood of point } x \right\},$$

$$\underline{N}_B = \{x \in \mathbb{R} | \{0\} \in B(x)\},$$

$$\bar{N}_B = \{x \in \mathbb{R} | B(x) = \{0\}\}.$$

Remark 3. a) The sets $\underline{N}_B$ and $\bar{N}_B$ are Borel ones, that is easily seen from the equalities $\{x \in \mathbb{R} | \{0\} \in B(x)\} = \{x \in \mathbb{R} | b_{int} = 0\}$, $\{x \in \mathbb{R} | B(x) = \{0\}\} = \{x \in \mathbb{R} | b_{ext} = 0\}$ and Proposition 2.

b) The following relations are valid: $\bar{M}_B \subseteq \underline{M}_B$, $\bar{N}_B \subseteq \underline{N}_B$.

c) For the sets M_v and N_v defined in (4) and (5) corresponding to the selections b_{int}, b_{ext}, and b_{opt}, we can deduce

$$M_{b_{int}} = \underline{M}_B, \quad M_{b_{ext}} = \bar{M}_B, \quad N_{b_{int}} = \underline{N}_B, \quad N_{b_{ext}} = \bar{N}_B, \quad N_{b_{opt}} = N_{b_{int}},$$

$$M_{b_{ext}} = \bar{M}_B \cup (\underline{M}_B \cap \underline{N}_B) \subseteq \underline{M}_B, \quad \bar{M}_B \setminus \underline{N}_B = M_{b_{opt}} \setminus N_{b_{opt}},$$

which, particularly, implies that the sets $\underline{M}_B$ and $\bar{M}_B$ are closed.

d) For every explicit selection v of the mapping B,

$$\bar{N}_B \subseteq N_v \subseteq \underline{N}_B, \quad \bar{M}_B \subseteq M_v \subseteq \underline{M}_B, \quad (M_v \setminus N_v) \supseteq (\bar{M}_B \setminus \underline{N}_B).$$

Weak solutions of stochastic differential inclusions

Definitions and properties of weak solutions.

Let us recall (cf. [1]) that the stochastic process $(X, \mathbb{F})$ defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ and trajectories in $C(\mathbb{R}_+, \mathbb{R})$ is called a *weak solution* of SDI (1) with initial distribution X_0, if there exist a Wiener process $(\overline{W}, \mathbb{F})$ with $\overline{W}_0 = 0$ and a measurable $\mathbb{F}$-adapted $l_+ \times \mathbf{P}$-a.e. selection $u : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ of the composition $B(X) : \mathbb{R}_+ \times \Omega \to comp(\mathbb{R})$ such that the following holds $\mathbf{P}$- a.s. for all $t \geq 0$:

$$X_t = X_0 + \int_0^t u(s)\, d\overline{W}_s. \tag{9}$$

The following definition introduces a subset of the set of SDI weak solutions.

Definition 1. A stochastic process $(X, \mathbb{F})$ defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ and trajectories in $C(\mathbb{R}_+, \mathbb{R})$ is called an *explicit weak solution* of SDI (1) with initial distribution X_0, if there exist a Wiener process $(\overline{W}, \mathbb{F})$ with $\overline{W}_0 = 0$ and a Borel measurable explicit selection $v : \mathbb{R} \to \mathbb{R}$ of the mapping $B : \mathbb{R} \to comp(\mathbb{R})$ such that the following holds $\mathbf{P}$- a.s. for all $t \geq 0$:

$$X_t = X_0 + \int_0^t v(X_s)\, d\overline{W}_s. \tag{10}$$

Proposition 3. *An explicit weak solution of SDI (1) is a weak solution of SDI (1).*

Proof. Let the stochastic process $(X, \mathbb{F})$ with trajectories in $C(\mathbb{R}_+, \mathbb{R})$ on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ and the Wiener process $(\overline{W}, \mathbb{F})$ be an explicit weak solution of SDI (1) with respect to some explicit selection v. Then the same process X is a weak solution on the same probability space with the same filtration and Wiener process. Namely, we can take the selection $u = v(X)$ which is a $(\mathcal{B}(\mathbb{R}_+) \times \mathcal{F})$-measurable $\mathbb{F}$-adapted process due to the facts that the function v is Borel measurable, and X is $(\mathcal{B}(\mathbb{R}_+) \times \mathcal{F})$-measurable and $\mathbb{F}$-adapted. For the selection, the following holds: $u(t, \omega) = v(X(t, \omega)) \in B(X(t, \omega))$, $\forall t \geq 0, \omega \in \Omega$.

The next statement is evident, but it should be emphasized due to its importance.

Proposition 4. *Every explicit weak solution of the stochastic differential inclusion (1) with respect to some selection v is a weak solution of the stochastic differential equation (2) with diffusion coefficient v. On the other hand, every weak solution of the stochastic differential equation (2) with some diffusion coefficient v is an explicit weak solution of the stochastic differential inclusion (1) with the right side that possesses the explicit selection v.*

Proof. The proof is easily accomplished from the definitions of an SDI explicit weak solution and an SDE weak solution.

The stochastic process $(X, \mathbb{F})$ is *trivial* if $X_t = X_0, \forall t \geq 0, \mathbf{P}$-a.s. Otherwise, the process is *non-trivial*.

Proposition 5. *The stochastic differential inclusion (1) has trivial weak solutions with initial distribution X_0 if and only if* $0 \in B(X(0, \omega))$ $\quad \mathbf{P}$*-a.s., e.c.* $b_{int}(X_0) = 0 \quad \mathbf{P}$*-a.s..*

Proof. The sufficiency is easily accomplished if we take the selection $u(t) \equiv 0$. To prove the necessity, let us take an arbitrary trivial weak solution X with respect to some selection u with initial distribution X_0. Then $u(t, \omega) = 0 \quad l_+ \times \mathbf{P}$-a.e.; hence, the definition of a SDI weak solution implies $0 \in B(X(t, \omega)) \quad l_+ \times \mathbf{P}$-a.e.. But

$$B(X(t, \omega)) = B(X(0, \omega)),\ \forall t \geq 0, \mathbf{P}\text{-a.s.},$$

therefore, $0 \in B(X(0, \omega)) \quad \mathbf{P}$-a.s. or, in other words, $b_{int}(X_0) = 0 \quad \mathbf{P}$-a.s..

Necessary and sufficient conditions for the existence of weak solutions of stochastic differential inclusions.

Theorem 1. *The stochastic differential inclusion (1) has weak and explicit weak solutions for every initial distribution if and only if*

$$\overline{M}_B \subseteq \underline{N}_B. \tag{11}$$

Proof. We will start with the necessity. Our objective is to prove that condition (11) implies the existence of weak solutions for every initial distribution with respect to the optimal characteristic selection. For the simplicity of notations, let us denote the selection $v \equiv b_{opt}$.

Point a) of Remark 2 shows that the selection v satisfies the conditions of the definition of an explicit weak solution of SDI (1). On the other hand, since the selection v is single-valued, we can consider a stochastic differential equation of type (2) with the diffusion coefficient $b \equiv v$. From Proposition 4, all the weak solutions of SDE (2) will be the explicit weak solutions of SDI (1).

We will use the method of random time change.

Let us take an arbitrary Wiener process $(\tilde{W}, \tilde{\mathbb{F}})$ on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with filtration $\tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_t)_{t \geq 0}$ and an arbitrary probability measure $\bar{P}$ on $\mathbb{R}$ as the initial distribution of the Wiener process. Using $\tilde{W}$, we can introduce the processes A^v and T^v and the set $U(M_v)$, as defined in (6), (7), and (8).

The process $A^v = (A^v_t)_{t \geq 0}$ is a continuous increasing family of $\tilde{\mathbb{F}}$-stopping times. From Lemma 2, the process A^v_t is finite $\mathbf{P}$-a.s. for all $t \geq 0$. We can define a process $X_t = \tilde{W}_{A^v_t}, \forall t \geq 0$ and a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0} = (\tilde{\mathcal{F}}_{A^v_t})_{t \geq 0}$ and can conclude that $(X, \mathbb{F})$ is a continuous local martingale with square characteristic $\langle X \rangle = A^v$ (cf. [16]).

Let us give the explicit form of A^v. The set $(N_v \setminus M_v)$ has the zero Lebesgue measure (from the definition of the set M_v). Hence, $\tilde{W} \in M_v \cup (\mathbb{R} \setminus N_v)$ $l_+ \times \mathbf{P}$-a.e. Using Lemma 2, we can conclude that $v(\tilde{W}_s) \neq 0$ for all $s < A^v_\infty$ $l_+ \times \mathbf{P}$-a.e.. Lemma 2 also implies $\tilde{W}_{A^v_\infty} \in M_v$ if $\{A^v_\infty < +\infty\}$. Due to condition (11) and point c) of Remark 3, we have $M_v \subseteq N_v$, and if $\{A^v_\infty < +\infty\}$, then $\mathbf{P}$-a.s.

$$v^2(\tilde{W}_{A^v_\infty}) = 0. \tag{12}$$

Using the property and the definition of the process T^v, we obtain that $\mathbf{P}$-a.s.

$$A^v_t = \int_0^{A^v_t} v^2(\tilde{W}_s) dT^v_s = \int_0^{T^v_{A^v_t}} v^2(X_s) ds,$$

where the last equality follows from Lemma 1. Hence, $\mathbf{P}$-a.s. $A^v_t = \int_0^t v^2(X_s)ds$ if $A^v_t < A^v_\infty$ and $\int_0^\infty v^2(X_s)ds$ if $A^v_t = A^v_\infty$. On the other hand, if $A^v_t = A^v_\infty$, then $X_s = W_{A^v_\infty}$ for all $s \geq t$, and we can conclude from (12) that $v^2(X_s) = 0$ for all $s \geq t$, which implies $\mathbf{P}$-a.s.

$$A^v_t = \int_0^t v^2(X_s) ds, \quad \forall t \geq 0.$$

From the Doob theorem (cf. [15], Theorem II.7.1′), there exists a Wiener process $(\bar{W}, \mathbb{F})$ (on a possibly extended probability space) such that $(X, \mathbb{F})$ is a weak solution of SDE (2) with initial distribution $\bar{P}$.

This process X is an explicit weak solution of SDI (1) from Proposition 4. From Proposition 3, this process is a weak solution of SDI (1).

Let $(X, \mathbb{F})$ be an arbitrary weak solution of inclusion (1) with respect to some selection u, e.c. (9) holds. Then a square variation $\langle X\rangle = A^u$ has the form

$$A_t^u = \int_0^t u^2(s)ds, \forall t \geq 0 \quad \mathbf{P} - \text{a.s.}$$

Let us define a process $T_t^u = \inf\{s \geq 0 : A_s^u > t\}$, that is right-inverse to the process A^u, and let us introduce a process $(W_t^u)_{t\geq 0} = (X_{T_t^u})_{t\geq 0}$ with filtration $\mathbb{F}^u = (\mathcal{F}_t^u)_{t\geq 0} = (\mathcal{F}_{T_t^u})_{t\geq 0}$, where $X_\infty = \lim_{t\to\infty} X_t$ if $\{A_\infty^u < +\infty\}$. We can conclude that $(W^u, \mathbb{F}^u)$ is a continuous local martingale such that $\langle W^u\rangle_t = t \wedge A_\infty^u$ and the initial distribution $W_0^v = X_0$. Hence, $(W^u, \mathbb{F}^u)$ is the Wiener process stopped in A_∞^u (cf. the proof of Lemma 2 in [8]). Using Lemma 1, we get

$$\int_0^{t\wedge A_\infty^u} b_{ext}^{-2}(W_s^u)ds = \int_0^{A_{T_t^u}^u} b_{ext}^{-2}(W_s^u)ds =$$

$$\int_0^{T_t^u} b_{ext}^{-2}(X_s)dA_s^u = \int_0^{T_t^u} b_{ext}^{-2}(X_s)u^2(s)ds \leq T_t^u. \tag{13}$$

The last inequality follows from the fact that since $u(t,\omega) \in B(X(t,\omega))$ for $l_+ \times \mathbf{P}$-almost all $(t,\omega) \in \mathbb{R}_+ \times \Omega$, $b_{ext}(X(t,\omega)) \geq u(t,\omega)$ for $l_+ \times \mathbf{P}$-almost all $(t,\omega) \in \mathbb{R}_+ \times \Omega$.

Let us take any point $x_0 \notin \underline{N}_B$ and consider a weak solution X of SDI (1) with initial distribution $X_0 = x_0$ that is non-trivial from Proposition 5. The non-triviality of the solution implies $\mathbf{P}(\{A_\infty^u > 0\}) > 0$. Therefore, there exists $t > 0$ such that $\mathbf{P}(\{A_\infty^u > t\}) > 0$. Taking into consideration $T_t^u < +\infty$ if $\{A_\infty^u > t\}$ and inequality (13), we can conclude

$$\mathbf{P}(\{\int_0^t b_{ext}^{-2}(W_s^u)ds < +\infty, A_\infty^u > t\}) > 0. \tag{14}$$

Hence, from Theorem 1 in [9], there exists an open neighborhood G of the point x_0 such that the function b_{ext}^{-2} is integrable over G, and $x_0 \notin \bar{M}_B$.

From the proof of Theorem 1, it is seen that condition (11) is valid if weak solutions exist only for constant initial distributions $x_0 \in \mathbb{R}$.

Corollary 1. *SDI (1) has weak and explicit weak solutions for every constant initial distribution $x_0 \in \mathbb{R}$ if and only if the right side of the inclusion satisfies condition (11).*

Corollary 2. *If stochastic differential inclusion (1) has weak solutions for every initial distribution, then the inclusion has explicit weak solutions with respect to the selection b_{opt} for every initial distribution.*

Conducting the proof of the sufficiency for the selections b_{int} and b_{ext}, one can conclude

Theorem 2. *If*

$$\underline{M}_B \subseteq \underline{N}_B \tag{15}$$

or

$$\bar{M}_B \subseteq \bar{N}_B, \tag{16}$$

then the stochastic differential inclusion (1) has weak and explicit weak solutions for every initial distribution.

Corollary 3. *For every initial distribution, condition (15) guarantees the existence of an explicit weak solution of the stochastic differential inclusion (1) with respect to the internal characteristic selection b_{int}. At the same time, for every initial distribution, condition (16) guarantees the existence of an explicit weak solution of the stochastic differential inclusion (1) with respect to the external characteristic selection b_{ext}.*

Lemma 3. *If the right side of inclusion (1) is closed (its graph is a closed subset of* $\mathbb{R}^2$*), then it satisfies condition (15).*

Proof. We will denote $U(x, C)$ - open ball around $x \in \mathbb{R}$ with radius $C > 0$. Since the family of such open balls is the fundamental system of neighborhoods of $\mathbb{R}$, we can use the open balls in the definitions of the sets $\underline{M}_B$ and $\overline{M}_B$. Let $x^* \in \underline{M}_B$. We use the rule of contraries. Assume that $x^* \notin \underline{N}_B$, e.c. $(x^*, 0)$ does not belong to the graph. The closure of the graph implies $\exists\, U(0, C_1), U(x^*, C_2)$ such that $U(0, C_1) \cap \{y | y \in B(x), x \in U(x^*, C_2)\} = \emptyset$ and, for all $y \in B(x), x \in U(x^*, C_2)$, the inequality $|y| \geq C_1$ holds. Therefore, $\beta(0, B(x)) \geq C_1$ or $\beta(0, B(x))^{-2} \leq C_1^{-2}$. Hence, $\int_{U(x^*, C_2)} \beta(0, B(x))^{-2}\, dx < \infty$. Thus, we have a conflict with the definition of the set $\underline{M}_B$.

Corollary 4. *The stochastic differential inclusion (1) has weak and explicit weak solutions for every initial distribution, if its right side B is closed.*

The next theorem clarifies the existence of explicit solutions for every explicit Borel measurable selection of the SDI right side. It is known that the selections always exist for every multi-valued Borel measurable mapping (cf. [4], Theorem III.6).

Theorem 3. *The stochastic differential inclusion (1) has explicit weak solutions with respect to every Borel measurable selection for every initial distribution if and only if*

$$\underline{M}_B \subseteq \bar{N}_B.$$

Proof. The proof of the sufficiency can be conducted similarly to that in Theorem 1. One needs to take an arbitrary explicit Borel measurable selection v of the mapping B and use the fact that $M_v \subseteq \underline{M}_B \subseteq \bar{N}_B \subseteq N_v$, which implies $\tilde{W} \in M_v \cup (\mathbb{R} \setminus N_v) \quad l_+ \times \mathbf{P}$-a.e. and if $\{A^v_\infty < +\infty\}$, then $\mathbf{P}$-a.s. $v^2(\tilde{W}_{A^v_\infty}) = 0$.

The proof of the sufficiency can be conducted by the steps of the necessity proof of Theorem 1. Having taken an arbitrary point $x_0 \notin \bar{N}_B$, one can investigate the behavior of explicit weak solutions with respect to the selection

$$v(x) = \begin{cases} b_{ext}(x), & x = x_0; \\ b_{int}(x), & x \neq x_0. \end{cases}$$

Using the arguments of the proof of Theorem 1, one can conclude that a solution with initial distribution x_0 with respect to the selection v is non-trivial. Therefore,

$$\mathbf{P}(\{\int_0^t b_{ext}^{-2}(W_s^v) ds < +\infty, A^v_\infty > t\}) > 0.$$

Hence, Lemma 1 in [9] implies the existence of an open neighborhood G of x_0 such that the function b_{ext}^{-2} is integrable over G, and $x_0 \notin \overline{M}_B$.

Remark 4. If the right side of SDI (1) is single-valued, then the existence conditions from Theorems 1, 2, and 3 are coincide and equal to condition (3) (the Engelbert—Schmidt theorem).

The next definition will use the set $\mathcal{P}(\mathbb{R} \cup \{\pm\infty\})$ of all non-empty subsets from $\mathbb{R} \cup \{\pm\infty\}$.

Definition 2. The multi-valued mapping $B : \mathbb{R} \to \mathcal{P}(\mathbb{R} \cup \{\pm\infty\})$ is *locally integrable in the wide sense (locally integrable in the narrow sense)*, if its every Borel measurable selection is locally integrable (there exists its Borel measurable locally integrable selection).

The upcoming statements will be dealing with the local integrability of the multi-valued mapping B^{-2} which can be defined for every $x \in \mathbb{R}$ by

$$B^{-2}(x) := \{z = \frac{1}{y^2} \in \mathbb{R}_+ \cup \{+\infty\}, \forall y \in B(x)\}.$$

Remark 5. The local integrability in the narrow sense of a multi-valued mapping is equivalent to the local integrability of its internal characteristic selection. At the same time, the local integrability in the wide sense of a multi-valued mapping is equivalent to the local integrability of its external characteristic selection.

Theorem 4. *The stochastic differential inclusion (1) has weak and explicit weak non-trivial solutions for every initial distribution if and only if the mapping B^{-2} is locally integrable over $\mathbb{R}$ in the narrow sense.*

Proof. The proof mostly repeats the steps of the proof of Theorem 1, that is why we will stop on the principal points about the non-triviality of solutions. The construction of a solution can be conducted for the selection $v(x) \equiv b_{ext}(x)$. By Remark 5, the function v^{-2} is locally integrable, that implies $M_v = \overline{M}_B = \emptyset$ and, hence, $A^v_\infty = U(M_v) = \infty$. Therefore, the constructed solution cannot be trivial, otherwise would be: $v(X_t) = 0, \forall t \geq 0, \mathbf{P}$-a.s. and $A^v_t = 0, \forall t \geq 0, \mathbf{P}$-a.s., that is, we have a conflict with $A^v_\infty = \infty$.

The proof of the necessity is easily accomplished due to the existence of non-trivial solutions for every initial distribution, which implies the fulfilment of (13) and, hence, (14).

Combining the proofs of Theorems 3 and 4, we can formulate the following:

Theorem 5. *The stochastic differential inclusion (1) has explicit weak non-trivial solutions with respect to every Borel measurable selection for every initial distribution if and only if the mapping B^{-2} is locally integrable over $\mathbb{R}$ in the wide sense.*

Examples.

1. The theorems of this paper allow us to investigate SDIs of type (1) with locally unbounded right sides. For instance, let the right side of SDI (1) be the countable union of straight strophoids

$$B(x) = \pm x\sqrt{\frac{2a+x}{2a-x}}, \quad a \in \mathbb{Z}, \quad x \in \mathbb{R}.$$

It can be easily verified that the right side satisfies the conditions of Theorem 1, hence, the inclusion has explicit weak solutions (from Proposition 3, they are weak solutions as well) for every initial distributions. Furthermore, the right side satisfies the conditions of Theorem 3. Hence, the inclusion has weak and explicit weak solutions with respect to every Borel measurable selection of B for every initial distributions. On the other hand, by Theorem 4, there exists a probabilistic measure $\bar{P}$ such that there is no non-trivial weak solution of the inclusion with $\bar{P}$ as its initial distribution.

2. SDI (1) with the right side

$$B(x) = \begin{cases} [|x|^{\frac{1}{4}}, |x|^{\frac{1}{5}}], & |x| \leq 1; \\ [|x|^{\frac{1}{4}}, |x|^{\frac{1}{3}}], & |x| > 1; \end{cases}$$

possesses non-trivial explicit weak solutions (by Proposition 3, they are weak solutions as well) for every initial distribution from Theorem 4 due to the local integrability of the mapping B^{-2} in the narrow sense. Its external characteristic selection

$$b_{ext}(x) = \begin{cases} |x|^{\frac{1}{5}}, & |x| \leq 1; \\ |x|^{\frac{1}{3}}, & |x| > 1; \end{cases}$$

is locally integrable in power (-2). Moreover, the inclusion possesses non-trivial explicit weak solutions with respect to every Borel measurable selection of B for every initial distributions from Theorem 5, because the internal characteristic selection $b_{int}(x) = |x|^{\frac{1}{4}}$ is locally integrable in power (-2).

3. Consider SDE of type (2) with the diffusion coefficient

$$b(x) = \begin{cases} Arcth(x), & x \in (-\infty, -1) \cup (1, +\infty); \\ Arth(x), & x \in (-1, 0) \cup (0, +1); \\ 1, & x \in \{-1, 0, +1\}; \end{cases}$$

where $Arth, Arcth$ - area hyperbolic tangent and cotangent, correspondingly. This equation does not have a weak solution for the initial distribution $x_0 = 0$ from Theorem 1 in [2].

Using one of the standard procedures (cf. [12,21]), we can generalize the equation. For instance, let us construct the right part B of the corresponding inclusion as a minimal close hull of the diffusion coefficient b excluding values $\{\pm\infty\}$. As a result, we obtain SDI with the right side

$$B(x) = \begin{cases} Arcth(x), & x \in (-\infty, -1) \cup (1, +\infty); \\ Arth(x), & x \in (-1, 0) \cup (0, +1); \\ \{0, 1\}, & x = 0; \\ 1, & x \in \{-1, 1\}; \end{cases}$$

Weak solutions of the initial SDE will be treated in the sense of weak solutions of the constructed SDI. The weak solutions exist for every initial distribution, since, for this right side, $\overline{M}_B = \underline{N}_B = \{0\}$.

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