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CH. A. AGAYEVA

STOCHASTIC OPTIMAL CONTROL PROBLEM WITH DELAY

A stochastic optimal control problem with variable delay on phase and on control is considered. The maximum principle for a nonlinear stochastic control system with controlled diffussion coefficient is proved.

Introducton

The stochastic differential equations with delay find much exhibits at the description of real systems, which are subjected, in one or another degree, to the influence of a random noise. Many problems in theories of the automatic control, in self-oscillating system, and so on are described by such equations. Therefore, the problems of optimal control for systems described by stochastic differential equations with delay are actual at present [1,2]. Earlier, the problems of stochastic optimal control with variable delay on phase [3,4] and with delay on control [5] were considered. This work is devoted to the problem of stochastic optimal control with variable delay both on phase and on control at the restriction on a right endpoint constraint. Our aim is to obtain a necessary condition for optimal control, when the diffusion coefficient contains the control variable.

Statement of the control problem

Let (Ω, F, P) be a probability space with filtration$\{F^t, t \in [t_0, t_1]\}$. Let w_t be an n-dimensional Wiener process. We assume that $F^t = \bar{\sigma}(w_s, t_0 \le s \le t)$. $L_F^2(t_0, t_1)$ is the space of all square integrable processes adapted to the family F^t. $R^{m\times n}$ is the space of linear transformations from R^m to R^n.

Consider the following stochastic optimal control problem with variable delay both on phase and on control:

$$dx_t = g(x_t, x_{t-h(t)}, u_t, u_{t-h(t)}, t)dt + f(x_t, x_{t-h(t)}, u_t, t)dw_t \;\; t \in (t_0, t_1], \tag{1}$$

$$x_t = \Phi(t), \;\; t \in [t_0 - h(t_0), t_0), \quad x_{t_0} = x_0, \tag{2}$$

$$u_t = Q(t), t \in [t_0 - h(t_0), t_0,) \tag{3}$$

$$u_t \in U_d \equiv \{u(\cdot,\cdot) \in L_F^2(t_0, t_1; R^m) | u(t,\cdot) \in U \subset R^m, \text{a.c.}\}, \tag{4}$$

where $\Phi(t)$, and $Q(t)$ are given non-random functions, $h(t) \ge 0$ is a continuously differentiable non-random function such that $\dfrac{dh(t)}{dt} < 1$.

The problem consists in the minimization of the cost functional

$$J(u) = E\left\{p(x_{t_1}) + \int_{t_0}^{t_1} l(x_t, u_t, t)dt\right\} \tag{5}$$

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which is defined on the decisions of system (1)-(4), generated by all admissible controls under the condition

$$Eq(x_{t_1}) \in G \subset R^k. \tag{6}$$

Our assumptions are:

I. $l(x,u,t),\ g(x,y,u,v,t),\ f(x,y,u,t)$ are continuous in the total arguments and

$$l(x,u,t) : R^n \times R^m \times [t_0,t_1] \to R^1,$$

$$g(x,y,u,v,t) : R^n \times R^n \times R^m \times R^m \times [t_0,t_1] \to R^n,$$
$$f(x,y,u,t) : R^n \times R^n \times R^m \times [t_0,t_1] \to R^{n\times n}.$$

II. When (t,u) are fixed, the functions l, g, and f are continuously differentiable with respect to (x,y), and their derivatives are continuous in (x,y,u,v).

$$(1+|x|+|y|)^{-1}\left(|g(x,y,u,v,t)| + |f(x,y,u,t)| + |g_x(x,y,u,v,t)| + |f_x(x,y,u,t)| + \right.$$

$$\left. + |g_y(x,y,u,t)| + |f_y(x,y,u,t)|\right) \le N,$$
$$(1+|x|)^{-1}\left(|l(x,u,t)|\right) + \left(l_x(x,u,t)\right) \le N.$$

III. Functions $p : R^n \to R; q : R^n \to R^k$ are continuously differentiable with respect to x:

$$|p(x)| + |p_x(x)| \le N\,(1+|x|)\,; \quad |q(x)| + |q_x(x)| \le N\,(1+|x|)\,.$$

Necessary conditions of optimality

Below, we will consider the stochastic control problem (1)-(5) without constraint (6). Let us present a definition that will be used later on.

Definition 1. [6]. $\lambda(x,X)$ is a star-shaped neighborhood of the point x with respect to the set X, if

$$\lambda(x,X) = \{y : y \in X, x+\varepsilon(y-x) \in X,\ \forall\varepsilon < \varepsilon_0, \varepsilon > 0\} \tag{7}$$

In what follows, we obtain the necessary condition of optimality that is called the maximum principle.

Theorem 1. *Let I-III hold, and let* (x_t^0, u_t^0) *be a solution of problem* (1)-(5). *Assume that there exist the stochastic processes* $(\psi_t,\beta_t) \in L_F^2(t_0,t_1;R^n) \times L_F^2(t_0,t_1;R^{nxn})$ *which are a solution of the following adjoint equation:*

(8)
$$\begin{cases} d\psi_t = -\Big[H_x(\psi_t,x_t^0,y_t^0,u_t^0,v_t^0,t) + H_y(\psi_z,x_z^0,y_z^0,u_z^0,v_z^0,z)\big|_{z=s(t)} s'(t)\Big]dt + \beta_t dw_t, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t_0 \le t < t_1 - h(t_1), \\ d\psi_t = -H_x(\psi_t,x_t^0,y_t^0,u_t^0,v_t^0,t)dt + \beta_t dw_t, t_1 - h(t_1) \le t < t_1, \\ \psi_{t_1} = -p_x(x_{t_1}^0). \end{cases}$$

Then we have $\forall u_t \in \Lambda(u^0)$ *a.c.:*

$$\begin{cases} H(\psi_\theta,x_\theta^0,y_\theta^0,u_\theta,v_\theta,\theta) - H(\psi_\theta,x_\theta^0,y_\theta^0,u_\theta^0,v_\theta^0,\theta) + \\ + \left[H(\psi_z,x_z^0,y_z^0,u_z,v_z,z) - H(\psi_z,x_z^0,y_z^0,u_z^0,v_z^0,z)\right]\big|_{z=s(\theta)} s'(\theta) \le \\ \qquad\qquad\qquad \le 0, a.e.\theta \in [t_0, t_1 - h(t_1)), \\ H(\psi_\theta,x_\theta^0,y_\theta^0,u_\theta,v_\theta,\theta) - H(\psi_\theta,x_\theta^0,y_\theta^0,u_\theta^0,v_\theta^0,\theta) \le 0,\ a.e.\ \theta \in [t_1 - h(t_1), t_1)\,. \end{cases} \tag{9}$$

Here, $t = s(\tau)$ *is a solution of the equation* $\tau = t - h(t)$*;* $y_t = x_{t-h(t)}$*;*$v_t = u_{t-h(t)}$*;*

$$H(\psi_t, x_t, y_t, u_t, v_t, t) = \psi_t^* g(x_t, y_t, u_t, v_t t) + \beta_t^* f(x_t, y_t, u_t, t) - l(x_t, u_t, t);$$

(10)

$$\Lambda(u^0) = \Big\{ u \in U : f(x_t^0, x_{t-h(t)}^0, u, , t) \in \lambda(f(x_t^0, x_{t-h(t)}^0, u^0, t),\, f(x_t^0, x_{t-h(t)}^0, U, t)) \Big\};$$

Proof. Let $\overline{u}_t = u_t^0 + \Delta u_t$ be some admissible control and $\overline{x}_t = x_t^0 + \Delta x_t$ be the corresponding trajectory of system (1)-(4). It is clear that

(11)

$$\begin{cases} d\Delta x_t = d(\bar{x}_t - x_t^0) = [g(\bar{x}_t, \bar{x}_{t-h(t)}, \bar{u}_t, \bar{v}_t, t) - g(x_t^0, x_{t-h(t)}^0, u_t^0, v_t^0, t)]dt + \\ +[f(\bar{x}_t, \bar{x}_{t-h(t)}, \bar{u}_t, , t) - f(x_t^0, x_{t-h(t)}^0, u_t^0, t)]dw_t = \; \{\Delta_{\overline{u}} g\,(x_t^0, x_{t-h(t)}^0, u_t^0, v_t^0, t) + \\ +\Delta_{\bar{v}} g(x_t^0, x_{t-h(t)}^0, u_t^0, v_t^0, t) + g_x(x_t^0, x_{t-h(t)}^0, u_t^0, v_t^0, t)\Delta x_t + \\ + \, g_y(x_t^0, x_{t-h(t)}^0, u_t^0, v_t^0, t)\Delta x_{t-h(t)} \Big\} dt + \Big\{ f_x(x_t^0, x_{t-h(t)}^0, u_t^0, t)\Delta x_t + \\ +f_y(x_t^0, x_{t-h(t)}^0, u_t^0, t)\Delta x_{t-h(t)} + \Delta_{\overline{u}} f(x_t^0, x_{t-h(t)}^0, u_t^0, t) \Big\} dw_t + \eta_t^1, \quad t \in (t_0, t_1] \\ \Delta x_t = 0, t \in [t_0 - h(t_0), t_0], \end{cases}$$

where

$$\eta_t^1 = \Big\{ \int_0^1 \Big[g_x^*(x_t^0 + \mu\Delta x_t, \bar{x}_{t-h(t)} \,, \bar{u}_t, \bar{v}_t, t) - \, g_x^*(x_t^0, x_{t-h(t)}^0, u_t^0, v_t^0, t) \Big] \Delta x_t d\mu +$$

$$+ \int_0^1 \Big[g_y^*(x_t^0, x_{t-h(t)}^0 + \mu\Delta x_{t-h(t)}, \bar{u}_t, \bar{v}_t, t) - \, g_y^*(x_t^0, x_{t-h(t)}^0, u_t^0, v_t^0, t) \Big] \Delta x_{t-h(t)} d\mu \Big\} dt +$$

$$+ \Big\{ \int_0^1 \Big[f_x^*(x_t^0 + \mu\Delta x_t, \bar{x}_{t-h(t)} \,, \bar{u}_t, t) - \, f_x^*(x_t^0, x_{t-h(t)}^0, u_t^0, t) \Big] \Delta x_t d\mu +$$

$$+ \int_0^1 \Big[f_y^*(x_t^0, x_{t-h(t)}^0 + \mu\Delta x_{t-h(t)}, \bar{u}_t, t) - \, f_y^*(x_t^0, x_{t-h(t)}^0, u_t^0, t) \Big] \; \Delta x_{t-h(t)} d\mu \Big\} dw_t.$$

According to the Itô formula, we have:

(12)

$$\begin{cases} d(\psi_t^* \Delta x_t) = d\psi_t^* \cdot \Delta x_t + \psi_t^* d\Delta x_t + \big\{ \beta_t^* \Delta_{\bar{u}} f(x_t^0, x_{t-h(t)}^0, u_t^0, t) + \\ +\beta_t^* f_x(x_t^0, x_{t-h(t)}^0, u_t^0, t)\Delta x_t + \beta_t^* f_y(x_t^0, x_{t-h(t)}^0, u_t^0, t)\Delta x_{t-h(t)} + \\ +\beta_t^* \int_0^1 [f_x(x_t^0 + \mu\Delta x_t, \bar{x}_{t-h(t)}, \bar{u}_t, t) - f_x(x_t^0, x_{t-h(t)}^0, u_t^0, t)]\Delta x_t d\mu + \\ +\beta_t^* \int_0^1 [f_y(x_t^0, x_{t-h(t)}^0 + \mu\Delta x_{t-h(t)}, \bar{u}_t, t) - f_y(x_t^0, x_{t-h(t)}^0, u_t^0, t)]\, \Delta x_{t-h(t)} d\mu \big\} dt. \end{cases}$$

Let

$$\psi_{t_1} = -p_x(x_{t_1}^0). \tag{13}$$

The increment of functional (5) along the admissible control $\bar{u}_t$ looks like

$$\Delta_{\bar{u}} J(u^0) = J(\bar{u}_t) - J(u_t^0) = E \Big\{ p(\bar{x}_{t_1}) - p(x_{t_1}^0) + \int_{t_0}^{t_1} \big[l(\bar{x}_t, \bar{u}_t, t) - l(x_t^0, u_t^0, t) \big] dt \Big\}$$

Taking (11), (12), and (13) into account, we obtain the following formula for the increment of the functional:

$$\Delta_{\bar{u}} J(u^0) = -E \int_{t_0}^{t_1} \Big[\psi_t^* \Delta_{\bar{u}} g(x_t^0, x_{t-h(t)}^0, u_t^0, v_t^0, t) + \psi_t^* \Delta_{\bar{v}} g(x_t^0, x_{t-h(t)}^0, u_t^0, v_t^0, t) +$$

$$+\beta_t^* \Delta_{\bar{u}} f(x_t^0, x_{t-h(t)}^0, u_t^0, t) - \Delta_{\bar{u}} l(x_t^0, u_t^0, t)\big] dt - E\int_{t_0}^{t_1} \big[d\psi_t^* + \psi_t^* g_x(x_t, x_{t-h(t)}, u_t^0, v_t^0, t) +$$

$$+\beta_t^* f_x(x_t^0, x_{t-h(t)}^0, u_t^0, t) - l_x(x_t^0, u_t^0, t)]\Delta x_t dt - E\int_{t_0}^{t_1} \Big[\psi_t^* g_y(x_t^0, x_{t-h(t)}^0, u_t^0, v_t^0, t) +$$

$$+\beta_t^* f_y(x_t^0, x_{t-h(t)}^0, u_t^0, t)\Big] \Delta x_{t-h(t)} dt + \eta_{t_0,t_1}, \tag{14}$$

where

$$\eta_{t_0,t_1} = E\int_0^1 \left[p_x^*(x_{t_1}^0 + \mu\Delta x_{t_1}) - p_x^*(x_{t_1}^0)\right] \Delta x_{t_1} d\mu +$$

$$+E\int_{t_0}^{t_1} \left\{\int_0^1 \left[l_x^*(x_t^0 + \mu\Delta x_t, \bar{u}_t, t) - l_x^*(x_t^0, \bar{u}_t, t)\right] \Delta x_t d\mu\right\} dt +$$

$$+E\int_{t_0}^{t_1} \left\{\int_0^1 \left[\psi_t^*(g_x(x_t^0 + \mu\Delta x_t, \bar{x}_{t-h(t)}, u_t^0, v_t^0, t) - g_x(x_t^0, \bar{x}_{t-h(t)}, u_t^0, v_t^0, t))\right] \Delta x_t d\mu_1 +\right.$$

$$\left.+\int_0^1 [\psi_t^*(g_y(x_t^0, x_{t-h(t)}^0 + \mu\Delta x_{t-h(t)}, u_t^0, v_t^0, t) - -g_y(x_t^0, x_{t-h(t)}^0, u_t^0, v_t^0, t)]\Delta x_{t-h(t)} d\mu dt\right\} +$$

$$+E\int_{t_0}^{t_1} \left\{\int_0^1 \beta_t^* \left[f_x(x_t^0 + \mu\Delta x_t, \bar{x}_{t-h(t)}, u_t^0, t) - f_x(x_t^0, \bar{x}_{t-h(t)}, u_t^0, t)\right] \Delta x_t d\mu +\right.$$

$$\left.+\int_0^1 \beta_t^* \left[f_y(x_t^0, x_{t-h(t)}^0 + \mu\Delta x_{t-h(t)}, u_t^0, t) - f_y(x_t^0, x_{t-h(t)}^0, u_t^0, t)\right] \Delta x_{t-h(t)} d\mu\right\} dt. \tag{15}$$

Let the random processes $\psi_t \in L_F^2(t_0, t_1; R^n)$ and $\beta_t \in L_F^2(t_0, t_1; R^{n\times n})$ be a solution of the adjoint equation (8). Assume that (9) is not fulfilled, i.e., for some $\theta \in [t_0, t_1)$ and $u_\theta \in \Lambda(u_\theta^0)$,

$$H\left(\psi_\theta, x_\theta^0, y_\theta^0, u_\theta, v_\theta, \theta\right) - H\left(\psi_\theta, x_\theta^0, y_\theta^0, u_\theta^0, v_\theta^0, \theta\right) = a > 0. \tag{16}$$

According to the definition of the set $\Lambda(u_\theta^0)$, there are the sequence of numbers $\{\varepsilon_i\}$, $\varepsilon_i \to 0$, $\varepsilon_i > 0$, and the sequence of vectors $\left\{u_t^i\right\}$, $u_t^i \in U$, such that

$$\Delta_{u_t^i} f\left(x_\theta^0, x_{\theta-h(\theta)}^0, u_\theta^0, \theta\right) = \varepsilon_i \Delta_{u_\theta} f\left(x_\theta^0, x_{\theta-h(\theta)}^0, u_\theta^0, \theta\right) \tag{17}$$

Let's consider the following needle-shaped variation:

$$\Delta_{u_i \alpha_i \theta} u_t^0 = \begin{cases} u_t^i - u_t^0, t \in [\theta, \theta+\alpha_i), u_i \in L_F^2(\theta, \theta+\alpha_i; R^m) \\ \\ 0, t \notin [\theta, \theta+\alpha_i), \end{cases} \tag{18}$$

where α_i is a sufficiently small positive number ($i \geq 1$), $r = \inf_{\theta \leq t \leq \theta+\alpha_i} h(t)$.

By $x_t^i = x_t^0 + \Delta_i x_t^0$, we denote the trajectories corresponding to variations (18). We need the estimation of $E\left|\Delta_i x_t^0\right|^2$. It is clear that, for $\Delta_i x_t^0 = 0$, $\forall t \in [t_0, \theta)$ and for $\forall t \in [s(t_0), s(\theta)]$, $\Delta_i y_t^0 = \Delta_i x_{t-h(t)}^0 = 0$.

Let $\forall \tau \in [\theta, \theta+\alpha_i)$. Then

$$\Delta_i x_\tau^0 = \int_\theta^\tau [g(x_t^0, x_{t-h(t)}^0, u_t^i, v_t^0, t) - g(x_t^0, x_{t-h(t)}^0, u_t^0, v_t^0, t)]dt +$$

$$+\int_\theta^\tau \Delta_{u_t^i} f(x_t^0, x_{t-h(t)}^0, u_t^0, t) dw_t + \int_0^1 \int_\theta^\tau g_x(x_t^0 + \mu\Delta_i x_t^0, x_{t-h(t)}^0, u_t^i, v_t^0, t)\Delta_i x_t^0 dt d\mu +$$

$$+\int_0^1 \int_\theta^\tau f_x(x_t^0 + \mu\Delta_i x_t^0, x_{t-h(t)}^0, u_i, t)\Delta_i x_t^0 dw_t d\mu$$

and

$$E|\Delta_i x_\tau^0|^2 \le E\left|\int_\theta^{\theta+\alpha_i}\left[g(x_t^0, x_{t-h(t)}^0, u_t^i, v_t^0, t) - g\left(x_t^0 x_{t-h(t)}^0, u_t^0, v_t^0, t\right)\right]dt\right|^2 +$$

$$+E\int_\theta^{\theta+\alpha_i}\left|\Delta_{u_t^i} f(x_t^0, x_{t-h(t)}^0, u_t^0, t)\right|^2 dt +$$

$$+E\int_\theta^\tau \left|\int_0^1 g_x(x_t^0 + \mu\Delta_i x(t), x_{t-h(t)}^0, u_t^i, v_t^0, t)d\mu\right|^2 |\Delta_i x_t^0|^2 dt +$$

$$+E\int_\theta^\tau \left|\int_0^1 f_x(x_t^0 + \mu\Delta_i x_t^0, x_{t-h(t)}^0,\ u_t^i,\ t)d\mu\right|^2 \left|\Delta_i x_t^0\right|^2 dt.$$

According to (17), we have

$$E\int_\theta^{\theta+\alpha_i} |\Delta_{u_t^i} f(x_t^0, x_{t-h(t)}^0, u_t^0, t)|^2 dt \le K\varepsilon_i^2\alpha_i, \mathrm{K} > 0.$$

The numbers $\{\varepsilon_i\}$ are fixed. Then, to account the choice of the numbers $\{\alpha_i\}$, we have

$$E|\Delta_i x_\tau^0|^2 \le N\varepsilon_i^2\alpha, \forall\tau \in [\theta, \theta+\alpha_i)$$

from the Gronwall inequality. For $\forall\tau \in [\theta+\alpha_i, \theta+r]$, we have

$$E\left|\Delta_i x_\tau^0\right|^2 \le E\left|\Delta_i x_{\theta+\alpha_i}^0\right|^2$$
$$+ E\int_{\theta+\alpha_i}^\tau \left|\int_0^1 g_x(x_t^0 + \mu\Delta_i x_t^0, x_{t-h(t)}^0, u_t^0, v_t^0, t)d\mu\right|^2 \left|\Delta_i x_t^0\right|^2 dt$$
$$+ \int_{\theta+\alpha_i}^\tau E\left|\int_0^1 f_x(x_t^0 + \mu\Delta_i x_t^0, x_{t-h(t)}^0, u_t^0, t)d\mu\right|^2 \left|\Delta_i x_t^0\right|^2 dt \le N\varepsilon_i^2\alpha_i.$$

Thus,

$$E|\Delta_i x_t^0|^2 \le N\varepsilon_i^2\alpha, \quad \text{for } \ \forall\tau \in [\theta, \theta+r].$$

We now consider the segment $[\theta+r, \theta+2r]$. We divide it into the parts

$$[\theta+r, \theta+r+\alpha_i) \quad \text{and} \quad [\theta+r+\alpha_i, \theta+2r)$$

and estimate the values $E|\Delta_i x_t|^2$ for $\forall t \in [\theta+r, \theta+2r]$. For $\forall\tau \in [\theta+r, \theta+r+\alpha_i]$, we obtain

$$\Delta_i x_\tau^0 = \int_{\theta+r}^\tau [g(x_t^0, x_{t-h(t)}^0, u_t^i, v_t^i, t) - g(x_t^0, x_{t-h(t)}^0, u_t^0, v_t^0, t)]dt +$$

$$+\int_{\theta+r}^\tau \Delta_{u_t^i} f(x_t^0, x_{t-h(t)}^0, u_t^0, t)dw_t + \int_{\theta+r}^\tau \int_0^1 g_x\left(x_t^0 + \mu\Delta_i x_t^0, x_{t-h(t)}^i, u_t^i, v_t^i, t\right)\Delta_i x_t d\mu dt +$$

$$+\int_{\theta+r}^\tau \int_0^1 f_x(x_t^0 + \mu\Delta_i x_t^0, x_{t-h(t)}^0, u_t^i, t)\Delta_i x_t d\mu dw_t +$$

$$+\int_{\theta+r}^\tau \int_0^1 g_y(x_t^0, x_{t-h(t)}^0 + \mu\Delta_i x_{t-h(t)}^0, u_t^i, v_t^i, t)\Delta_i y_t d\mu dt +$$

$$+\int_{\theta+r}^\tau \int_0^1 f_y(x_t^0, x_{t-h(t)}^0 + \mu\Delta_i x_{t-h(t)}^0, u_t^i, t)\Delta_i y_t d\mu dw_t.$$

Since

$$E|\Delta_i y_t^0|^2 \le N\varepsilon_i^2\alpha, t \in [\theta+r, \theta+2r],$$

we obtain that

$$E\left|\Delta_i x_\tau^0\right|^2 \le E\left(\left|\int_{\theta+r}^{\tau}\left[g(x_t^0,x_{t-h(t)}^0,u_t^i,v_t^i,t)-g(x_t^0,x_{t-r}^0,u_t^0,v_t^0,t)\right]dt\right|^2+\right.$$

$$+\int_{\theta+r}^{\tau}\left|\Delta_{u_t^i} f(x_t^0,\ x_{t-h(t)}^0,u_t^0,t)\right|^2 dt+\int_{\theta+r}^{\tau}\left[\left|\int_0^1 g_x(x_t^0+\mu\Delta_i x_t^0,x_{t-h(t)}^0,u_t^i,v_t^i,t)d\mu\right|^2+\right.$$

$$+\int_0^1\left|f_x(x_t^0+\mu\Delta_i x_t^0,x_{t-h(t)}^0,u_t^i,t)d\mu\right|^2\Bigg]\left|\Delta_i x_t\right|^2 dt+\int_{\theta+r}^{t}\left[\left|\int_0^1 g_y(x_t^0,x_{t-h(t)}^0+\right.\right.$$

$$\left.\left.+\mu\Delta_i x_{t-h(t)}^0,u_t^i,v_t^i,t)d\mu\right|^2+\int_0^1\left|f_y(x_t^0,x_{t-h(t)}^0+\mu\Delta_i x_{t-h(t)}^0,u_t^i,t)d\mu\right|^2\right]\left|\Delta_i y_t\right|^2 dt\Bigg).$$

Whence $E|\Delta_i x_t^0|^2 \le N\varepsilon_i^2\alpha_i, t\in[\theta+r,\theta+r+\alpha_i)$.

Having executed the similar transformations, we obtain

$$E|\Delta_i x_t^0|^2 \le N\varepsilon_i^2\alpha_i,\quad \forall t\in[\theta+r+\alpha_i,\theta+2r]$$

Thus, we obtain

$$E|\Delta_i x_t^0|^2 \le N\varepsilon_i^2\alpha_i,\quad \forall t\in[\theta+r,\theta+2r].$$

Hence, for $\forall t\in[\theta,\theta+2r]$, the relation

$$E\left|\Delta_i x_t^0\right|^2 \le N\varepsilon_i^2\alpha_i$$

is fulfilled. Then, dividing each of the segments

$$[\theta+(j-1)r,\theta+jr], j=\overline{1,m}, \theta+mr\ge t_1,$$

into the segments $[\theta+(j-1)r,\theta+(j-1)r+\alpha_i)$ and $[\theta+(j-1)r+\alpha_i,\theta+jr]$, we can prove the correctness of the following estimations:

$$E\left|\Delta_i x_t^0\right|^2 \le N\alpha_i^2, \mathrm{t}\in[\theta+(j-1)r,\theta+jr], \mathrm{j}=\overline{1,\mathrm{m}}.$$

Thus, we have proved the correctness of the following estimation for almost all $t\in[t_0,t_1]$:

$$E\left|\Delta_i x_t^0\right|^2 \le N\varepsilon_i^2\alpha_i. \tag{19}$$

Since the random processes ψ_t,β_t are a solution of system (8), using the obtained estimation (19) and taking formulas (15) and (14) into account, we have

$$\Delta_{u_t^i} J(u^0) = -E\int_\theta^{\theta+\alpha_i}[\psi_t^* g\left(x_t^0,x_{t-h(t)}^0,u_t^i,v_t^i,t\right)-\psi_t^* g\left(x_t^0,x_{t-h(t)}^0,u_t^0,v_t^0,t\right)+$$

$$+\beta_t^* f(x_t^0,x_{t-h(t)}^0,u_t^i,t)-\beta_t^* f(x_t^0,x_{t-h(t)}^0,u_t^0,t)-l(x_t^0,u_t^i,t)+l(x_t^0,u_t^0,t)\Big]dt+o(\alpha_i)=$$

$$=-\alpha_i[\psi_t^* g\left(x_t^0,x_{t-h(t)}^0,u_t^i,v_t^i,t\right)-\psi_t^* g\left(x_t^0,x_{t-h(t)}^0,u_t^0,v_t^0,t\right)+$$

$$+\beta_t^* f(x_t^0,x_{t-h(t)}^0,u_t^i,t)-\beta_t^* f(x_t^0,x_{t-h(t)}^0,u_t^0,t)-l(x_t^0,u_t^i,t)+l(x_t^0,u_t^0,t)\Big]+o(\alpha_i). \tag{20}$$

Now, at the expense of the selection of numbers α_i according to (16), for sufficiently large i, relation (20) yields $\Delta_{u_i} J(u^0)<0$, which disagrees with the optimality of u_t^0. The proof is completed.

Problem with endpoint constraint

Now we consider the stochastic optimal control problem with final state constraint. We apply the Ekeland's variational principle [7].

Theorem 2. *Let I-III hold, and let* (x_t^0, u_t^0) *be a solution of problem* (1)-(6)*. Assume that there exist nonzero* $(\lambda_0, \lambda_1) \in R^{k+1}$ *such that* $\lambda_0 \geq 0, \lambda_1$ *is a normal to the set* G *at the point* $Eq(x_{t_1}^0), and \lambda_0^2 + |\lambda_1|^2 = 1$*, and the stochastic processes* $(\psi_t, \beta_t) \in L_F^2(t_0, t_1; R^n) \times L_F^2(t_0, t_1; R^{nxn})$*, which are a solution of the following adjoint equation:*

$$(21) \quad \begin{cases} d\psi_t = -\Big[H_x(\psi_t, x_t^0, y_t^0, u_t^0, v_t^0, t) + H_y(\psi_z, x_z^0, y_z^0, u_z^0, v_z^0, z)\big|_{z=s(t)} s'(t)\Big]dt + \\ \qquad +\beta_t dw_t, t_0 \leq t < t_1 - h(t_1), \\ d\psi_t = -H_x(\psi_t, x_t^0, y_t^0, u_t^0, v_t^0, t)dt + \beta_t dw_t, t_1 - h(t_1) \leq t < t_1, \\ \psi_{t_1} = -\lambda_0 p_x(x_{t_1}^0) - \lambda_1 q_x(x_{t_1}^0). \end{cases}$$

Then we have $\forall u_t \in \Lambda(u^0)$ *a.s.:*

$$(22) \quad \begin{cases} H(\psi_\theta, x_\theta^0, y_\theta^0, u_\theta, v_\theta, \theta) - H(\psi_\theta, x_\theta^0, y_\theta^0, u_\theta^0, v_\theta^0, \theta) + \\ +\big[H(\psi_z, x_z^0, y_z^0, u_z, v_z, z) - H(\psi_z, x_z^0, y_z^0, u_z^0, v_z^0, z)\big]\big|_{z=s(\theta)} s'(\theta) \leq 0, \\ \qquad a.e.\ \theta \in [t_0, t_1 - h(t_1)), \\ H(\psi_\theta, x_\theta^0, y_\theta^0, u_\theta, v_\theta, \theta) - H(\psi_\theta, x_\theta^0, y_\theta^0, u_\theta^0, v_\theta^0, \theta) \leq 0, a.e.\ \theta \in [t_1 - h(t_1),\ t_1)\,. \end{cases}$$

Here,

$$(23) \qquad H(\psi_t, x_t, y_t, u_t, v_t, t) = \psi_t^* g(x_t, y_t, u_t, v_t t) + \beta_t^* f(x_t, y_t, u_t, t) - l(x_t, u_t, t).$$

Proof. For any natural j, we introduce the functional

$$J_j(u) = S_j(Ep(x_{t_1}) + E\int_{t_0}^{t_1} l(x_t, u_t, t)dt, Eq(x_{t_1})) =$$

$$(24) \qquad = \min_{(c,y)\in\mathcal{E}} \sqrt{\left|c - \frac{1}{j} - Ep(x_{t_1}) - E\int_{t_0}^{t_1} l(x_t, u_t, t)dt\right|^2 - |y - Eq(x_{t_1})|^2},$$

where $\mathcal{E} = \big\{(c, y) : c \leq J^0, y \in G\big\}$. Here, J^0 is the minimal value of the functional in problem (1)-(6). Let V be the space of controls with the metric

$$d(u, v) = (l \otimes P)\left\{(t, \omega) \in [t_0, t_1] \times \Omega : \nu_t \neq u_t\right\}.$$

We use the following results which can be proved by the scheme in [3].

Lemma 1. *Let us assume that conditions I-III are fulfilled. If* $d(u_t^{(n)}, u_t) \to 0, n \to \infty$*, then* $\lim_{n\to\infty}\left\{\sup E\left|x_t^{(n)} - x_t\right|^2\right\} = 0$.

Due to continuity of the functional $J_j : V \to R^n$, according to the Ekeland's variational principle, we have that $\exists u_t^j : d(u_t^j, u_t^0) \leq \sqrt{\varepsilon_j}$ and, for $\forall u \in V$,

$$J_j(u^j) \leq J_j(u) + \sqrt{\varepsilon_j} d(u^j, u), \quad \varepsilon_j = \frac{1}{j}.$$

This inequality means that (x_t^j, u_t^j) is a solution of the following problem:

$$(25)\qquad \begin{cases} I_j(u) = J_j(u) + \sqrt{\varepsilon_j} E\int_{t_0}^{t_1} \delta(u_t, u_t^j)dt \to \min, \\ dx_t = g(x_t, x_{t-h(t)}, u_t, v_t, t)dt + f(x_t, x_{t-h(t)}, u_t, t)dw_t, \\ x_t = \Phi(t)\,, t \in [-h(t_0), t_0]\,, \quad h(t) \geq 0\,, \\ u_t = Q(t)\,, t \in [-h(t_0), t_0]\,, \\ u_t^j \in U_d. \end{cases}$$

Let (x_t^j, u_t^j) be a solution of (25), and let there exist the random processes $\psi_t^j \in L_F^2(t_0, t_1; R^n)$, $\beta_t^j \in L_F^2(t_0, t_1; R^{n\times n})$, and non-zero $(\lambda_0^j, \lambda_1^j) \in R^{k+1}$ such that

$$(26)\qquad \begin{cases} d\psi_t^j = -\Big[H_x\left(\psi_t^j, x_t^j, y_t^j, u_t^j, v_t^j, t\right) + H_y\left(\psi_z^j, x_z^j, x_z^j, u_z^j, v_z^j, z\right)\big|_{z=s(t)} s'(t)\Big]dt + \\ +\beta_t^j dw_t, t_0 \leq t \leq t_1 - h(t_1) \\ d\psi_t^j = -H_x(\psi_t^j, x_t^j, y_t^j, u_t^j, v_t^j, t)dt + \beta_t^j dw_t, t_1 - h(t_1) \leq t < t_1 \\ \psi_{t_1}^j = -\lambda_0^j p_x(x_{t_1}^j) - \lambda_1^j q_x(x_{t_1}^j), \end{cases}$$

$$(27)\qquad (\lambda_0^j, \lambda_1^j) = \left(\frac{-c_j + \frac{1}{j} + Ep\left(x_{t_1}^j\right) + E\int_{t_0}^{t_1} l(x_t^j, u_t^j, t)dt}{J_j^0}, \frac{-y_j + Eq\left(x_{t_1}^j\right)}{J_j^0}.\right)$$

Then, according to the previously proved Theorem 1 for $\forall u_t \in \Lambda\left(u^j\right)$, we get

$$(28)\qquad \begin{cases} H(\psi_\theta, x_\theta^0, y_\theta^0, u_\theta, v_\theta, \theta) - H(\psi_\theta, x_\theta^0, y_\theta^0, u_\theta^j, v_\theta^j, \theta) + \\ + \left[H(\psi_z, x_z^0, y_z^0, u_z, v_z, z) - H(\psi_z, x_z^0, y_z^0, u_z^j, v_z^j, z)\right]\big|_{z=r(\theta)} s'(\theta) \leq 0, \\ \qquad \text{a.e.}\theta \in [t_0, t_1 - h(t_1)), \\ H(\psi_\theta, x_\theta^0, y_\theta^0, u_\theta, v_\theta, \theta) - H(\psi_\theta, x_\theta^0, y_\theta^0, u_\theta^j, v_\theta^j, \theta) \leq 0, \ \text{a.e. } \theta \in [t_1 - h(t_1), t_1)\,. \end{cases}$$

Since $|(\lambda_0^j, \lambda_1^j)| = 1$, we can consider

$$(\lambda_0^j, \lambda_1^j) \to (\lambda_0, \lambda_1), j \to \infty.$$

It is known that S_j is a convex function differentiable in terms of Gato at the point $(Ep(x_{t_1}^j) + E\int_{t_0}^{t_1} l(x_t^j, u_t^j, t)dt, Eq(x_{t_1}^j))$. Then, for all $(c, y) \in \mathcal{E}$, we obtain

$$\left(\lambda_0^j, c - \frac{1}{j} - Ep(x_{t_1}^j) - E\int_{t_0}^{t_1} l(x_t^j, u_t^j, t)dt +\right) + \left(\lambda_1^j, y - Eq(x_{t_1}^j)\right) \leq \frac{1}{j}.$$

Going to the limit in the last inequality, we obtain that $\lambda_0 \geq 0$, and λ_1 is a normal to the set G at the point $Eq(x_{t_1}^0)$.

Since $\psi_{t_1}^j = -\lambda_j^0 p(x_{t_1}^j) - \lambda_j q(x_{t_1}^j)$, we get $\psi_{t_1}^j \to -\lambda_0 p(x_{t_1}^0) - \lambda_1 q(x_{t_1}^0)$ i.e. $\psi_{t_1}^j \to \psi_{t_1}$ in $L_F^2(t_0, t_1; R^n)$, $j \to \infty$.

Lemma 2. *Let the random processes ψ_t^j, β_t^j be a solution of system* (26), *and let ψ_t, β_t be a solution of system* (21). *Then*

$$E\int_{t_0}^{t_1} |\psi_t^j - \psi_t|^2 dt + E\int_{t_0}^{t_1} |\beta_t^j - \beta_t|^2 dt \to 0, \ if \ d(u_t^j, u_t) \to 0, \ j \to \infty.$$

Due to Lemma 2 and assumptions I, II, it follows that we can go to the limit in (26) and (27). We got (21) and (22), respectively. Theorem 2 is proved.

Corollary 1. *If $f\,(x, y, U, t)$ is convex, we can deduce that* (22) *is true for $\forall\, u \in U$. In other words, we obtain the maximum principle in the global form in this case.*

Corollary 2. *If the shift coefficient does not depend on the delay on control,* $g = g(x, y, u, t)$*, we obtain the result given in [3].*

Bibliography

1. Kolmanovskii V.B., Myshkis A.D., *Applied Theory of Functional Differential Equations*, Kluwer, N.Y., 1992.
2. Chernousko F.L., Kolmanovskii V.B., *Optimal Control under Random Perturbations*, Nauka, Moscow, 1978 (in Russian).
3. Agayeva Ch.A., Allahverdiyeva J.J., *The maximum principle for stochastic system with variable delay*, Reports of NAS of Azerbaijan **LIX** (2003), no. 5–6, 61–65 (in Russian).
4. Agayeva Ch.A., Allahverdiyeva J.J., *The necessary condition of optimality for one stochastic control system with variable delay*, Abstracts of the International Conference: "Dynamical Systems: Stability, Control, Optimization" Minsk **I** (1998), 11–12 (in Russian).
5. Agayeva Ch.A., *The necessary condition of optimality for one stochastic control problem with a constant delay on control*, Transactions of NAS of Azerbaijan, Information science and control problems (2005), no. 2, 117–123 (in Russian).
6. Gabasov R., Kirillova F.M., *The Maximum Principle in the Theory of Optimal Control*, Nauka i Tekhnika, Minsk, 1974 (in Russian).
7. Ekeland I., *Nonconvex minimization problems*, Bull.Amer. Math.Soc. (NS) **1** (1979), 443–474.

Institute of Cybernetics, Azerbaijan National Academy of Sciences, 9, F. Agayev Str., Baku, AZ1141, Azerbaijan
E-mail: cher.agayeva@rambler.ru