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ERDÖS-RÉNYI LAW FOR THE LOCAL TIME OF THE HYBRID PROCESS

Our aim in this paper is to study the Erdös-Rényi law for the local time of the hybrid of empirical and partial sums process. The corresponding local time can be see as a modified version of the local time of the symmetric random walk by introducing a time t and a sequence of independent with the same distribution random variables X_i's, independent of the random walk.

1. Introduction

The Erdös-Rényi law states the convergence properties of the maximal average gain of a player over a short period in a fair game, just as the ordinary law of large numbers does for the average gain over a long period. In the following theorem we give the general form of this result (cf. [20]).

Theorem 1.1. *Let $\{X_i\}_{i\geq 1}$ be a sequence of independent, identically distributed, non-degenerate random variables on a probability space (Ω, A, P). Suppose that the moment generating function of X_1 given by $f(s) = E[e^{sX_1}]$ is finite for $s \in (0, s_0)$. Furthermore suppose that $E[X_1] = 0$. Let α be any number such that that the function $f(s)e^{-s\alpha}$ takes on its minimum at some point in the open interval $(0, s_0)$, and set*

$$\min_{s\in(0,s_0)} f(s)e^{-s\alpha} = e^{-1/c}.$$

Then $c > 0$, and setting $S(0) = 0$, $S(n) = X_1 + \cdots + X_n$, it follows that

$$\lim_{N\to\infty} \max_{0\leq n\leq N-[c\log N]} \frac{S(n + [c \log N]) - S(n)}{[c \log N]} = \alpha, a.s.$$

The list of references related to the Erdös-Rényi laws is very important. Then we give some references corresponding to our approach. We refer to [9] where almost sure limit Theorems was proved for maxima of functions of moving blocks of size $c \log n$ of independent random variables (rv's) and for maxima of functions of the empirical probability measures of these blocks, several examples was presented as corollaries for frequently used test statistics and point estimators, [22] where it was shown that the results of [9] can be extended to cover also situations of stochastic processes where stationarity and independence of increments are not generally available, but for randomly chosen subsequences of the process. In [14] (also see references therein) the Erdös-Rényi law for renewal processes was constructed from nonidentically distributed random variables, [20] where the Erdös-Rényi type law for cumulative processes in renewal theory was studied, [21], [23], [11] where the gap between the Erdös-Rényi and invariance principle for the partial sums process was studied, [5] where it was studied the exact convergence rates in Erdös-Rényi type theorems for renewal processes, [13] where functional version of the Erdös-Rényi law concerning increments of partial sum processes over subintervals of critical length $a_T \approx c \log T$ was studied, [15] where the Erdös-Rényi law for the iterated

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Poisson processes was obtained (cf. Lemma 2.4), [17] where the Erdös-Rényi law for the hybrid process was studied, [8] where the Erdös-Rényi law was stated for the local time of a recurrent random walk (cf. Theorem 2) and [7] where the Erdös-Rényi law was stated for the local time of a simple symmetric random walk.

In our case, we consider the process $\tilde{A} = \left\{ \tilde{A}_n(t), t \in \mathbb{R}, 1 \leq n < \infty \right\}$ given by

$$\tilde{A}_n(t) = \frac{\tilde{U}(n,t)}{\sqrt{n}}, \tag{1}$$

where

$$\tilde{U}(.,t) = \left\{ \tilde{U}(n,t) = \sum_{j=1}^{n} \epsilon_j 1_{\{X_j \leq t\}} \right\}_{n \geq 1} \tag{2}$$

where 1_A denotes the set indicator function, and sequences $\{\epsilon_i\}_{i\geq 1}$ and $\{X_i\}_{i\geq 1}$ satisfies the following conditions

- (C1) the random variables $\epsilon_i = S(i) - S(i-1),\ (i \geq 1)$ are i.i.d., with $P(\epsilon_i = 1) = P(\epsilon_i = -1) = 1/2$,
- (C2) the random variables $X_i,\ (i \geq 1)$ are i.i.d. of same cumulative distribution function (cdf) F,
- (C3) the random variables X_i's and ϵ_i's are independent.

The process $\tilde{A}$ is known in the literature as the hybrids of empirical and partial sums processes, see for instance [16] and [3] where an approximation for the local time of the hybrid process by the local time of some Brownian motion with random time was stated.

Let us define the local time of $\tilde{A}_n(t)$ by

$$\xi_t^x(\tilde{A}_n) = \frac{1}{\sqrt{n}} \sum_{s \leq t} 1_{\{\tilde{A}_n(s)=x\}},\ t \in \mathbb{R}, x \in \mathbb{R}. \tag{3}$$

Our aim here is to study the behavior of

$$\lim_{n\to\infty} \max_{0 \leq t \leq n-[c\log n]} \frac{\xi_{t+[c\log n]}^x(\tilde{A}_n) - \xi_t^x(\tilde{A}_n)}{[c\log n]}.$$

Our paper is organized as follows, in the rest of this section we give some useful results of the processes that are in studying. In section 2, we state our main result, jointly with some technical Lemma's giving the proofs.

We will assume without loss of generality that all random variables and processes are defined on the same probability space.

Diebolt in [12] introduced and investigated nonparametric testing procedures for the autoregression function in a class of nonlinear autoregressive processes of order one, by considering $\tilde{\tilde{U}}(n,t) = \sum_{j=1}^n \epsilon_j H(X_j) 1_{\{X_j \leq t\}}$ where the function $H(\cdot)$ has bounded variation on the real line in the place of $\tilde{U}(n,t)$. The asymptotic properties of their procedures can be derived from the limiting behavior of the hybrids of empirical and partial sums processes. The time transformation for the limiting Wiener process given in [12] is given by $\bar{J}_n(t) = \int_{-\infty}^t H^2(s)dF_n(s)$, where

$$F_n(t) = \frac{1}{n} \sum_{1 \leq i \leq n} 1_{\{X_i \leq t\}}, \quad \text{for } -\infty < t < \infty.$$

Later, Horváth in [16], showed that the random time change "$\bar{J}_n(t)$" can be replaced with a non-random time change says $\bar{J}(t) = \int_{-\infty}^t H^2(s)dF(s)$, where $F(\cdot)$ is the common distribution function of $\{X_1, X_2, \ldots\}$, without reducing the rates of the approximations given in [12]. He also investigated the almost sure approximation of the two-parameter process

$\{\overline{A}_n(t) : -\infty < t < \infty,\ 1 \le n < \infty\}$ by a two parameter Wiener process. Horváth et al. in [17] studied the weighted bootstrap processes $\{\beta_n(t) = n^{-1/2}\sum_{1\le i\le n}(\epsilon_i - \bar{\epsilon_n})1_{\{X_i\le t\}} : -\infty < t < \infty,\ 1 \le n < \infty\}$ in order to perform statistical test to detect a possible change in the distribution of independent observations. The complete convergence for the process $\{\tilde{A}_n(t) : -\infty < t < \infty,\ 1 \le n < \infty\}$ was deeply investigated, in a general setting, in [2] and [1].

Remark 1.1. By [16], p.5, we have without loss of generality, there is a sequence of i.i.d. random variables $\{Y_i\}_{i\ge1}$ uniform on $(0,1)$ such that $X_i = Q(Y_i)$, with $Q(y) = \inf\{x :\ F(x) \ge y\}$ i.e. the quantile function of F, then we can consider $U(.,t)(0 \le t \le 1)$ in the place of $\tilde{U}(.,t)(-\infty < t < \infty)$.

Remark 1.2. From [17], p.66, we have, for any $c > 0$

$$\lim_{n\to\infty}\frac{1}{[c\log[n^{1/3}]]}\max_{0\le i\le[n^{1/3}]} U(N(i+[c\log[n^{1/3}]])) - U(N(i)) = \alpha(c), a.s.$$

where $\{N(t), 0 \le t < \infty\}$ denote an homogeneous Poisson process and $\alpha(c)$ is in a one to one correspondence with the moment generation function of $U(N(1))$.

Let $f(x), x \in \mathbb{Z}$ be a real valued function. It is well known that we have the following relation

$$\sum_{i=1}^{n} f(S(i)) = \sum_{x=-\infty}^{\infty} f(x)L_n^x(S),\ n = 1, 2, \ldots$$

where $L_n^x(S)$ denotes the local time of S given by

$$L_n^x(S) = \sum_{i=1}^{n} 1_{\{S(i)=x\}}. \tag{4}$$

Let us mention that one of the best recent references on local times for random walks on lattices of $\mathbb{R}^d$ and for Brownian motions is the book of [6], see also [18] and [19].

We will restrict ourselves to the one-dimensional case i.e. the random walk on $\mathbb{Z}$ and we will denote by $\log t = \log(t \vee e)$, $\log_2 t$ the two-iterated logarithm i.e. $\log_2 t = \log\log t$ and $[x]$ denote the integer part of some real x.

Now, we consider that

$$\xi_t^x(\tilde{A}_n) = \int_0^1 \delta_t^s(F_n)d_s\nu_s^x(V_n)$$

this representation of the local time of the hybrid process was given by the author on [3] and was obtained by using the following arguments : $\tilde{A}_n(t) \stackrel{d}{=} V_n(F_n(t))$, where $F_n(t) = n^{-1}\sum_{i=1}^{n} 1_{\{X_i\le t\}}$ and $V_n(t) = \sum_{i=1}^{[nt]}\epsilon_i/\sqrt{n}$. The local time time at a level x up to t of the empirical distribution function $F_n(t)$ was defined by

$$\delta_t^x(F_n) = \frac{1}{\sqrt{n}}\sum_{s\le t} 1_{\{F_n(s)=x\}}. \tag{5}$$

and the local time for V_n was given by

$$\nu_t^x(V_n) = \sum_{j=1}^{[nt]} 1_{\left\{\sum_{i=1}^{j}\epsilon_i=[\sqrt{n}x]\right\}},\quad x \in \mathbb{R}. \tag{6}$$

It is not difficult to see that $\nu_1^x(V_n)$, can be written as

$$\nu_1^x(V_n) = L_n^{[\sqrt{n}x]}(S) \tag{7}$$

where $L_n^{[\sqrt{n}x]}(S)$ is the local time of the simple symmetric random walk defined by the random variables $\epsilon_i's$ (see (4)).

Notice that for the symmetric random walk, there are well known distribution properties of $L_n^x(S)$, for $x > 0$ and $x = 0$ (see Theorem 9.4 of [19]).

2. Results and proofs

We have to deal with the Erdös-Rényi type increments for the local time of the hybrid process. It is well know that for this kind of increments we can not use invariance principle and the limit denoted by α is in correspondence with $c = c(\alpha)$.

Theorem 2.1. *Under conditions* (C_1), (C_2) *and* (C_3), *we have with probability one*

$$\lim_{n\to\infty} \max_{0\le t\le n-[c\log n]} \frac{|\xi^x_{t+[c\log n]}(\tilde{A}_n) - \xi^x_t(\tilde{A}_n)|}{c\log n} = \alpha$$

where $\alpha(c)$ *is such that* $\rho(\alpha) = e^{-c/2}$ *with*

$$\rho(\alpha) = \frac{(1-\alpha)^{(1-\alpha)}}{(1-2\alpha)^{(1/2-\alpha)}} \ (1/2 < \alpha < 1).$$

Notice that $\alpha(c)$ and $\rho(\alpha)$ corresponds to the case of the Erdös-Rényi law for the local time of the symmetric random walk, see [8].

Proof of Theorem 2.1. The proof of our Theorem 2.1 will be based on some previous remark and the following two technical Lemma's.

Remark that we have

$$\begin{aligned}
\xi^x_{t+[c\log n]}(\tilde{A}_n) - \xi^x_t(\tilde{A}_n) &= \int_0^1 \delta^s_{t+[c\log n]}(F_n) d_s\nu^x_s(V_n) - \int_0^1 \delta^s_t(F_n) d_s\nu^x_s(V_n) \\
&= \int_0^1 \left(\delta^s_{t+[c\log n]}(F_n) - \delta^s_t(F_n)\right) d_s\nu^x_s(V_n)
\end{aligned}$$

$$= \sum_{y=1}^{n} \left(\delta^y_{t+[c\log n]}(M_n) - \delta^y_t(M_n)\right)\left(L_y^{[\sqrt{n}x]}(S) - L_{y-1}^{[\sqrt{n}x]}(S)\right) \tag{8}$$

where $M_n(t) = nF_n(t)$ and $y = [ns]$.

From [17] (cf. proof of Theorem 1.4) and by using Remark 1.1, we have

$$\{nF_n(s),\ 0 \le s \le 1\} \stackrel{d}{=} \{N(s\tau(n+1)),\ 0 \le s \le 1\} \tag{9}$$

where $0 = \tau(0) < \tau(1) < \cdots < \tau(n+1)$ are the renewal times of the homogeneous Poisson process with intensity parameter 1 denoted by $\{N(t), 0 \le t < \infty\}$, which is independent of $\{\epsilon_i, 1 \le i < \infty\}$.

Now, roughly speaking replace s in (9) by $\frac{i}{\tau(n+1)} + \frac{[c\log n]}{\tau(n+1)}$, then we have

$$\{N\left(\left(\frac{i}{\tau(n+1)} + \frac{[c\log n]}{\tau(n+1)}\right)\tau(n+1)\right), 0 \le i \le n - [c\log n]\}$$
$$\stackrel{d}{=} \{N\left(i + [c\log n]\right), 0 \le i \le n - [c\log n]\}\ .$$

From the just mentioned arguments, in the place of (8) we consider

$$\sum_{y=1}^{n} \left(\delta^y_{i+c\log n}(N(\cdot)) - \delta^y_i(N(\cdot))\right)\left(L_y^{[\sqrt{n}x]}(S) - L_{y-1}^{[\sqrt{n}x]}(S)\right),\ 0 \le i \le n - [c\log n]$$

$$= \sum_{y=1}^{n} \sum_{j=i}^{i+[c\log n]} 1_{\{N(j)=y\}} \left(L_y^{[\sqrt{n}x]}(S) - L_{y-1}^{[\sqrt{n}x]}(S)\right),\ 1 \le i \le n - [c\log n]. \tag{10}$$

Notice that $y-1$ could be replaced by $y-2$ (for $y \ge 2$) in the last relation, because y and $y-2$ corresponds to return times at the level $[\sqrt{n}x]$ for the symmetric random walk (i.e. y and $y-2$ are both even integers or odd integers). Now, remark that (10), can be written as

$$\sum_{j=i}^{i+[c\log n]}\sum_{y=1}^{n} 1_{\{N(j)=y\}}1_{\{S(y)=[\sqrt{n}x]\}},\ 1\le i\le n-[c\log n]$$

$$(11)\qquad = \sum_{j=i}^{i+[c\log n]} 1_{\{S(N(j))=[\sqrt{n}x]\}},\ 1\le i\le n-[c\log n].$$

From (11), we have

$$(12)\ \xi^x_{t+[c\log n]}(\tilde{A}_n)-\xi^x_t(\tilde{A}_n) = \sum_{j=[t]}^{[t]+[c\log n]} 1_{\{S(N(j))=[\sqrt{n}x]\}},\ 1\le t\le n-[c\log n].$$

Now, we state two technical Lemma's, their proof will be based on (12).

Lemma 2.1. *Under the same conditions of Theorem 2.1, we have,*

$$\limsup_{n\to\infty}\max_{0\le t\le n-[c\log n]}\frac{\xi^x_{t+[c\log n]}(\tilde{A}_n)-\xi^x_t(\tilde{A}_n)}{[c\log n]}\le \alpha, a.s.$$

Proof. Let us define $\mathcal{A}_n=\left\{\xi^x_{t+[c\log n]}(\tilde{A}_n)-\xi^x_t(\tilde{A}_n)\ge\alpha(1+3\epsilon)[c\log n]\right\}$ for all $\epsilon>0$, then

$$\mathbb{P}\left(\sup_{0\le t\le n-[c\log n]}\left(\xi^x_{t+[c\log n]}(\tilde{A}_n)-\xi^x_t(\tilde{A}_n)\right)\ge\alpha(1+3\epsilon)[c\log n]\right)\le\sum_{n=0}^{\infty}\mathbb{P}(\mathcal{A}_n).$$

Our first aim is to evaluate $\mathbb{P}(\mathcal{A}_n)$. Let us define $\alpha^+_\epsilon=\alpha(1+3\epsilon)$ and $\tau=[t]$. Without loss of generality we can consider that $N(\tau)$ and $N(\tau+[c\log n])$ are both even integers and from (12), we consider

$$\begin{aligned}\mathbb{P}(\mathcal{A}_n) &= \mathbb{P}\left(L^x_{N(\tau+[c\log n])}(S)-L^x_{N(\tau)}(S)\ge\alpha^+_\epsilon[c\log n]\right)\\
&= \mathbb{P}\left(L^x_{N(\tau)+(N(\tau+[c\log n])-N(\tau))}(S)-L^x_{N(\tau)}(S)\ge\alpha^+_\epsilon[c\log n]\right)\\
&= \mathbb{P}\Big(L^x_{N(\tau)+(N(\tau+[c\log n])-N(\tau))}(S)-L^x_{N(\tau)}(S)\\
&\ge\alpha^+_\epsilon[c\log n]\cap(N(\tau+[c\log n])-N(\tau))>2[c\log n]\Big)\\
&+\mathbb{P}\Big(L^x_{N(\tau)+(N(\tau+[c\log n])-N(\tau))}(S)-L^x_{N(\tau)}(S)\\
&\ge\alpha^+_\epsilon[c\log n]\cap(N(\tau+[c\log n])-N(\tau))\le 2[c\log n]\Big)\\
&\le\mathbb{P}(N(\tau+[c\log n])-N(\tau)\ge 2[c\log n])\\
&\quad+\mathbb{P}\left(L^x_{N(\tau)+2[c\log n]}(S)-L^x_{N(\tau)}(S)\ge\alpha^+_\epsilon[c\log n]\right).\end{aligned}$$

From the independence between S and $N(\cdot)$ and stationarity of the increments of the Poisson process, we have

$$\mathbb{P}(\mathcal{A}_n)\le\mathbb{P}(N([c\log n])>2[c\log n])$$
$$+\sum_{l\ge0}\mathbb{P}\left(L^x_{2l+2[c\log n]}(S)-L^x_{2l}(S)\ge\alpha^+_\epsilon[c\log n]\right)\right)\mathbb{P}(N(\tau)=2l),$$

given

$$(13)\qquad \mathbb{P}(\mathcal{A}_n)\le\mathbb{P}(N(c\log n)\ge 2c\log n)$$

$$+\sum_{l\geq 0}\mathbb{P}\left(L^x_{2l+2[c\log n]}(S)-L^x_{2l}(S)\geq \alpha^+_\epsilon[c\log n]\right)e^{-\tau}\frac{(\tau)^{2l}}{(2l)!}.$$

Recall that

$$\mathbb{P}\left(N(\lambda)-\lambda\geq a\right)\leq e^{a-(\lambda+a)\ln(1+\frac{a}{\lambda})}.$$

Then for the first term in the right hand side of (13), we have

$$\text{(14)}\qquad \mathbb{P}\left(N(c\log n)\geq 2c\log n\right)\leq \mathbb{P}\left(N(c\log n)-c\log n\geq c\log n\right)$$
$$\leq e^{c\log n((1-2\log(2))}\ (c>1/\log 2).$$

Now, from Lemma 3.1 of [8] for $l>0$, we have

$$\mathbb{P}\left(L^x_{2l+2[c\log n]}(S)-L^x_{2l}(S)\geq \alpha^+_\epsilon[c\log n]\right)$$
$$\leq C\left(\frac{[c\log n]}{l}\right)^{1/2}\mathbb{P}\left(L^x_{2[c\log n]}(S)\geq \alpha^+_\epsilon[c\log n]\right)$$

with C some constant depending only on the distribution of ϵ_i's and from Lemma 3.2 of [8] we have for $l\geq 0$ and $l^*=\max(1,l)$

$$\mathbb{P}\left(L^x_{2l+2[c\log n]}(S)-L^x_{l}(S)\geq \alpha^+_\epsilon[c\log n]\right)\leq C\left(\frac{2[c\log n]}{l^*}\right)^{1/2}\left(\rho(\alpha^+_\epsilon(1-\epsilon))\right)^{\frac{[c\log n]}{2}}$$

and from in (3.24) of [8], we have finally that $\rho(\alpha^+_\epsilon(1-\epsilon))\leq\rho(\alpha(1+\epsilon))$. Given finally in our case

$$\mathbb{P}\left(L^x_{l+[c\log n]}(S)-L^x_{l}(S)\geq \alpha^+_\epsilon[c\log n]\right)\leq C\left([c\log n]\right)^{1/2}\left(\rho(\alpha(1+\epsilon))\right)^{\frac{[c\log n]}{2}}$$

Tacking in account (13) and (14), we have

$$\mathbb{P}\left(\mathcal{A}_n\right)\leq e^{-c\log n(2\log(2)-1)}+C\left([c\log n]\right)^{1/2}\left(\rho(\alpha(1+\epsilon))\right)^{\frac{[c\log n]}{2}}.$$

Recall that in the case of the Erdös-Rényi law for the local time of the symmetric random walk, we have

$$\rho(\alpha)=\frac{(1-\alpha)^{(1-\alpha)}}{(1-2\alpha)^{(1/2-\alpha)}}\ (1/2<\alpha<1).$$

with $\alpha(c)$ such that $\rho(\alpha)=e^{-1/2c}$ and for some $\delta>0$, we have

$$\rho(\alpha(1+\epsilon))\leq e^{-(1+\delta)/(2c)}$$

see (3.24) of [8]. Then,

$$\mathbb{P}\left(\mathcal{A}_n\right)=\mathbb{P}\left(\xi^x_{t+[c\log n]}(\tilde{A}_n)-\xi^x_t(\tilde{A}_n)\geq \alpha^+_\epsilon[c\log n]\right)$$
$$\leq n^{-(2\log(2)-1)c}+(\log n)^{1/2}n^{-(1+\delta)/2}.$$

Let k be an integer such that $k(2\log(2)-1)/\log(2)>1$, then

$$\sum_{n=1}^{\infty}\mathbb{P}\left(\mathcal{A}_{n^k}\right)<\infty$$

Then, we can conclude by the Borel-Cantelli Lemma. □

Lemma 2.2. *Under the same conditions of Theorem 2.1, we have*

$$\liminf_{n\to\infty}\max_{0\leq t\leq n-[c\log n]}\frac{\xi^x_{t+c\log n}(\tilde{A}_n)-\xi^x_t(\tilde{A}_n)}{[c\log n]}\geq\alpha, a.s.$$

Proof. Define for all $\epsilon > 0$

$$\mathcal{B}_n = \{ \sup_{0\le t\le n-[c\log n]} \{\xi^x_{t+[c\log n]}(\tilde{A}_n) - \xi^x_t(\tilde{A}_n)\} \le \alpha(1-\epsilon)[c\log n]\}.$$

Without loss of generality, we can consider that $N(t)$ and $N(t+[c\log n])$ are both even integers. Let us define $\alpha^-_\epsilon = \alpha(1-\epsilon)$. As in the proof of Lemma 2.1 , we have

$$\mathbb{P}(\mathcal{B}_n) = \mathbb{P}\left(\sup_{0\le t\le n-[c\log n]} \{L^x_{N(t+[c\log n])}(S) - L^x_{N(t)}(S)\} \le \alpha^-_\epsilon [c\log n]\right)$$

then we must give an upper bound for the last probability. In the same way as in [21] (see also [4]), we have that

$$\sup_{0\le t\le n-[c\log n]} \{N(t+[c\log n]) - N(t)\} \ge \max_{i=0,\cdots,n-[c\log n]-1} \{N(i+[c\log n]) - N(i)\}$$

given

$$\mathbb{P}(\mathcal{B}_n) \le \mathbb{P}\left(\max_{i=0,\cdots,[n/(c\log n)]-1} \{L^x_{N(i[c\log n])}(S) - L^x_{N((i-1)[c\log n])}(S)\} \le \alpha^-_\epsilon [c\log n]\right)$$

$$= \prod_{i=1}^{[n/(c\log n)]-1} \mathbb{P}\left(L^x_{N(i[c\log n])}(S) - L^x_{N((i-1)[c\log n])}(S) \le \alpha^-_\epsilon [c\log n]\right)$$

$$= \prod_{i=1}^{[n/(c\log n)]-1} \mathbb{P}\left(L^x_{N([c\log n])}(S) \le \alpha^-_\epsilon [c\log n]\right), \tag{15}$$

where we have used that

$$L^x_{N(i[c\log n])}(S) - L^x_{N((i-1)[c\log n])}(S) \overset{d}{=} L^x_{N([c\log n])}(S).$$

Now,

$$\mathbb{P}\left(L^x_{N([c\log n])}(S) \le \alpha^-_\epsilon [c\log n]\right)$$

$$\le \mathbb{P}\left(L^x_{N([c\log n])}(S) \le \alpha^-_\epsilon [c\log n], N([c\log n]) > [c\log n]\right) + \mathbb{P}\left(N(\delta[c\log n]) \le [c\log n]\right)$$

$$\le \mathbb{P}\left(L^x_{[c\log n]}(S) \le \alpha^-_\epsilon [c\log n]\right) + \mathbb{P}\left(N(\delta[c\log n]) \le [c\log n]\right)$$

for some $0 < \delta < 1$.
The last inequality jointly with (15) gives

$$\mathbb{P}(\mathcal{B}_n) \le \prod_{i=1}^{[n/(c\log n)]-1} \mathbb{P}\left(L^x_{N([c\log n])}(S) \le \alpha^-_\epsilon [c\log n]\right)$$

$$\le \mathbb{P}\left(L^x_{[c\log n]}(S) \le \alpha^-_\epsilon [c\log n], N([c\log n]) > [c\log n]\right)^{[n/(c\log n)]-1} \tag{16}$$

$$+\mathbb{P}\left(N(\delta[c\log n]) \le [c\log n]\right)^{[n/(c\log n)]-1}.$$

For to evaluate the second term in the right hand-side of the precedent inequality (i.e. (16)), let us recall that for the Poisson process

$$\{N(t) > n\} = \{T_n \le t\}$$

where T_n is the sum of n independent random variables with common law exponential with parameter 1, then

$$\mathbb{P}\left(N(\delta[c\log n]) \le [c\log n]\right) = \mathbb{P}\left(T_{[c\log n]} > \delta[c\log n]\right)$$

$$= \mathbb{P}\left(\frac{T_{[c\log n]}}{[c\log n]} > \delta\right) = O(e^{-[c\log n]I(\delta)}) \tag{17}$$

where $I(x) = x - 1 - \log x$ with $I(x) \uparrow \infty$, as $x \downarrow 0$.

The first term in the right hand-side of (16) can be evaluated by

$$\begin{aligned}\mathbb{P}\left(L^x_{N([c\log n])}(S) \leq \alpha^-_\epsilon [c\log n], N([c\log n]) > [c\log n]\right)^{[n/(c\log n)]-1} \\ \leq \mathbb{P}\left(L^x_{[c\log n]}(S) \leq \alpha^-_\epsilon [c\log n]\right)^{[n/(c\log n)]-1} \\ = 1 - \mathbb{P}\left(L^x_{[c\log n]}(S) > \alpha^-_\epsilon [c\log n]\right)^{[n/(c\log n)]-1}.\end{aligned}$$

The last term can be evaluated by recalling that for $0 < \eta < 1/2$, we have

$$\mathbb{P}\left(L^0_{2n}(S) > [2\eta n]\right) \geq c_3(\eta)\frac{1}{\sqrt{n}}\rho^n(\eta) \tag{18}$$

and

$$\rho(\alpha(c) - \epsilon_1) \geq e^{-\frac{1-\delta}{c}}, \tag{19}$$

obtained in the same way as in [7].

From (18) and (19) replacing $2n$ by $[c\log n]$ and in the same way as in [7], we have

$$1 - \mathbb{P}\left(L^x_{[c\log n]}(S) > \alpha^-_\epsilon [c\log n]\right)^{[n/(c\log n)]-1} \leq e^{\{-C_1 cn^{\frac{\delta-\epsilon}{2}}(c\log n)^{-1}\}}. \tag{20}$$

By choosing $\epsilon = \delta/2$ in (19) giving (20) and tacking in account (17) jointly with (16), we have

$$\mathbb{P}(\mathcal{B}_n) < e^{\{-c_4 n^{\frac{\delta}{4}}(c\log n)^{-1}\}}$$

then

$$\sum_{n=1}^{\infty} \mathbb{P}(\mathcal{B}_n) < \infty$$

which means that with probability 1, only finitely many of the events $\mathcal{B}_n$ can occur. Then by using the Borel-Cantelli lemma we can conclude. □

The proof of Theorem 2.1 is obtained from Lemma's 2.1 and 2.2. □

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