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GEOMETRIC ENTROPY IN BANACH SPACES

We introduce and study two notions of entropy in a Banach space X with a normalized Schauder basis $\mathcal{B} = (e_n)$. The geometric entropy $\mathbf{E}(A)$ of a subset A of X is defined to be the infimum of radii of compact bricks containing A, where a brick $K_{\mathcal{B},\mathcal{E}}$ is the set of all sums of convergent series $\sum a_n e_n$ with $|a_n| \le \varepsilon_n$, $\mathcal{E} = (\varepsilon_n)$, $\varepsilon_n \ge 0$. The unconditional entropy $\mathbf{E}_0(A)$ is defined similarly, with respect to 1-unconditional bases of X. We obtain several compactness characterizations for bricks (Theorem 3.7) useful for main results. If $X = c_0$ then the two entropies of a set coincide, and equal the radius of a set. However, for $X = \ell_2$ the entropies are distinct. The unconditional entropy of the image $T(B_H)$ of the unit ball of a separable Hilbert space H under an operator T is finite if and only if T is a Hilbert-Schmidt operator, and moreover, $\mathbf{E}_0\big(T(B_H)\big) = \|T\|_{HS}$, the Hilbert-Schmidt norm of T. We also obtain sufficient conditions on a set in a Hilbert space to have finite unconditional entropy. For Banach spaces without a Schauder basis we offer another entropy, called the Auerbach entropy. Finally, we pose some open problems.

1. Introduction

In this paper, we discuss the structure of compact sets in Banach spaces and introduce related geometric entropies. The main characteristic of a set is its size in various directions. This investigation was initiated by the first-named author in [7]. Originally it is motivated by the theory of stochastic flows. To describe the problem more precisely, consider the following example. Let $x(t), t \in [0;1]$ be a continuous in square mean centered Gaussian process. When studying the filtration problem related to this process [8] or the existence of the local time for it [3], it is natural to ask of how much independency does it have. This question can be formulated more precisely if we remind that, for jointly Gaussian centered random variables the orthogonality in L_2-sense means their independency. Consequently, one can try to evaluate the independency in the process x as follows. For the set of random variables $x(t_1), \dots, x(t_n)$ define new variables $\pi x(t_1), \dots, \pi x(t_n)$ as the orthogonal complements of $x(t_k)$ to the linear span of $x(t_i), i \ne k$. Then

$$\sup_{t_1 \le \ldots \le t_n, n \ge 1} \sum_{k=1}^{n} E(\pi x(t_k))^2$$

can be considered as a total amount of independency in x. Noting that the set $\{x(t), t \in [0;1]\}$ is compact in the Hilbert space of square integrable random variables and keeping

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in mind more general sets of parameters, one can consider the same procedure for an arbitrary compact K in a real separable Hilbert space H. For a sequence $\{x_n; n \geq 1\}$ of elements of K define the new sequence $\{\pi x_n; n \geq 1\}$, where πx_k is the orthogonal complement of x_k to the closed linear span of $x_i, i \neq k$.

Definition 1.1 ([7])**.** The *quadratic entropy* of $K \subseteq H$ is defined by

$$\mathcal{H}_2(K) = \sup_{\{x_n; n\geq 1\}} \sum_{k=1}^{\infty} \|\pi x_n\|^2,$$

where the sup is taken over all sequences of elements of K.

In [7] some basic properties of the quadratic entropy were studied. In particular, it was shown that a compact set need not have finite quadratic entropy, and sufficient conditions on a set to have finite quadratic entropy were given. It turned out that the condition $\mathcal{H}_2(K) < +\infty$ is closely related to the following value

$$\sigma_2(K) = \sup_{(e_n)_{n=1}^{\infty}} \sum_{n=1}^{\infty} \sup_{x\in K} (x, e_n)^2,$$

where the exterior sup is taken over all orthonormal bases $(e_n)_{n=1}^{\infty}$ in H. The condition $\sigma_2(K) < +\infty$ leads to the conclusion $\mathcal{H}_2(K) < +\infty$. The interest to compact sets with finite σ_2-characteristics was inspired by the consideration of admissible shifts for Gaussian measures in [18]. In the mentioned paper it was shown that $\sigma_2(K)$ is finite if and only if there exists a Hilbert–Schmidt operator A on H such that $K \subseteq A(B_H)$ (here B_H is the closed unit ball of H). In this way one can suggest that the finiteness of $\mathcal{H}_2(K)$ depends on the possibility to cover K by the some special sets in H. As such sets one can take the infinite-dimensional bricks with finite main diagonal. The ideas described above were essentially based on the geometry of a Hilbert space H. But the same considerations for sets of random variables which are non-Gaussian or (and) do not have the second moment naturally lead to similar objects in a Banach space. Although there is no analogue of the orthogonal complement in a Banach space, it is still possible to define bricks relatively to a Schauder basis. Using this approach, one can define a geometric entropy in Banach spaces. It occurs that there are different types of bricks in general (depending on the properties of a basis). This is the reason for the existence of different quantities related to $\mathcal{H}_2$ for the case of a Hilbert space.

In the present paper we introduce and provide a systematic study of the geometric entropies in Banach spaces. The paper is organized as follows. In Section 1 we summarize necessary definitions and facts on Banach spaces. Section 2 contains different definitions of a brick in a Banach space, statements about the relationships between them and examples. In Section 3 we define and study different radii of a brick, all equal the maximum norm of elements for compact bricks. The main result here (Theorem 3.9) gives several characterizations of compact bricks. In Section 4 we introduce and study the geometric entropies. Proposition 4.7 gives another equivalent definition of the entropies, and Proposition 4.8 shows that in the Banach space c_0 both entropies of a set equal its radius (the last observation is very natural, because the closed unit ball of c_0 is a brick). Section 5 is devoted to a study of the entropies in an infinite dimensional separable Hilbert space H. Theorem 5.1 asserts that the unconditional entropy of the image of the closed unit ball under an operator $T : H \to H$ is finite if and only if T is a Hilbert-Schmidt operator. Moreover, the unconditional entropy equals the Hilbert-Schmidt norm of T. In particular, this gives a simple method to evaluate the unconditional entropy of concrete sets. Fir instance, $\mathbf{E}_0(B_X) = \sqrt{n}$, where B_X is the closed unit ball of an n-dimensional subspace X of H. Theorem 5.5 establishes sufficient conditions on a compact set in a Hilbert space to have finite unconditional entropy, and Theorem 5.6 gives examples of sets in ℓ_2 with distinct entropy and unconditional entropy. In Section 6 we offer another type of a geometric entropy, called the Auerbach entropy, for Banach spaces without a Schauder basis. In the last Section 7 we pose several problems.

Remark that our notions of entropies have no connections with the other ones, like metric entropy [6], topological entropy [4], dynamic entropy [12]

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1.1. **Necessary information on bases in Banach spaces.** All linear (e.g., normed, Banach, Hilbert) spaces are considered over the reals.

A subset A of a Banach space X is called *precompact* if for every $\varepsilon > 0$, A contains a finite ε-net, that is, a finite collection $x_1, \dots, x_m \in A$ with $(\forall x \in A)(\exists k \in \{1, \dots, m\})\|x - x_k\| \le \varepsilon$. By the well known Hausdorff criterion (which is valid for metric spaces), A is precompact if and only if its norm closure $\overline{A}$ is compact in X.

1.1.1. *Bases in Banach spaces.* We follow mainly [13] (see also [1], [17]). Recall that a sequence $(e_n)_{n=1}^\infty$ in X is called a *basis* (more precisely, a *Schauder basis*) of X if for every $x \in X$ there is a unique sequence of scalars $(a_n)_{n=1}^\infty$ such that $x = \sum_{n=1}^\infty a_n e_n$. In this case, the coefficients $a_n = e_n^*(x)$ are continuous linear functionals of x and called *biorthogonal functionals*. So, $x = \sum_{n=1}^\infty e_n^*(x)\, e_n$ for each $x \in X$. The biorthogonal functionals possess the following property: $e_i^*(e_j) = \delta_{i,j}$. Moreover, this property determines the biorthogonal functionals: for every sequence $(f_n)_{n=1}^\infty$ in X^* the condition $f_i(e_j) = \delta_{i,j}$ for all i, j implies that $f_i = e_i^*$ for all i. The partial sums projections P_n of X defined by $P_n x = \sum_{k=1}^n e_k^*(x)\, e_k$, $x \in X$, calling the *basis projections*, are uniformly bounded in n, and the number $K = \sup_n \|P_n\| < \infty$ is called the *basis constant* of $(e_n)_{n=1}^\infty$. A *basic sequence* is any sequence $(e_n)_{n=1}^\infty$ in X which is a basis of some subspace X_0 of X (more precisely, a basis of its closed linear span $[e_n]_{n=1}^\infty$). A sequence $(e_n)_{n=1}^\infty$ of nonzero elements of X is a basic sequence if and only if there is a number $K \in [1, +\infty)$ such that

$$\Big\|\sum_{k=1}^n a_k e_k\Big\| \le K \Big\|\sum_{k=1}^m a_k e_k\Big\|;$$

for all $1 \le n < m$ and all collections of scalars $(a_k)_{k=1}^m$. A basis (basic sequence) $(e_n)_{n=1}^\infty$ is said to be *normalized* provided that $\|e_n\| = 1$ for all n. If $(e_n)_{n=1}^\infty$ is a basic sequence then $(e_n/\|e_n\|)_{n=1}^\infty$ is a normalized basic sequence. Let $(e_n)_{n=1}^\infty$ be a basic sequence in X; $(a_n)_{n=1}^\infty$ a sequence of scalars and $0 \le k_1 < k_2 < \dots$ integers. A sequence $(u_n)_{n=1}^\infty$ of nonzero vectors in X of the form

$$u_n = \sum_{i=k_n+1}^{k_{n+1}} a_i e_i$$

is called a *block basis* of $(e_n)_{n=1}^\infty$. Every block basis (in particular, every subsequence) of a basic sequence is itself a basic sequence with a basis constant which does not exceed that of $(e_n)_{n=1}^\infty$.

1.1.2. *Unconditional bases.* A series $\sum_{n=1}^\infty x_n$ of elements of a Banach space X is said to be *unconditionally convergent* if for any permutation[1] of the positive integers $\varphi : \mathbb{N} \to \mathbb{N}$ the series $\sum_{n=1}^\infty x_{\varphi(n)}$ converges. We need the next criterion of unconditional convergence [17, Lemma 16.1].

Lemma 1.2. *For any sequence $(x_n)_{n=1}^\infty$ in a Banach space X the following assertions are equivalent*

[1]i.e., a bijection

(*i*) *the series* $\sum_{n=1}^{\infty} x_n$ *unconditionally converges;*
(*ii*) *for any sequence of signs* $\theta_n = \pm 1$ *the series* $\sum_{n=1}^{\infty} \theta_n x_n$ *converges;*
(*iii*) *for any sequence of scalars* $(a_n)_{n=1}^{\infty}$ *such that* $|a_n| \le 1$, $n = 1, 2, \dots$ *the series* $\sum_{n=1}^{\infty} a_n x_n$ *converges.*

A basis $(e_n)_{n=1}^{\infty}$ of a Banach space X with the biorthogonal functionals $(e_n^*)_{n=1}^{\infty}$ is called an *unconditional basis* if the series $\sum_{n=1}^{\infty} e_n^*(x)\, e_n$ converges unconditionally for every $x \in X$. In this case, for any subset $I \subseteq \mathbb{N}$ the projection $P_I x = \sum_{n \in I} e_n^*(x)\, e_n$ is well defined on X and bounded, as well as for any sequence of signs $\Theta = (\theta_n)_{n=1}^{\infty}$, $\theta_n = \pm 1$ the operator $M_\Theta x = \sum_{n=1}^{\infty} \theta_n e_n^*(x)\, e_n$. Moreover, $\sup_I \|P_I\| \le \sup_\Theta \|M_\Theta\| \le 2 \sup_I \|P_I\| < \infty$ and the number $\sup_\Theta \|M_\Theta\|$ is called the *unconditional constant* of the unconditional basis $(e_n)_{n=1}^{\infty}$. An unconditional basis with unconditional constant 1 is said to be 1-*unconditional*. A basis which is not unconditional is called a *conditional* basis. A sequence which is an unconditional (respectively, conditional) basis in its closed linear span is called an *unconditional basic sequence* (respectively, *conditional basic sequence*).

Every infinite dimensional Banach space contains a basic sequence, however, not every infinite dimensional separable Banach space contains a basis. The classical Banach spaces $L_1[0,1]$ and $C[0,1]$ contain bases, however they cannot be isomorphically embedded in a Banach space with an unconditional basis. The standard basis $e_n = (\underbrace{0, \dots, 0}_{n-1}, 1, 0, 0, \dots)$ of the spaces c_0 and ℓ_p for $1 \le p < \infty$ is 1-unconditional.

We remark that every 1-unconditional basic sequence $(e_n)_{n=1}^{\infty}$ in a Hilbert space H is orthogonal, because the inequality $\|e_n + e_m\| = \|e_n - e_m\|$ yields

$$(e_n, e_m) = \frac{1}{4}\Big(\|e_n + e_m\| - \|e_n - e_m\|\Big) = 0$$

if $n \ne m$.

We also need the following statement from [13, Proposition 1.c.7] which is true for real Banach spaces.

Lemma 1.3. *Let* $(e_n)_{n=1}^{\infty}$ *be an unconditional basic sequence in a Banach space* X *with the unconditional constant* M. *Let* $(a_n)_{n=1}^{\infty}$ *be any sequence of scalars for which the series* $\sum_{n=1}^{\infty} a_n e_n$ *converges. Then for any bounded sequence of scalars* $(\lambda_n)_{n=1}^{\infty}$ *one has*

$$\Big\|\sum_{n=1}^{\infty} \lambda_n a_n e_n\Big\| \le M \sup_n |\lambda_n| \Big\|\sum_{n=1}^{\infty} a_n e_n\Big\|.$$

Furthermore, we need the following finite dimensional version of Lemma 1.3.

Lemma 1.4. *Let* $(x_n)_{n=1}^{N}$ *be a finite sequence of elements in a real Banach space* X. *Then for any collection of scalars* $(\lambda_n)_{n=1}^{N}$ *one has*

$$\Big\|\sum_{n=1}^{N} \lambda_n x_n\Big\| \le \max_n |\lambda_n| \max_{\theta_n = \pm 1} \Big\|\sum_{n=1}^{N} \theta_n x_n\Big\|.$$

Formally Lemma 1.4 does not follow from 1.3, however its proof provided in the Lindenstrauss-Tzafriri book could be modified to prove Lemma 1.4. Besides, Lemma 1.4 follows from Lemma 2.3 of [15].

1.1.3. *Boundedly complete bases.* A basis $(e_n)_{n=1}^{\infty}$ of a Banach space X is called *boundedly complete* if for any sequence of scalars $(a_n)_{n=1}^{\infty}$ the boundedness of the partial sums $\sup_n \big\|\sum_{k=1}^{n} a_k e_k\big\| < \infty$ implies the convergence of the series $\sum_{n=1}^{\infty} a_n e_n$. Every basis of a reflexive Banach space is boundedly complete [13, Theorem 1.b.5]. The standard basis of the nonreflexive space ℓ_1 is evidently boundedly complete as well. However, every Banach space with a boundedly complete basis is isomorphic to a conjugate space

[13, Theorem 1.b.4]. A kind of converse statement is also true: by a deep result of Johnson, Rosenthal and Zippin [9], if a conjugate Banach space X^* has a basis then X^* contains a boundedly complete basis. Finally, an unconditional basis of a Banach space X is boundedly complete if and only if X contains no subspace isomorphic to c_0 [13, Theorem 1.c.10].

1.2. **A characterization of precompactness of sets in a Banach space with a basis.** We provide below a convenient characterization of precompactness in terms of biorthogonal functionals. Informally speaking, it asserts that the precompactness of a subset A of a Banach space X is equivalent to the uniform convergence of the Fourier series of elements of A with respect to a given basis. This statement is not new (see e.g. [11, 11.2.2]).

Lemma 1.5. *Let X be a Banach space with a basis $(e_n)_{n=1}^\infty$ and the biorthogonal functionals $(e_n^*)_{n=1}^\infty$. A bounded set $A \subset X$ is precompact if and only if*

$$\lim_{N\to\infty} \sup_{x\in A} \Big\| \sum_{n>N} e_n^*(x)\, e_n \Big\| = 0. \tag{1.1}$$

2. Bricks

The notion of entropy is based on the concept of bricks in a Banach space, that is, a box with sides that are parallel to the coordinate hyperplanes with respect to a given basis. The latter concept we develop in this section.

2.1. **Definition and properties.** Let X be a Banach space with a basis $\mathcal{B} = (e_n)_{n=1}^\infty$ and biorthogonal functionals $e_k^* \in X^*$, and let $\mathcal{E} = (\varepsilon_n)_{n=1}^\infty$ be a sequence of nonnegative numbers.

Definition 2.1. A *brick* (more precisely, the *brick corresponding to the pair* $(\mathcal{B}, \mathcal{E})$) is defined to be the following set

$$K_{\mathcal{B},\mathcal{E}} = \big\{ x \in X : \ (\forall n \in \mathbb{N})\ |e_n^*(x)| \le \varepsilon_n \big\}.$$

The numbers ε_n are called the *half-hight* of the brick $K_{\mathcal{B},\mathcal{E}}$.

In other words, $K_{\mathcal{B},\mathcal{E}}$ consists of all sums of convergent series $x = \sum_{n=1}^\infty a_n e_n$ with coefficients satisfying $|a_n| \le \varepsilon_n$ for all n.

A simple observation: any brick $K_{\mathcal{B},\mathcal{E}}$ coincides with the brick $K_{\mathcal{B}',\mathcal{E}'}$, where $\mathcal{B}' = (e_n')_{n\in M}$ is the normalized basis $e_n' = \|e_n\|^{-1} e_n$, $n = 1, 2, \ldots$, and $\mathcal{B}' = (e_n')_{n\in M}$ the half-height $\varepsilon_n' = \varepsilon_n \|e_n\|$, $n = 1, 2, \ldots$. So, **we consider bricks constructed by normalized bases only.**

Definition 2.2. A brick constructed by an unconditional basis, 1-unconditional basis, or boundedly complete basis is called an *unconditional,* 1-*unconditional* or respectively, a *boundedly complete* brick.

Recall that a subset A of a linear space X is called *absolutely convex* provided for all $m \in \mathbb{N}$, $x_1, \ldots, x_m \in A$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{K}$ the inequality $|\lambda_1| + \ldots + |\lambda_m| \le 1$ implies $\lambda_1 x_1 + \ldots + \lambda_m x_m \in A$.

Proposition 2.3. *Every brick in a Banach space X is an absolutely convex closed subset of X.*

Proof. Let $\mathcal{B} = (e_n)_{n=1}^\infty$ be a normalized basis of X with the biorthogonal functionals $e_k^* \in X^*$ and $\mathcal{E} = (\varepsilon_n)_{n=1}^\infty$. Assume $m \in \mathbb{N}$, $x_1, \ldots, x_m \in K_{\mathcal{B},\mathcal{E}}$, $\lambda_1, \ldots, \lambda_m \in \mathbb{K}$ and $|\lambda_1| + \ldots + |\lambda_m| \le 1$. Then for every $n \in \mathbb{N}$ one has

$$\begin{aligned} |e_n^*(\lambda_1 x_1 + \ldots + \lambda_m x_m)| &\le |\lambda_1||e_n^*(x_1)| + \ldots + |\lambda_m||e_n^*(x_m)| \\ &\le |\lambda_1|\,\varepsilon_n + \ldots + |\lambda_m|\,\varepsilon_n \le \varepsilon_n. \end{aligned}$$

Thus, $K_{\mathcal{B},\mathcal{E}}$ is absolutely convex. By continuity of e_n^*'s, $K_{\mathcal{B},\mathcal{E}}$ is closed. □

One can deduce from Lemma 1.5 that if $K_{\mathcal{B},\mathcal{E}}$ is compact then $\lim_{n\to\infty} \varepsilon_n = 0$, and also if $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, then $K_{\mathcal{B},\mathcal{E}}$ is compact. We are not going to provide details because of a more general characterization of compactness for bricks below (Theorem 3.9).

Definition 2.4. A brick $K_{\mathcal{B},\mathcal{E}}$ is said to be *solid* if for each $x \in K_{\mathcal{B},\mathcal{E}}$ and each numbers $a_1, a_2, \ldots \in \mathbb{K}$ such that $|a_n| \le |e_n^*(x)|$ for all $n \in \mathbb{N}$ the series $\sum_{n=1}^{\infty} a_n e_n$ converges[2].

Evidently, if the series $\sum_{n=1}^{\infty} \varepsilon_n e_n$ converges unconditionally then the brick $K_{\mathcal{B},\mathcal{E}}$ is solid. In particular, every of the following two conditions is sufficient for $K_{\mathcal{B},\mathcal{E}}$ to be solid

(1) $K_{\mathcal{B},\mathcal{E}}$ is unconditional;
(2) $\sum_{n=1}^{\infty} \varepsilon_n < \infty$.

Recall that an element $x_0 \in A$ of a subset A of a linear space X is called an *extreme point* of A if there is no segment of A centered at x_0, e.i. for every $x \in X$ there exists $\lambda \in [-1, 1]$ such that $x_0 + \lambda x \notin A$. The next statement easily follows from the definitions.

Proposition 2.5. *Let X be a Banach space with a normalized basis $\mathcal{B} = (e_n)_{n=1}^{\infty}$ and biorthogonal functionals $e_k^* \in X^*$, and let $\mathcal{E} = (\varepsilon_n)_{n=1}^{\infty}$. A vector $x_0 \in K_{\mathcal{B},\mathcal{E}}$ is an extreme point of $K_{\mathcal{B},\mathcal{E}}$ if and only if $|e_n^*(x_0)| = \varepsilon_n$ for all $n \in \mathbb{N}$.*

It is immediate that if $K_{\mathcal{B},\mathcal{E}}$ has an extreme point then $\lim_{n\to\infty} \varepsilon_n = 0$. As we will see below, the existence of an extreme point of a brick is unrelated to its boundedness.

Observe that every element $x_0 \in X$ generates a brick corresponding to a normalized basis $\mathcal{B} = (e_n)_{n=1}^{\infty}$ of X and the sequence $\varepsilon_n = |e_n^*(x_0)|$, an extreme point of which x_0 is.

First we show that the existence of an extreme point of a brick does not imply its boundedness.

Example 2.6. There exists an unbounded brick with en extreme point.

Proof. Let $X = c$ be the space of all converging sequences with the supremum norm. Consider the summing basis [13, p. 20] $e_n = (\underbrace{0, \ldots, 0}_{n-1}, 1, 1, \ldots)$, and the brick generated by the element

$$x_0 = e_1 - \frac{e_2}{2} + \frac{e_3}{3} - \ldots + \frac{(-1)^{n+1} e_n}{n} + \ldots,$$

which therefore is an extreme point of it (the convergence of the series in c follows from that of the Leibniz series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$). The unboundedness of the brick $K_{\mathcal{B},\mathcal{E}}$ with the half-height $\varepsilon_n = \frac{1}{n}$ is guaranteed by the equality

$$\left\| e_1 + \frac{e_2}{2} + \frac{e_3}{3} + \ldots + \frac{e_n}{n} \right\| = \sum_{k=1}^{n} \frac{1}{k}$$

and the divergence of the harmonic series. □

In contrast to this, every unconditional brick with an extreme point is bounded. Moreover, the norm of any extreme point (and hence, of an arbitrary element) is estimated by the unconditional constant of the basis and the norm of any fixed extreme point.

Proposition 2.7. *Let X be a Banach space with a normalized unconditional basis $\mathcal{B} = (e_n)_{n=1}^{\infty}$ and $\mathcal{E} = (\varepsilon_n)_{n=1}^{\infty}$. Let x_0 be an extreme point of the brick $K_{\mathcal{B},\mathcal{E}}$. Then $K_{\mathcal{B},\mathcal{E}}$ is bounded by $M \|x_0\|$, where M is the unconditional constant of $\mathcal{B}$. In particular, if $\mathcal{B}$ is 1-unconditional then $\|x\| \le \|x_0\|$ for every $x \in K_{\mathcal{B},\mathcal{E}}$.*

[2]and hence, its sum belongs to $K_{\mathcal{B},\mathcal{E}}$

Proposition 2.7 follows from Lemma 1.3.

Now we show that the boundedness of a brick does not imply the existence of an extreme point, even of a 1-unconditional brick.

Example 2.8. There exists a bounded 1-unconditional brick without an extreme point.

Proof. Such a brick, for example, is the closed unit ball of the space c_0. Indeed, consider the 1-unconditional standard basis of c_0 and take $\varepsilon_n = 1$, $n = 1, 2, \ldots$. Then the absence of extreme points is obvious. ☐

Proposition 2.9. *Every bounded boundedly complete brick contains an extreme point.*

Proof. Let X be a Banach space with a normalized boundedly complete basis $\mathcal{B} = (e_n)_{n=1}^{\infty}$ and biorthogonal functionals $e_k^* \in X^*$, and let $\mathcal{E} = (\varepsilon_n)_{n=1}^{\infty}$. Assume $\|x\| \leq L$ for all $x \in K_{\mathcal{B},\mathcal{E}}$ and some number L. Observe that for $a_n = \varepsilon_n$ the condition $\big\|\sum_{k=1}^{n} a_k e_k\big\| \leq L$ holds for every $n \in \mathbb{N}$, because $\sum_{k=1}^{n} a_k e_k \in K_{\mathcal{B},\mathcal{E}}$. Since the basis is boundedly complete, the series $x_0 = \sum_{n=1}^{\infty} \varepsilon_n e_n$ converges, and hence there is an extreme point x_0. ☐

3. Radii and a compactness characterization for bricks

By the radius $r(A)$ of a bounded subset of a Banach space we mean the number $r(A) = \sup\limits_{x \in A} \|x\|$.

We consider two more radii of a brick: the extreme radius and the unconditional radius. In the case where the unconditional radius of a brick is finite all three radii coincide. Moreover, in this case (and only in this case) the brick is compact. If an extreme radius is finite then it equals the radius.

Definition 3.1. Let X be a Banach space with a normalized basis $\mathcal{B} = (e_n)_{n=1}^{\infty}$, and let $\mathcal{E} = (\varepsilon_n)_{n=1}^{\infty}$. The *extreme radius* $r^{\text{ext}}(K_{\mathcal{B},\mathcal{E}})$ and the *unconditional radius* $r^{\text{unc}}(K_{\mathcal{B},\mathcal{E}})$ of the brick $K_{\mathcal{B},\mathcal{E}}$ is defined to be either a number or a symbol ∞ by (1) and (2) respectively.

(1) $r^{\text{ext}}(K_{\mathcal{B},\mathcal{E}}) = \sup\big\{\|x_0\| : x_0 \text{ is an extreme point of } K_{\mathcal{B},\mathcal{E}}\big\}$, if an extreme point exists, and $r^{\text{ext}}(K_{\mathcal{B},\mathcal{E}}) = \infty$ otherwise.

(2) $r^{\text{unc}}(K_{\mathcal{B},\mathcal{E}}) = \sup\limits_{\theta_n = \pm 1} \Big\|\sum\limits_{n=1}^{\infty} \theta_n \varepsilon_n e_n\Big\|$ (the norm of a divergent series is ∞).

The difference between the defined radii could be demonstrated using Example 2.8 where as a brick we take the closed unit ball B_{c_0} of the space c_0. By the definitions, $r(B_{c_0}) = 1$, however $r^{\text{ext}}(B_{c_0}) = r^{\text{unc}}(B_{c_0}) = \infty$. Below we construct a brick $K_{\mathcal{B},\mathcal{E}}$ (Example 3.4), for which $r^{\text{ext}}(K_{\mathcal{B},\mathcal{E}}) = r(K_{\mathcal{B},\mathcal{E}}) < \infty$, however $r^{\text{unc}}(B_{c_0}) = \infty$. On the other hand, Example 2.6 may mislead the reader by hinting that a brick with a finite extreme radius need not be bounded. Actually, we have the following statement on the connection between the radii.

3.1. **The connection between radii.**

Theorem 3.2. *For an arbitrary brick $K_{\mathcal{B},\mathcal{E}}$ in a Banach space X the following assertions hold.*

(1) $r^{\text{ext}}(K_{\mathcal{B},\mathcal{E}}) \leq r^{\text{unc}}(K_{\mathcal{B},\mathcal{E}})$.

(2) *If* $r^{\text{ext}}(K_{\mathcal{B},\mathcal{E}}) < \infty$ *then* $r^{\text{ext}}(K_{\mathcal{B},\mathcal{E}}) = r(K_{\mathcal{B},\mathcal{E}})$.

(3) *If* $r^{\text{unc}}(K_{\mathcal{B},\mathcal{E}}) < \infty$ *then* $r^{\text{ext}}(K_{\mathcal{B},\mathcal{E}}) = r^{\text{unc}}(K_{\mathcal{B},\mathcal{E}}) = r(K_{\mathcal{B},\mathcal{E}})$.

Proof. Item (1) follows immediately from the definitions.

(2) Assume $r^{\text{ext}}(K_{\mathcal{B},\mathcal{E}}) < \infty$. By the definitions, $r^{\text{ext}}(K_{\mathcal{B},\mathcal{E}}) \leq r(K_{\mathcal{B},\mathcal{E}})$. The inequality $r(K_{\mathcal{B},\mathcal{E}}) \leq r^{\text{ext}}(K_{\mathcal{B},\mathcal{E}})$ is quite thin; its proof we present separately (see Lemma 3.3 below).

(3) Assume $r^{\text{unc}}(K_{\mathcal{B},\mathcal{E}}) < \infty$. The equality $r^{\text{ext}}(K_{\mathcal{B},\mathcal{E}}) = r^{\text{unc}}(K_{\mathcal{B},\mathcal{E}})$ follows from the definitions as well, and the equality $r(K_{\mathcal{B},\mathcal{E}}) = r^{\text{ext}}(K_{\mathcal{B},\mathcal{E}})$ follows from item (2). □

Lemma 3.3. *If $r^{\text{ext}}(K_{\mathcal{B},\mathcal{E}}) < \infty$ then the brick $K_{\mathcal{B},\mathcal{E}}$ is bounded by $r^{\text{ext}}(K_{\mathcal{B},\mathcal{E}})$.*

Proof. First we prove that for every $n_0 \in \mathbb{N}$ and every $\varepsilon > 0$ there is $N \geq n_0$ such that for all signs $\theta_1, \ldots, \theta_N \in \{-1, 1\}$ one has

$$\Big\| \sum_{n=1}^{N} \theta_n \varepsilon_n e_n \Big\| < r^{\text{ext}}(K_{\mathcal{B},\mathcal{E}}) + \varepsilon. \tag{3.1}$$

Indeed, fix any extreme point $x_0 = \sum_{n=1}^{\infty} \alpha_n e_n$ of $K_{\mathcal{B},\mathcal{E}}$, $|\alpha_n| = \varepsilon_n$, $n = 1, 2, \ldots$ (an extreme point exists because $r^{\text{ext}}(K_{\mathcal{B},\mathcal{E}}) < \infty$). Choose $N \geq n_0$ so that

$$\Big\| \sum_{n>N} \alpha_n e_n \Big\| < \varepsilon. \tag{3.2}$$

Let $\theta_1, \ldots, \theta_N \in \{-1, 1\}$ be any signs. Observe that

$$x = \sum_{n=1}^{N} \theta_n \varepsilon_n e_n + \sum_{n>N} \alpha_n e_n$$

is an extreme point of $K_{\mathcal{B},\mathcal{E}}$, hence $\|x\| \leq r^{\text{ext}}(K_{\mathcal{B},\mathcal{E}})$. Taking into account (3.2), we obtain

$$\Big\| \sum_{n=1}^{N} \theta_n \varepsilon_n e_n \Big\| \leq \|x\| + \Big\| \sum_{n>N} \alpha_n e_n \Big\| < r^{\text{ext}}(K_{\mathcal{B},\mathcal{E}}) + \varepsilon.$$

Thus, (3.1) is proved.

Let $\hat{x} \in K_{\mathcal{B},\mathcal{E}}$ be any element. Show that $\|\hat{x}\| \leq r^{\text{ext}}(K_{\mathcal{B},\mathcal{E}})$. Fix any $\varepsilon > 0$ and pick $n_0 \in \mathbb{N}$ so that for each $m \geq n_0$

$$\Big\| \sum_{n>m} e_n^*(\hat{x})\, e_n \Big\| < \varepsilon. \tag{3.3}$$

Then by the above, choose $N \geq n_0$ so that for all signs $\theta_1, \ldots, \theta_N \in \{-1, 1\}$ one has (3.1). Then

$$\begin{aligned} \|\hat{x}\| = \Big\| \sum_{n=1}^{\infty} e_n^*(\hat{x})\, e_n \Big\| &\overset{(3.3)}{\leq} \Big\| \sum_{n=1}^{N} e_n^*(\hat{x})\, e_n \Big\| + \varepsilon \\ &\overset{\text{Lemma 1.4}}{\leq} \max_{\theta_n = \pm 1} \Big\| \sum_{n=1}^{N} \theta_n \varepsilon_n e_n \Big\| + \varepsilon \overset{(3.1)}{<} r^{\text{ext}}(K_{\mathcal{B},\mathcal{E}}) + 2\varepsilon. \end{aligned}$$

By the arbitrariness of $\varepsilon > 0$, $\|\hat{x}\| \leq r^{\text{ext}}(K_{\mathcal{B},\mathcal{E}})$. Thus, $K_{\mathcal{B},\mathcal{E}}$ is bounded by $r^{\text{ext}}(K_{\mathcal{B},\mathcal{E}})$. □

3.2. **Bricks with finite extreme radius.** In this subsection we study the question of the compactness of a brick with finite extreme radius.

Example 3.4. There exists a noncompact brick in c_0 of finite extreme radius.

Proof. This example is a modification of Example 2.6. We choose integers $0 = n_0$, $2 = n_1 < n_2 < \ldots$ such that

$$\frac{1}{n_{k-1} + 1} + \ldots + \frac{1}{n_k} \in [1, 2], \quad k = 1, 2, \ldots. \tag{3.4}$$

Then for $X = c_0$ we define a basis $\mathcal{B} = (f_n)_{n=1}^{\infty}$ by

$$f_1 = (1, 1, 0, 0, \ldots),$$
$$f_2 = (0, 1, 0, 0, \ldots),$$

$$f_3 = (0, 0, \underbrace{1, 1, \dots, 1, 1}_{n_2 - n_1}, 0, 0, \dots),$$
$$f_4 = (0, 0, \underbrace{0, 1, \dots, 1, 1}_{n_2 - n_1}, 0, 0, \dots),$$
$$\dots$$
$$f_{n_2} = (0, 0, \underbrace{0, 0, \dots, 0, 1}_{n_2 - n_1}, 0, 0, \dots),$$
$$f_{n_2+1} = (\underbrace{0, 0, \dots, 0}_{n_2}, \underbrace{1, 1, \dots, 1, 1}_{n_3 - n_2}, 0, 0, \dots),$$
$$\dots .$$

Using standard arguments, one can prove that the above system $\mathcal{B} = (f_n)_{n=1}^\infty$ is a basis of c_0. More precisely, first we need to prove the inequality $\big\|\sum_{k=1}^n a_k f_k\big\| \le \big\|\sum_{k=1}^m a_k f_k\big\|$ for all $n < m$ and any collection of scalars $(a_k)_{k=1}^m$. Then we prove that the linear span of $(f_n)_{n=1}^\infty$ is dense in c_0 (because the standard basis of c_0 is contained in that linear span). We omit the details which are straightforward.

Then we define half-height by $\varepsilon_n = \frac{1}{n}$, $n = 1, 2, \dots$, set $\mathcal{E} = (\varepsilon_n)_{n=1}^\infty$ and prove that the brick $K_{\mathcal{B},\mathcal{E}}$ is as desired. First we show that $K_{\mathcal{B},\mathcal{E}}$ contains an extreme point. Indeed, the series $f_0 = \sum_{n=1}^\infty (-1)^{n+1} \varepsilon_n f_n$ converges in c_0, because the series $\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}$ converges, and hence, possesses the Cauchy condition. By (3.4) we obtain that the brick is norm bounded by 2, hence, $r^{\rm ext}(K_{\mathcal{B},\mathcal{E}}) \le 2$. The noncompactness of $K_{\mathcal{B},\mathcal{E}}$ follows from Lemma 1.5 and the fact that the sequence

$$g_k = \frac{1}{n_{k-1}+1} f_{n_{k-1}+1} + \dots + \frac{1}{n_k} f_{n_k}, \quad k = 1, 2, \dots .$$

satisfies $g_k \in K_{\mathcal{B},\mathcal{E}}$ and $\|g_k\| \ge 1$ by (3.4). $\square$

Now we show that in most natural cases (unconditional or boundedly complete basis) a brick of finite extreme radius is compact.

Theorem 3.5. *Every unconditional or boundedly complete brick of finite extreme radius is compact.*

For the proof we need the following lemma which is also be used later.

Lemma 3.6. *Let X be a Banach space with a normalized basis $\mathcal{B} = (e_n)_{n=1}^\infty$, and let $\mathcal{E} = (\varepsilon_n)_{n=1}^\infty$ be a sequence of nonnegative real numbers such that the brick $K_{\mathcal{B},\mathcal{E}}$ is not compact. Then there are $\delta > 0$ and a block basis $(u_k)_{k=1}^\infty$ of $\mathcal{B}$ such that $u_k \in K_{\mathcal{B},\mathcal{E}}$ and $\|u_k\| \ge \delta$ for $k = 1, 2, \dots$.*

Proof. By Proposition 2.3, $K_{\mathcal{B},\mathcal{E}}$ is not precompact and by Lemma 1.5, (1.1) does not hold for $A = K_{\mathcal{B},\mathcal{E}}$. Let $e_k^* \in X^*$ be the biorthogonal functionals. Choose $\delta > 0$ and a sequence $x_{N_k} \in K_{\mathcal{B},\mathcal{E}}$ so that $N_1 < N_2 < \dots$, $\|x_{N_k}\| \ge 2\delta$ and $e_n^*(x_{N_k}) = 0$ as $n \le N_k$, that is, $x_{N_k} = \sum_{n > N_k} e_n^*(x_{N_k})\, e_n$ for $k = 1, 2, \dots$. Set $n_1 = 0$ and choose $n_2 > n_1$ so that

$$\Big\| \sum_{n > n_2} e_n^*(x_{N_1})\, e_n \Big\| < \delta.$$

Then for $u_1 = \sum_{n=1}^{n_2} e_n^*(x_{N_1})\, e_n$ one gets that $u_1 \in K_{\mathcal{B},\mathcal{E}}$ and

$$\|u_1\| \ge \Big\| \sum_{n=1}^\infty e_n^*(x_{N_1})\, e_n - \sum_{n > n_2} e_n^*(x_{N_1})\, e_n \Big\| \ge 2\delta - \delta = \delta.$$

At the second step we choose $j_2 > j_1 = 1$ so that $N_{j_2} > n_2$. Thus,

$$x_{N_{j_2}} = \sum_{n>N_{j_2}} e_n^*(x_{N_{j_2}})\, e_n = \sum_{n>n_2} e_n^*(x_{N_{j_2}})\, e_n.$$

Choose $n_3 > n_2$ so that

$$\Big\| \sum_{n>n_3} e_n^*(x_{N_{j_2}})\, e_n \Big\| < \delta.$$

Then for $u_2 = \sum_{n=n_2+1}^{n_3} e_n^*(x_{N_{j_2}})\, e_n$ we obtain that $u_2 \in K_{\mathcal{B},\mathcal{E}}$ and

$$\|u_2\| \geq \Big\| \sum_{n=n_2+1}^{\infty} e_n^*(x_{N_{j_2}})\, e_n - \sum_{n>n_3} e_n^*(x_{N_{j_2}})\, e_n \Big\| \geq 2\delta - \delta = \delta.$$

Proceeding like that step by step, we construct the desired block basis $(u_k)_{k=1}^{\infty}$. □

Proof of Theorem 3.5. Let X be a Banach space with a normalized unconditional or boundedly complete basis $\mathcal{B} = (e_n)_{n=1}^{\infty}$ and biorthogonal functionals $e_k^* \in X^*$, and let $\mathcal{E} = (\varepsilon_n)_{n=1}^{\infty}$. Assume $r^{\text{ext}}(K_{\mathcal{B},\mathcal{E}}) < \infty$, and show that for $A = K_{\mathcal{B},\mathcal{E}}$ we have (1.1).

The case where $\mathcal{B}$ is unconditional. Let M be the unconditional constant of $\mathcal{B}$ and x_0 any extreme point of A (an extreme point exists, because $r^{\text{ext}}(K_{\mathcal{B},\mathcal{E}}) < \infty$). Fix any $\varepsilon > 0$ and choose $n_0 \in \mathbb{N}$ so that for every $N \geq n_0$ one has $\big\|\sum_{n>N} e_n^*(x_0)\, e_n\big\| < M^{-1}\varepsilon$. Then for each $x \in K_{\mathcal{B},\mathcal{E}}$ and each $N \geq n_0$, taking into account $|e_n^*(x)| \leq \varepsilon_n = |e_n^*(x_0)|$ and Lemma 1.3, we obtain

$$\Big\| \sum_{n>N} e_n^*(x)\, e_n \Big\| \leq M \Big\| \sum_{n>N} e_n^*(x_0)\, e_n \Big\| < \varepsilon.$$

The case where $\mathcal{B}$ is boundedly complete. Assume (1.1) is false. by Lemma 1.5, the brick $K_{\mathcal{B},\mathcal{E}}$ is not compact. Using Lemma 3.6, we choose $\delta > 0$ and a block basis $u_k = \sum_{j=n_k+1}^{n_{k+1}} a_j e_j$ of $\mathcal{B}$ such that $u_k \in K_{\mathcal{B},\mathcal{E}}$ and $\|u_k\| \geq \delta$ for $k = 1, 2, \ldots$, where $0 = n_1 < n_2 < \ldots$. Since $S_n = \sum_{i=1}^{n} a_i e_i \in K_{\mathcal{B},\mathcal{E}}$ for all $n \in \mathbb{N}$, we get $\|S_n\| \leq r^{\text{ext}}(K_{\mathcal{B},\mathcal{E}})$ by (2) of Theorem 3.2. Since the basis $(e_n)_{n=1}^{\infty}$ is boundedly complete, the series $\sum_{n=1}^{\infty} a_n e_n$ converges. However, this is impossible, because the Cauchy condition for its convergence contradicts the inequalities $\|u_k\| \geq \delta$, $k = 1, 2, \ldots$. So, (1.1) is valid.

Thus, (1.1) holds anyway. By Lemma 1.5, the brick $K_{\mathcal{B},\mathcal{E}}$ is precompact. By Proposition 2.3, $K_{\mathcal{B},\mathcal{E}}$ is compact. □

The following theorem shows that examples like 3.4 are possible for Banach spaces containing an isomorph of c_0 only.

Theorem 3.7. *For a Banach space with a Schauder basis the following assertions are equivalent.*

(1) *X contains a noncompact brick with finite extreme radius.*
(2) *X contains a subspace isomorphic to c_0.*

Proof. (1) ⇒ (2). Let $\mathcal{B} = (e_n)_{n=1}^{\infty}$ be a normalized basis, $\mathcal{E} = (\varepsilon_n)_{n=1}^{\infty}$ a sequence of nonnegative real numbers such that the brick $K_{\mathcal{B},\mathcal{E}}$ is noncompact and has finite extreme radius. By (2) of Theorem 3.2, $r(K_{\mathcal{B},\mathcal{E}}) < \infty$. Choose by Lemma 3.6 $\delta > 0$ and a block basis $(u_k)_{k=1}^{\infty}$ of $\mathcal{B}$ such that $u_k \in K_{\mathcal{B},\mathcal{E}}$ and $\|u_k\| \geq \delta$ for $k = 1, 2, \ldots$.

Now we show that the sequence (u_k) is equivalent to the unit vector basis of c_0. Let $(a_k)_{k=1}^{n}$ be any scalars with $\alpha = \max_k |a_k| > 0$. On the one hand, since $\sum_{k=1}^{n} \alpha^{-1} a_k u_k \in K_{\mathcal{B},\mathcal{E}}$, we obtain

$$\Big\| \sum_{k=1}^{n} a_k u_k \Big\| = \alpha \Big\| \sum_{k=1}^{n} \alpha^{-1} a_k u_k \Big\| \leq \alpha \cdot r(K_{\mathcal{B},\mathcal{E}}).$$

On the other hand, for $j \in \{1, \dots, n\}$, so that $|a_j| = \alpha$ the well known estimate $|a_j| \le 2K \Big\| \sum_{k=1}^n a_k u_k \Big\|$, where K is the basis constant of $(u_k)_{k=1}^\infty$, see [13, p. 7], implies $\Big\| \sum_{k=1}^n a_k u_k \Big\| \ge (2K)^{-1}\alpha$.

(2) ⇒ (1). By Sobczyk's theorem [13, Theorem 2.f.5], a subspace X_1 of X which is isomorphic to c_0 is complemented in X, that is, $X = Y \oplus X_1$ for some subspace Y of X. Hence, X is isomorphic to $X \oplus c_0$[3], because

$$X \simeq Y \oplus c_0 \simeq Y \oplus (c_0 \oplus c_0) \simeq (Y \oplus c_0) \oplus c_0 \simeq X \oplus c_0,$$

where $E \simeq F$ means that E is isomorphic to F. Observe that, if $(e_n)_{n=1}^\infty$ and $(f_n)_{n=1}^\infty$ are Schauder bases of E and F respectively then the mixed system

$$g_n = \begin{cases} (e_k, 0), & \text{if } n = 2k-1, \\ (0, f_k), & \text{if } n = 2k \end{cases}$$

is a Schauder basis in the direct sum $X \oplus c_0$. So, since X has a basis, due to the isomorphism $X \simeq X \oplus c_0$, X has a basis $\mathcal{B} = (e_n)_{n=1}^\infty$ such that the subsequence $(e_{2k})_{k=1}^n$ is equivalent to the unit vector basis of c_0. Using this subsequence and Example 3.4, one can construct a noncompact brick in X with finite extreme radius. □

3.3. **A compactness characterization for bricks.** In this subsection we characterize compactness for bricks, partially in terms of the following notion.

Definition 3.8. Let X be a Banach space with a normalized basis $\mathcal{B} = (e_n)_{n=1}^\infty$ and biorthogonal functionals $e_k^* \in X^*$, and let $\mathcal{E} = (\varepsilon_n)_{n=1}^\infty$. The brick $K_{\mathcal{B},\mathcal{E}}$ is called *holistic* if for any sequence of scalars $(a_n)_{n=1}^\infty$ such that $|a_n| \le \varepsilon_n$ for all $n \in \mathbb{N}$ the series $\sum_{n=1}^\infty a_n e_n$ converges.

In other words, a holistic brick is a solid brick with an extreme point (cf. Definition 2.4).

The following result is important for the concept of entropy.

Theorem 3.9 (A compactness test for bricks)**.** *Let X be a Banach space with a normalized basis $\mathcal{B} = (e_n)_{n=1}^\infty$, and let $\mathcal{E} = (\varepsilon_n)_{n=1}^\infty$. Then the following assertions are equivalent.*

(1) *The brick $K_{\mathcal{B},\mathcal{E}}$ is compact.*
(2) *The brick $K_{\mathcal{B},\mathcal{E}}$ is holistic.*
(3) *The series $\sum_{n=1}^\infty \varepsilon_n e_n$ converges unconditionally.*
(4) $r^{\mathrm{unc}}(K_{\mathcal{B},\mathcal{E}}) < \infty$.
(5) *$K_{\mathcal{B},\mathcal{E}}$ is homeomorphic to the Tikhonov cube $[-1,1]^\omega$*

Proof. The equivalence (2) ⇔ (3) follows from Lemma 1.2.

(1) ⇒ (2). Let $(a_n)_{n=1}^\infty$ be a sequence of scalars such that $|a_n| \le \varepsilon_n$ for all $n \in \mathbb{N}$. Set $x_N = \sum_{n=1}^N a_n e_n$ and show that the series $\sum_{n=1}^\infty a_n e_n$ converges. Assume, on the contrary, that this is false. Then the series does not meet the Cauchy condition, and hence, there are $\delta > 0$, sequences of integers $1 = n_0 < n_1 < \dots$ and $(\ell_k)_{k=1}^\infty$ such that $\ell_k \le n_{k+1} - n_k$ and $\|u_k\| \ge \delta$ for $k = 1, 2, \dots$, where $u_k = \sum_{j=n_k+1}^{n_k+\ell_k} a_j e_j$. Observe that $u_k \in K_{\mathcal{B},\mathcal{E}}$ for all k. Denote by K the basis constant of $\mathcal{B}$ and prove that $\|u_k - u_m\| \ge \delta/K$ for all $k < m$, which contradicts the compactness of $K_{\mathcal{B},\mathcal{E}}$. Indeed, $\delta \le \|u_k\| \le K\|u_k - u_m\|$.

(2) ⇒ (1). We prove that

$$\lim_{N\to\infty} \sup_{|a_n| \le \varepsilon_n} \Big\| \sum_{n>N} a_n e_n \Big\| = 0. \tag{3.5}$$

[3]we do not care what norm one considers on the direct sum $X \oplus c_0$, because all the norms are equivalent and the corresponding Banach spaces are isomorphic

Indeed, if this were false, we would choose $\delta > 0$, a sequence of numbers $0 = n_0 < n_1 < \ldots$ and a sequence $(a_n)_{n=1}^{\infty}$, $|a_n| \leq \varepsilon_n$ so that

$$\Big\| \sum_{j=n_{k-1}+1}^{n_k} a_j e_j \Big\| \geq \delta,$$

which contradicts the Cauchy condition for $\sum_{n=1}^{\infty} a_n e_n$. Thus, (3.5) is proved. By Lemma 1.5, the brick $K_{\mathcal{B},\mathcal{E}}$ is precompact, and then by Proposition 2.3 is compact.

It is left to show the equivalence of (4) to the other conditions. Indeed, the implication (4) $\Rightarrow$ (3) is obvious. On the other hand, (2) together with (1) implies (4).

(1) $\Leftrightarrow$ (5). By Tikhonov's theorem, T is compact. Thus, $K_{\mathcal{B},\mathcal{E}}$ is compact whenever it is homeomorphic to T. Conversely, if $K_{\mathcal{B},\mathcal{E}}$ is compact then the map $F(x) = \big(\varepsilon_1^{-1} e_1^*(x), \varepsilon_2^{-1} e_2^*(x), \ldots\big)$ is a homeomorphism of $K_{\mathcal{B},\mathcal{E}}$ onto T (the continuity of T^{-1} follows from Lemma 1.5). □

Remark that one can provide a topological proof of the implication (1) $\Rightarrow$ (2). Indeed, the function$F$ from $K_{\mathcal{B},\mathcal{E}}$ to the cube $T_{\mathcal{E}} = [-\varepsilon_1, \varepsilon_1] \times [-\varepsilon_2, \varepsilon_2] \times \ldots$ endowed with the Tikhonov topology, defined by $F(x) = \big(e_1^*(x), e_2^*(x), \ldots\big)$, is continuous. Then the image $F(K_{\mathcal{B},\mathcal{E}})$ is compact in $T_{\mathcal{E}}$. Since any point $y = (\eta_1, \eta_2, \ldots) \in T_{\mathcal{E}}$ is the limit of $y_n = (\eta_1, \ldots, \eta_n, 0, 0, \ldots) \in F(K_{\mathcal{B},\mathcal{E}})$, we obtain that $y \in F(K_{\mathcal{B},\mathcal{E}})$. So, the brick $K_{\mathcal{B},\mathcal{E}}$ is holistic.

As a consequence of theorems 3.2 and 3.9 we get the following result.

Corollary 3.10. *All the radii of any compact brick $K_{\mathcal{B},\mathcal{E}}$ are finite and coincide. Moreover, all of them equal the norm $\|x_0\|$ of some extreme point x_0 of $K_{\mathcal{B},\mathcal{E}}$.*

The second statement of Corollary 3.10 follows from (3) of Theorem 3.2 and the general well known fact that a continuous real valued function on a compact set attains its maximum.

The implication (3) $\Rightarrow$ (1) of Theorem 3.9 gives the following Gelfand theorem [10, Theorem 1.3.4].

Corollary 3.11 (Gelfand's theorem)**.** *If a series $\sum_{n=1}^{\infty} x_n$ of elements of a Banach space X unconditionally converges then the set of all sums $\sum_{n=1}^{\infty} \theta_n x_n$, $\theta_n = \pm 1$ is compact.*

Remark also that by the compactness of the closed convex hull of a compact set [5, p. 364], the implication (3) $\Rightarrow$ (1) one can deduce from the above Gelfand theorem.

4. Geometric entropy

The geometric entropy of a set A is going to be the infimum of the radii of compact bricks containing A. Depending on a type of bricks, we get different types of geometric entropy.

Definition 4.1. Let X be a Banach space. The *geometric entropy* and the *unconditional entropy* of a subset $A \subseteq X$ is a number or the symbol ∞, defined, respectively, by

(1) $\mathbf{E}(A) = \inf\limits_{A \subseteq K_{\mathcal{B},\mathcal{E}}} r(K_{\mathcal{B},\mathcal{E}})$ (the infimum is taken over all compact bricks $K_{\mathcal{B},\mathcal{E}}$ in X);

(2) $\mathbf{E}_0(A) = \inf\limits_{A \subseteq K_{\mathcal{B},\mathcal{E}}} r(K_{\mathcal{B},\mathcal{E}})$ (the infimum is taken over all 1-unconditional compact bricks $K_{\mathcal{B},\mathcal{E}}$ in X).

In the case where no compact brick (of the corresponding type) contains A, we set the corresponding entropy to be equal ∞. In particular, if X has no basis then there is no brick in X, and hence all subsets of X has infinite geometric entropy.

4.1. **Common properties.** The following statements collect simple properties of the entropies.

Proposition 4.2. *Let X be a Banach space. Then*

(1) *for any $A \subseteq X$ and $\lambda > 0$ one has $\mathbf{E}(\lambda A) = \lambda \mathbf{E}(A)$ and $\mathbf{E}_0(\lambda A) = \lambda \mathbf{E}_0(A)$;*
(2) *if $A \subseteq B \subseteq X$ then $\mathbf{E}(A) \le \mathbf{E}(B)$ and $\mathbf{E}_0(A) \le \mathbf{E}_0(B)$;*
(3) *if $A \subseteq X$ then $\mathbf{E}_0(A) \ge \mathbf{E}(A) \ge r(A)$;*
(4) *if $K_{\mathcal{B},\mathcal{E}}$ is a compact brick then $\mathbf{E}(K_{\mathcal{B},\mathcal{E}}) = r(K_{\mathcal{B},\mathcal{E}})$;*
(5) *if $K_{\mathcal{B},\mathcal{E}}$ is a 1-unconditional compact brick then $\mathbf{E}_0(K_{\mathcal{B},\mathcal{E}}) = r(K_{\mathcal{B},\mathcal{E}})$.*

The proof is obvious.

Given a subset A of a Banach space X, by $\overline{\mathrm{absconv}\,(A)}$ we denote the closure of an absolute convex hull of A, which by definition equals the least closed absolute convex set in X containing A. The next assertion follows from Proposition 2.3.

Proposition 4.3. *For any subset A of a Banach space X one has $\mathbf{E}\big(\overline{\mathrm{absconv}\,(A)}\big) = \mathbf{E}(A)$ and $\mathbf{E}_0\big(\overline{\mathrm{absconv}\,(A)}\big) = \mathbf{E}_0(A)$.*

4.2. **Sudakov's characteristic.** Following Sudakov [18] and generalizing his notions introduced for an orthonormal basis of a Hilbert space to a normalized basis of a Banach space, we give some definitions.

Let X be a Banach space with a normalized basis $\mathcal{B} = (e_n)_{n=1}^\infty$ and the biorthogonal functionals $(e_n^*)_{n=1}^\infty$.

Definition 4.4. The *clearances* of a subset $A \subset X$ relatively to the basis $\mathcal{B}$ are set to be the sequence $\gamma_{\mathcal{B},n}(A) \in [0, +\infty]$ defined by

$$\gamma_{\mathcal{B},n}(A) = \sup_{x \in A} |e_n^*(x)|, \ \ n = 1, 2, \ldots. \tag{4.1}$$

The following simple observation will be used to study of operator images of the closed unit ball.

Lemma 4.5. *Let X be a Banach space with a normalized basis $\mathcal{B} = (e_n)_{n=1}^\infty$ and biorthogonal functionals $(e_n^*)_{n=1}^\infty$. Let $T \in \mathcal{L}(X)$. Then the clearances of the image $T(B_X)$ of the closed unit ball of X relatively to the basis $\mathcal{B}$ are evaluated by $\gamma_{\mathcal{B},n}\big(T(B_X)\big) = \|T^* e_n^*\|$, $n \in \mathbb{N}$.*

Proof. Indeed, for each $n \in \mathbb{N}$ one has

$$\gamma_{\mathcal{B},n}\big(T(B_X)\big) = \sup_{x \in B_X} |e_n^*(Tx)| = \sup_{x \in B_X} |(T^* e_n^*)(x)| = \|T^* e_n^*\|.$$

□

Definition 4.6. The *radius* of a subset $A \subset X$ relatively to the basis $\mathcal{B}$ is the number or the symbol $r_{\mathcal{B}}(A) \in [0, +\infty]$ defined by

$$r_{\mathcal{B}}(A) = \sup_{\theta_n = \pm 1} \Big\| \sum_{n=1}^\infty \theta_n \gamma_{\mathcal{B},n}(A)\, e_n \Big\| \tag{4.2}$$

(here the norm of a divergent series is set to be ∞).

This latter radius of a set generalizes the unconditional radius of a brick. Indeed, if $K_{\mathcal{B},\mathcal{E}}$ is a brick in a Banach space X constructed by a basis $\mathcal{B} = (e_n)_{n=1}^\infty$ with biorthogonal functionals $e_k^* \in X^*$ and half-height $\mathcal{E} = (\varepsilon_n)_{n=1}^\infty$ then $\gamma_{\mathcal{B},n}(K_{\mathcal{B},\mathcal{E}}) = \varepsilon_n$ for all $n \in \mathbb{N}$, and hence, $r_{\mathcal{B}}(K_{\mathcal{B},\mathcal{E}}) = r^{\mathrm{unc}}(K_{\mathcal{B},\mathcal{E}})$. Thus, Theorem 3.9 and item (3) of Theorem 3.2 imply that if $K_{\mathcal{B},\mathcal{E}}$ is a compact brick then $r_{\mathcal{B}}(K_{\mathcal{B},\mathcal{E}}) = r(K_{\mathcal{B},\mathcal{E}})$.

In [18] the author used the sum of the series $\sum_{n=1}^\infty \gamma_{\mathcal{B},n}^2(A)$ instead of the radius of A introduced above, which corresponds to the square of the radius for the case of an orthonormal basis of a Hilbert space. Very likely, that in such cases the square root of the sum is not taken just for aesthetic reasons, however it is much more natural to consider the norm of an element as a characteristic of something than the square of the norm. Another observation is that, for an orthonormal basis of a Hilbert space (more generally, for a 1-unconditional basis of a Banach space in the real case) the norm of the sum that appears in the definition of the radius does not depend on the signs θ_n, and hence one may replace the right-hand side of (4.2) with the expression $\left\|\sum_{n=1}^{\infty} \gamma_{\mathcal{B},n}(A)\, e_n\right\|$.

Remark that the radius of a set $r_{\mathcal{B}}(A)$ does depend on the basis $\mathcal{B}$. Moreover, in [7] the first named author provided an example of a set in a separable Hilbert space the radius of which relatively to a certain basis is finite, and infinite relatively to another one.

The *Sudakov characteristic* of a subset A of a Banach space X with a basis is the number or symbol $s(A) \in [0, \infty]$, defined by

$$s(A) = \sup_{\mathcal{B}} r_{\mathcal{B}}(A),$$

where the supremum is taken over all normalized bases $\mathcal{B}$ of X.

The following statement shows that the geometric entropy of a set can be defined as the Sudakov characteristic, but replacing sup with inf.

Proposition 4.7. *For any subset A of a Banach space X the following equalities hold*

(1) $\mathbf{E}(A) = \inf_{\mathcal{B}} r_{\mathcal{B}}(A)$ *(here the infimum is taken over all normalized bases $\mathcal{B}$ of X);*

(2) $\mathbf{E}_0(A) = \inf\{r_{\mathcal{B}}(A) : \mathcal{B}$ *is a 1-unconditional basis of* $X\}$.

Proof. We prove (1) only; item (2) is proved similarly. We prove (1) under the assumption that the set of compact bricks containing A is nonempty (otherwise both sides of the equality equal ∞). So, let $\mathcal{B} = (e_n)_{n=1}^{\infty}$ be any normalized basis of X with the biorthogonal functionals $(e_n^*)_{n=1}^{\infty}$, $A \subseteq X$ any subset. Set $\Gamma_{\mathcal{B}} = \Gamma_{\mathcal{B}}(A) = \big(\gamma_{\mathcal{B},n}(A)\big)_{n=1}^{\infty}$, where $\gamma_{\mathcal{B},n}(A)$ are the clearances of A relatively to $\mathcal{B}$, defined by (4.1). By (4.2) and Definition 3.1 (2), $r_{\mathcal{B}}(A) = r(K_{\mathcal{B},\Gamma_{\mathcal{B}}})$. Hence, taking into account that $A \subseteq K_{\mathcal{B},\Gamma_{\mathcal{B}}}$, we obtain

$$\mathbf{E}(A) = \inf_{A \subseteq K_{\mathcal{B},\mathcal{E}}} r(K_{\mathcal{B},\mathcal{E}}) \le \inf_{\mathcal{B}} r(K_{\mathcal{B},\Gamma_{\mathcal{B}}}) = \inf_{\mathcal{B}} r_{\mathcal{B}}(A).$$

In order to prove the other side inequality, we fix any normalized basis $\mathcal{B}_0$ of X, and denote by $\Gamma_{\mathcal{B}_0}$ the clearances of A relatively to $\mathcal{B}_0$. Since $A \subseteq K_{\mathcal{B}_0,\Gamma_{\mathcal{B}_0}}$, one has

$$\inf_{\mathcal{B}} r_{\mathcal{B}}(A) \le r_{\mathcal{B}_0}(A) \le r_{\mathcal{B}_0}(K_{\mathcal{B}_0,\Gamma_{\mathcal{B}_0}}) = r(K_{\mathcal{B}_0,\Gamma_{\mathcal{B}_0}}).$$

By arbitrariness of $\mathcal{B}_0$, we get

$$\inf_{\mathcal{B}} r_{\mathcal{B}}(A) \le \inf_{A \subseteq K_{\mathcal{B},\Gamma_{\mathcal{B}}}} r(K_{\mathcal{B},\Gamma_{\mathcal{B}}}).$$

It remains to observe that, if $A \subseteq K_{\mathcal{B},\mathcal{E}}$ then $A \subseteq K_{\mathcal{B},\Gamma_{\mathcal{B}}} \subseteq K_{\mathcal{B},\mathcal{E}}$, and therefore,

$$\inf_{A \subseteq K_{\mathcal{B},\Gamma_{\mathcal{B}}}} r(K_{\mathcal{B},\Gamma_{\mathcal{B}}}) = \inf_{A \subseteq K_{\mathcal{B},\mathcal{E}}} r(K_{\mathcal{B},\mathcal{E}}) = \mathbf{E}(A).$$

□

4.3. **Is the geometric entropy of every precompact set finite?** Equivalently, is every precompact set in a Banach space X contained in a compact brick? We know very little concerning this question. By Proposition 4.8, the answer is affirmative if X embeds isomorphically in c_0. We also know that the answer is negative for the unconditional entropy if X is isomorphic to a Hilbert space (see Corollary 5.2 and Example 5.4).

Proposition 4.8. *Every precompact subset A of the Banach space c_0 has finite geometric entropy. Moreover,*

$$\mathbf{E}(A) = \mathbf{E}_0(A) = r(A).$$

Proof. Let A be a precompact subset of c_0. Let $\varepsilon_n = \gamma_n(A) = \sup_{x \in A} |e_n^*(x)|$, $n = 1, 2, \dots$ be the clearances of A with respect to the standard basis $\mathcal{B} = (e_n)_{n=1}^\infty$ of c_0. One can easily deduce from Lemma 1.5 that $\lim_{n \to \infty} \varepsilon_n = 0$. Hence, the series $x_0 = \sum_{n=1}^\infty \varepsilon_n e_n$ unconditionally converges in c_0. By Theorem 3.9, the brick $K_{\mathcal{B},\mathcal{E}}$ is compact, where $\mathcal{E} = (\varepsilon_n)_{n=1}^\infty$. By the definition of ε_n's, $A \subseteq K_{\mathcal{B},\mathcal{E}}$, and therefore A has finite geometric entropy. Thus, $\mathbf{E}_0(A) \le r(K_{\mathcal{B},\mathcal{E}}) = \|x_0\|$, because

$$\Big\| \sum_{n=1}^\infty \theta_n \varepsilon_n e_n \Big\| = \max_n \varepsilon_n = \|x_0\| = \varepsilon_{n_0}$$

for some $n_0 \in \mathbb{N}$. By the definition of ε_{n_0}, for any $\varepsilon > 0$ there is $x \in A$ with $|e_{n_0}^*(x)| \ge \varepsilon_{n_0} - \varepsilon$. In particular, $\|x\| \ge \|x_0\| - \varepsilon$. By arbitrariness of ε, $\|x_0\| \le r(A)$. So, $\mathbf{E}_0(A) \le r(A)$. By (3) of Proposition 4.2, the proof is completed. $\square$

5. Geometric entropy in a Hilbert space

Let H be a separable infinite dimensional Hilbert space. By the observation made just before Lemma 1.3, a sequence in a separable Hilbert space is a 1-unconditional normalized basis if and only if it is an orthonormal basis. So, one may replace 1-unconditional normalized bases with orthonormal bases in the definition of the unconditional entropy $\mathbf{E}_0(A)$ of a set $A \subseteq H$. For the same reason, a 1-unconditional brick in a Hilbert space we called a rectangular brick.

In this section we construct different classes of compact sets having infinite unconditional entropy, and provide a sufficient condition on a precompact set to have finite unconditional entropy. We also show that the unconditional entropy in a Hilbert space is different from the geometric entropy by constructing a corresponding example.

5.1. **The ranges of Hilbert-Schmidt operators.** Recall that an operator $T \in \mathcal{L}(H)$ is called a *Hilbert-Schmidt operator* if $\|T\|_{HS}^2 = \sum_{n=1}^\infty \|Te_n\|^2 < \infty$ for some (equivalently, every) orthonormal basis $(e_n)_{n=1}^\infty$ of H. It is known that the number $\|T\|_{HS}$ does not depend on the basis $(e_n)_{n=1}^\infty$, and is called the Hilbert-Schmidt norm of T. If T is a Hilbert-Schmidt operator then so is the conjugate operator T^* and $\|T^*\|_{HS} = \|T\|_{HS}$. On Hilbert-Schmidt operators, see [2].

By an *ellipsoid* in a Hilbert space H we mean the image $T(B_H)$ of the closed unit ball B_H of H under a Hilbert-Schmidt operator $T \in \mathcal{L}(H)$. The following theorem characterizes Hilbert-Schmidt operators in terms of unconditional entropy, and gives a formula for evaluating the unconditional entropy of ellipsoids.

Theorem 5.1. *Let $T \in \mathcal{L}(H)$ be a linear bounded operator. Then the image $T(B_H)$ of the closed unit ball B_H of H under T has finite unconditional entropy if and only if T is a Hilbert-Schmidt operator. Moreover, $\mathbf{E}_0\big(T(B_H)\big) = \|T\|_{HS}$.*

Proof. Let $\mathcal{B} = (e_n)_{n=1}^\infty$ be any orthonormal basis of H. Then for any choice of signs $\theta_n = \pm 1$ we obtain

$$\Big\| \sum_{n=1}^\infty \theta_n \gamma_{\mathcal{B},n}\big(T(B_H)\big)\, e_n \Big\|^2 = \sum_{n=1}^\infty \gamma_{\mathcal{B},n}^2\big(T(B_H)\big) \overset{\text{by Lemma 4.5}}{=} \sum_{n=1}^\infty \|T^* e_n\|^2 = \|T^*\|_{HS}^2 = \|T\|_{HS}^2.$$

Thus, $r_{\mathcal{B}}\big(T(B_H)\big) = \|T\|_{HS}$. By arbitrariness of the basis $(e_n)_{n=1}^\infty$ and Proposition 4.7, $\mathbf{E}_0\big(T(B_H)\big) = \|T\|_{HS}$. $\square$

As a consequence of Theorem 5.1, we obtain a class of examples of compact sets with infinite unconditional entropy.

Corollary 5.2. *If $T \in \mathcal{L}(H)$ is a compact operator on a Hilbert space H which is not a Hilbert-Schmidt operator then the image $T(B_H)$ of the unit ball of H under T is a precompact set with infinite unconditional entropy.*

Using Theorem 5.1, we easily evaluate the unconditional entropy of the unit ball of a finite dimensional subspace of a Hilbert space.

Proposition 5.3. *Let B_X denote the closed unit ball of a finite dimensional subspace X of H. Then $\mathbf{E}_0(B_X) = \sqrt{\dim X}$.*

Proof. For convenience, we choose an orthonormal basis $(e_n)_{n=1}^\infty$ of H such that $X = [e_n]_{n=1}^N$, where $N = \dim X$. Denote by P the orthogonal projection of H onto X. Then by Theorem 5.1,

$$E_0^2(B_X) = \sum_{n=1}^\infty \|Pe_n\|^2 = \sum_{n=1}^N \|e_n\| = N.$$

□

Using Proposition 5.3, we construct another class of examples of compact sets in H with infinite unconditional entropy.

Example 5.4. There exists a precompact set in H with infinite unconditional entropy.

Proof. Let $(m_n)_{n=1}^\infty$ be a sequence of integers and $(\delta_n)_{n=1}^\infty$ a sequence of positive reals satisfying $\delta_n \to 0$ and $\delta_n\sqrt{m_n} \to \infty$ as $n \to \infty$. For each $n \in \mathbb{N}$ choose a subspace X_n of H of dimension m_n and set $A = \bigcup_{n=1}^\infty \delta_n B_{X_n}$. Then by propositions 5.3 and 4.2, for each $n \in \mathbb{N}$ one has

$$\mathbf{E}_0(A) \geq \mathbf{E}_0(\delta_n B_{X_n}) = \delta_n \mathbf{E}_0(B_{X_n}) = \delta_n\sqrt{m_n},$$

and so, $\mathbf{E}_0(A) = \infty$ by arbitrariness on n and the condition $\delta_n\sqrt{m_n} \to \infty$ as $n \to \infty$. The condition $\delta_n \to 0$ as $n \to \infty$ guarantees the compactness of A. □

5.2. **Sufficient conditions on a set to have finite unconditional entropy.** Following [6], for a precompact set A of H and a number $\varepsilon > 0$ we consider the covering number of A by closed balls of radius ε, defined as follows

$$N(A,\varepsilon) = \min\Big\{N \in \mathbb{N} : \ (\exists\{x_1,\dots,x_N\} \subseteq H)(A \subseteq \bigcup_{k=1}^N B(x_k,\varepsilon)\Big\}.$$

Theorem 5.5. *Let $K \subseteq H$ be a compact set. If there is $\delta > 0$ such that*

$$\int_0^\delta \big(\ln N(K,\varepsilon)\big)^{1/2}\, d\varepsilon < \infty \tag{5.1}$$

then $\mathbf{E}_0(K) < \infty$.

Proof. Consider a white noise ξ in H is the sense of [16], that is, a family of joint Gaussian random variables $\big\{\langle u,\xi\rangle : u \in H\big\}$ such that for all $u, v \in H$ and $\alpha, \beta \in \mathbb{R}$ we have

(1) $E\langle u,\xi\rangle = 0$;
(2) $E\langle u,\xi\rangle^2 = \|u\|^2$;
(3) $\langle \alpha u + \beta v,\xi\rangle = \alpha\langle u,\xi\rangle + \beta\langle v,\xi\rangle$.

Given $\omega \in K$, we set $\eta(\omega) = \langle \omega,\xi\rangle$. Then η is a Gaussian random process defined on the compact set K such that for all $\omega_1, \omega_2 \in K$ one has

$$E\big(\eta(\omega_1) - \eta(\omega_2)\big)^2 = \|\omega_1 - \omega_2\|^2. \tag{5.2}$$

Then by [14, Theorem 6.1.2] and (5.1), there exists a continuous modification of the process η. So, η can be considered as a Gaussian random element in $C(K)$ of mean zero.

By the Kwapień-Szymanski theorem [19, Theorem 5.7], η admits a representation of the form

$$\eta(\omega) = \sum_{k=1}^{\infty} \eta_k f_k(\omega),$$

where $(\eta_k)_{k=1}^{\infty}$ are standard Gaussian random variables which are limits of linear combinations of values of η, and f_k, $k = 1, 2, \ldots$ are elements of $C(K)$ such that

$$\sum_{k=1}^{\infty} \|f_k\|^2 < \infty \tag{5.3}$$

(here $\|f\| = \sup_{\omega \in K} |f(\omega)|$ for $f \in C(K)$). The random variables $(\eta_k)_{k=1}^{\infty}$ correspond to some sequence $(e_k)_{k=1}^{\infty}$ in H such that $\eta_k = \langle e_k, \xi \rangle$, $k = 1, 2, \ldots$, which is an orthonormal sequence by (1) and (2). Then for each $\omega \in K$ we have

$$\eta(\omega) = \langle \omega, \xi \rangle = \sum_{k=1}^{\infty} \langle e_k, \xi \rangle f_k(\omega) \overset{\text{by (3) and (5.3)}}{=} \Big\langle \sum_{k=1}^{\infty} f_k(\omega)\, e_k, \xi \Big\rangle = \eta\Big(\sum_{k=1}^{\infty} f_k(\omega)\, e_k \Big).$$

Then by (5.2), for each $\omega \in K$ we obtain $\omega = \sum_{k=1}^{\infty} f_k(\omega)\, e_k$.

Assume $\mathcal{B} = (e_n)_{n=1}^{\infty}$ is a complete system in H, and hence, an orthonormal basis. For each $n \in \mathbb{N}$ we set $\varepsilon_n = \sup_{\omega \in K} |(\omega, e_n)|$, $\mathcal{E} = (\varepsilon_n)_{n=1}^{\infty}$ and observe that

$$\gamma_{\mathcal{B},n}(K) = \varepsilon_n = \sup_{\omega \in K} |f_n(\omega)| \le \|f_n\|$$

for all n, hence, $K \subseteq K_{\mathcal{B},\mathcal{E}}$, and $K_{\mathcal{B},\mathcal{E}}$ is a compact brick by (5.3).

It remains to say that, if $[e_n]_{n=1}^{\infty} \neq H$, then instead of $\mathcal{B}$ we consider an orthonormal basis $\mathcal{B}' = (e'_n)_{n=1}^{\infty}$ containing $\mathcal{B}$ as a subsequence, and a sequence $\mathcal{E}' = (\varepsilon'_n)_{n=1}^{\infty}$ instead of $\mathcal{E}$, containing $\mathcal{E}$ as a subsequence at the same indices as $\mathcal{B}'$ contains $\mathcal{B}$, and $\varepsilon'_n = 0$ at the rest of the indices n. ☐

5.3. **A set in a Hilbert space with distinct geometric and unconditional entropies.** Remind that $\mathbf{E}(A) \le \mathbf{E}_0(A)$ for any subset $A \subseteq H$ (see item (3) of Proposition 4.2).

Does there exist a subset $A \subseteq H$ with $\mathbf{E}(A) < \mathbf{E}_0(A)$? In other words, does there exist a compact brick K in H and a number $\delta > 0$ such that for every compact rectangular brick $K_{\mathcal{B},\mathcal{E}}$ in H containing K one has $r(K_{\mathcal{B},\mathcal{E}}) \ge r(K) + \delta$? A simple geometrical argument shows that this is impossible if K is a 2-dimensional brick, that is, if $\varepsilon_n = 0$ for $n \ge 3$. Indeed, any parallelogram in a plane centered at zero is obviously contained in a rectangle with the same radius. However, there exists a 3-dimensional brick which is not contained in a rectangular brick of a close radius.

To make the notation simpler, we introduce some new and very natural terminology. Given a finite linearly independent system $\mathcal{U} = (u_k)_{k=1}^{n}$ in a Hilbert space H, we say that the set

$$P_{\mathcal{U}} = \Big\{ \sum_{k=1}^{n} a_k u_k : (\forall k)\, |a_k| \le 1 \Big\}$$

is an (n-dimensional) *oblique parallelepiped.* The radius $r(P_{\mathcal{U}})$ of an oblique parallelepiped $P_{\mathcal{U}}$ is defined by $r(P_{\mathcal{U}}) = \max_{x \in P_{\mathcal{U}}} \|x\|$. To each oblique parallelepiped $P_{\mathcal{U}}$ in an infinite dimensional separable Hilbert space H with $\mathcal{U} = (u_k)_{k=1}^{n}$ there naturally corresponds a brick $K_{\mathcal{B},\mathcal{E}}$, where $\mathcal{B} = (e_k)_{k=1}^{\infty}$, $e_k = u_k / \|u_k\|$ for $k = 1, \ldots, n$ and $(e_k)_{k>n}$ is any extension of $(e_k)_{k=1}^{n}$ to a Schauder basis in H and $\mathcal{E} = (\varepsilon_k)_{k=1}^{\infty}$ are the numbers defined by $\varepsilon_k = \|u_k\|$ for $k = 1, \ldots, n$ and $\varepsilon_k = 0$ for $k > n$. It is immediate that $P_{\mathcal{U}} = K_{\mathcal{B},\mathcal{E}}$.

Theorem 5.6. *For every $\varepsilon > 0$ there exists a set K in ℓ_2 such that $\mathbf{E}(K) = 1$ and $\mathbf{E}_0(K) > \sqrt{2} - \varepsilon$.*

Proof. Let $n \geq 3$. For convenience of the calculations, we define an n-dimensional oblique parallelepiped $P_{\mathcal{U}_n}$ with $\mathcal{U}_n = (u_k)_{k=1}^n$ in ℓ_2 by a system which is linearly dependent (more precisely, of rank $n-1$). Of course, this is not acceptable for our purpose, however in this case all the parameters are easily found. Then one should somewhat perturb the system by adding to, say, u_n a vector to make the system linearly independent, however of small norm, not to spoil the needed estimates too much.

So, we set

$$\begin{aligned} u_1 &= (n-1, \underbrace{-1, -1, \dots, -1}_{n-1}, 0, 0, \dots); \\ u_2 &= (-1, n-1, \underbrace{-1, \dots, -1}_{n-2}, 0, 0, \dots); \\ &\dots; \\ u_n &= (\underbrace{-1, \dots, -1, -1}_{n-1}, n-1, 0, 0, \dots). \end{aligned}$$

Direct simple calculations show that $\|u_i\|^2 = n^2 - n$ for $i = 1, \dots, n$. By symmetry,

$$r(P_{\mathcal{U}_n}) = \max_{\theta_i = \pm 1} \Big\| \sum_{i=1}^n \theta_i u_i \Big\| = \max_{1 \leq k \leq n} \|u_1 + \dots + u_k - u_{k+1} - \dots - u_n\|.$$

One can easily evaluate

$$\|u_1 + \dots + u_k - u_{k+1} - \dots - u_n\|^2 = 4kn(n-k) \leq n^3.$$

Hence, $\mathbf{E}(P_{\mathcal{U}_n}) = r(P_{\mathcal{U}_n}) \leq n^{3/2}$.

Let $K_{\mathcal{B},\mathcal{E}}$ be any rectangular brick in ℓ_2 with an orthonormal basis $\mathcal{B} = (e_k)_{k=1}^\infty$ and a height $\mathcal{E} = (\varepsilon_k)_{k=1}^\infty$ such that $P_{\mathcal{U}_n} \subseteq K_{\mathcal{B},\mathcal{E}}$. Observe that if $i \neq j$ then

$$-n = (u_i, u_j) = \Big(\sum_{k=1}^\infty (u_i, e_k)\, e_k, \sum_{m=1}^\infty (u_j, e_m)\, e_m \Big) = \sum_{k=1}^\infty (u_i, e_k)(u_j, e_k).$$

Hence,

$$\sum_{k=1}^\infty |(u_i, e_k)||(u_j, e_k)| \geq n \ \text{ for all } \ i, j \in \{1, \dots, n\} \text{ with } i \neq j.$$

Fix any $k \in \mathbb{N}$ and choose $\theta_i = \pm 1$ for $i = 1, \dots, n$ so that $(\theta_i u_i, e_k) = |(u_i, e_k)|$. Since $u \stackrel{\text{def}}{=} \sum_{i=1}^n \theta_i u_i \in P_{\mathcal{U}_n} \subseteq K_{\mathcal{B},\mathcal{E}}$, we obtain that

$$\sum_{i=1}^n |(u_i, e_k)| = \sum_{i=1}^n (\theta_i u_i, e_k) = (u, e_k) \leq \varepsilon_k.$$

Then, taking into account everything above, we get

$$\begin{aligned} 2n^3 - 2n^2 &= n \cdot (n^2 - n) + (n^2 - n) \cdot n \\ &\leq \sum_{i=1}^n \|u_i\|^2 + \sum_{\substack{i,j=1 \\ i \neq j}}^n \sum_{k=1}^\infty |(u_i, e_k)||(u_j, e_k)| \\ &= \sum_{k=1}^\infty \Big(\sum_{i=1}^n |(u_i, e_k)| \Big)^2 \leq \sum_{k=1}^\infty \varepsilon_k^2 = r^2(K_{\mathcal{B},\mathcal{E}}). \end{aligned}$$

By the arbitrariness of $K_{\mathcal{B},\mathcal{E}}$, we get $\mathbf{E}_0(P_{\mathcal{U}_n})^2 \geq 2n^3 - 2n^2 > n^3 = \mathbf{E}(P_{\mathcal{U}_n})^2$ as $n > 2$. Moreover,

$$\frac{\mathbf{E}_0(P_{\mathcal{U}_n})}{\mathbf{E}(P_{\mathcal{U}_n})} \geq \sqrt{\frac{2n^3 - 2n^2}{n^3}} = \sqrt{2 - \frac{2}{n}}.$$

Fixing any $\varepsilon > 0$ and setting $K = \mathbf{E}(P_{\mathcal{U}_n})^{-1}P_{\mathcal{U}_n}$, we obtain the desired inequality for large enough n. □

6. Possible geometric entropy for spaces without a Schauder basis

Investigation of geometric entropy defined above becomes impossible if X has no basis. However, one can use finite Auerbach systems, which behave like orthonormal systems in a Hilbert space. Let X be a finite dimensional normed space of dimension $n \in \mathbb{N}$. A system of vectors $(e_k)_{k=1}^n$ in X is called an *Auerbach system* if $\|e_k\| = 1$ for $k = 1, \dots, n$ and there are functionals $(e_k^*)_{k=1}^n$ in X^* such that $\|e_k^*\| = 1$ for $k = 1, \dots, n$ and $e_i^*(e_j) = \delta_{i,j}$ for $i, j = 1, \dots, n$. Evidently, in this case $x = \sum_{k=1}^n e_k^*(x)\, e_k$ for each $x \in X$. Every finite dimensional normed space contains an Auerbach system [13, Proposition 1.c.3].

More generally, a system of elements $(e_k)_{k=1}^n$ in a (not necessary finite dimensional) Banach space is called an *Auerbach system* if $(e_k)_{k=1}^n$ is an Auerbach system in its linear span $[e_k]_{k=1}^n$. One can show that, in a Hilbert space a system $(e_k)_{k=1}^n$ is an Auerbach system if and only if $(e_k)_{k=1}^n$ in an orthonormal system.

Let X be a Banach space with an Auerbach system $\mathcal{B} = (e_k)_{k=1}^n$, $X_0 = [e_k]_{k=1}^n$ and biorthogonal functionals $(e_k^*)_{k=1}^n$, $e_k^* \in X_0^*$ and let $\mathcal{E} = (\varepsilon_k)_{k=1}^n$ be a collection of nonnegative numbers.

Definition 6.1. An *Auerbach brick* (more precisely, an n-dimensional Auerbach brick) is defined to be the following set

$$K_{\mathcal{B},\mathcal{E}} = \big\{x \in X_0 : \ |e_k^*(x)| \le \varepsilon_k \text{ for } k = 1, \dots, n\big\}.$$

Since we do not assume $\varepsilon_k > 0$ for all $k = 1, \dots, n$ in the above definition, we have that any n-dimensional Auerbach brick is an m-dimensional Auerbach brick with $m \ge n$.

Like bricks, Auerbach bricks are closed absolutely convex sets. Moreover, all are compact, and so, all the radii of an Auerbach brick mean the same.

Definition 6.2. Let X be a Banach space. The *Auerbach entropy* $\mathbf{E}_a(A)$ of a set $A \subseteq X$ is defined to be the infimum of the set of reals $\alpha \ge 0$ possessing the following property: for every $\varepsilon > 0$ there exists an Auerbach brick $K_{\mathcal{B},\mathcal{E}}$ of radius $r(K_{\mathcal{B},\mathcal{E}}) \le \alpha$ such that $A \subseteq K_{\mathcal{B},\mathcal{E}} + \varepsilon B_X$. If such a number α does not exist then we set $\mathbf{E}_a(A) = \infty$.

For the Auerbach entropy analogues of proposition 4.2, 4.3 and 4.7 are true. Moreover, the following inequality holds in the general case.

Proposition 6.3. *Let X be a Banach space and $A \subseteq X$. Then $\mathbf{E}_a(A) \le \mathbf{E}_0(A)$.*

Proof. If $\mathbf{E}_0(A) = \infty$ then there is nothing to prove. Let $\mathbf{E}_0(A) < \infty$. Fix any $\varepsilon > 0$ and choose a 1-unconditional basis $\mathcal{B} = (e_n)_{n=1}^\infty$ of X and a sequence of nonnegative numbers $\mathcal{E} = (\varepsilon_n)_{n=1}^\infty$ such that $r(K_{\mathcal{B},\mathcal{E}}) < \mathbf{E}_0(A) + \varepsilon$. Then choose $N \in \mathbb{N}$ so that $\big\|\sum_{n>N} \varepsilon_n e_n\big\| < \varepsilon$. Observe that $\mathcal{B}' = (e_n)_{n=1}^N$ is an Auerbach system. Indeed, set $X_0 = [e_n]_{n=1}^N$ and $f_n = e_n^*|_{X_0}$ for $n = 1, \dots, N$. Then $f_i(e_j) = e_i^*(e_j) = \delta_{i,j}$ for $i, j = 1, \dots, N$ and

$$\|f_n\| \le \|e_n^*\| = \sup_{x \in B_X} \|e_n^*(x)\, e_n\| \overset{\text{by Lemma 1.3}}{\le} \sup_{x \in B_X} \Big\|\sum_{k=1}^\infty e_k^*(x)\, e_k\Big\| = \sup_{x \in B_X} \|x\| = 1.$$

Since obviously $\|f_n\| \ge 1$, we obtain $\|f_n\| = 1$ for $n = 1, \dots, N$, completing the proof that $\mathcal{B}' = (e_n)_{n=1}^N$ is an Auerbach system. Since any $x \in A$ has a representation $x = y + z$, where $y = \sum_{n=1}^N e_n^*(x)\, e_n = \sum_{n=1}^N f_n(y)\, e_n$ and $z = \sum_{n>N} e_n^*(x)\, e_n$ with $|f_n(y)| = |e_n^*(x)| \le \varepsilon_n$ for $n = 1, \dots, N$ and $\|z\| \le \varepsilon$, we obtain that $A \subseteq K_{\mathcal{B}',\mathcal{E}'} + \varepsilon B_X$, where $\mathcal{E}' = (\varepsilon_n)_{n=1}^N$. It is immediate that $K_{\mathcal{B}',\mathcal{E}'} \subseteq K_{\mathcal{B},\mathcal{E}}$. Thus, $\mathbf{E}_a(A) \le \mathbf{E}_0(A) + \varepsilon$. By arbitrariness of $\varepsilon > 0$, we obtain that $\mathbf{E}_a(A) \le \mathbf{E}_0(A)$. □

7. Remarks and open problems

As mentioned above, not much is known about compact sets having infinite geometric entropy.

Problem 7.1. *Does there exist an infinite dimensional separable Banach space X which is not isomorphic to a subspace of c_0 such that all precompact sets in X have finite geometric entropy (unconditional entropy)?*

By Theorem 5.6, for every $\varepsilon > 0$ there exists a set K in ℓ_2 such that $\mathbf{E}(K) = 1$ and $\mathbf{E}_0(K) > \sqrt{2} - \varepsilon$. Is the latter estimate sharp?

Problem 7.2. *Evaluate the number*

$$\Upsilon = \sup_{\substack{K \subseteq \ell_2 \\ \mathbf{E}(K)=1}} \mathbf{E}_0(K) \in [\sqrt{2}, +\infty].$$

In particular, is Υ finite?

Problem 7.3. *Evaluate the geometric entropy of the closed unit ball of an n-dimensional subspace of ℓ_2.*

Problem 7.4. *Let $1 \le p < \infty$. Evaluate the geometric entropy (unconditional entropy) of the closed unit ball of the n-dimensional subspace spanned by $e_1, \dots, e_n$ in ℓ_p, where $(e_n)_{n=1}^{\infty}$ is the standard basis of ℓ_p.*

We also do not know if the basis constant can influence on the geometric entropy.

Problem 7.5. *Let X be a Banach space with a monotone*[4] *Schauder basis. Define the monotone entropy by*

$$\mathbf{E}_m(A) = \inf_{A \subseteq K_{\mathcal{B},\mathcal{E}}} r(K_{\mathcal{B},\mathcal{E}}),$$

where the infimum is taken over all compact monotone bricks $K_{\mathcal{B},\mathcal{E}}$ in X. Is $\mathbf{E}_m(A) = \mathbf{E}(A)$ for all subsets $A \subseteq X$?

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[4] that is, the basis constant equals one

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