Theory of Stochastic Processes
Vol. 19 (35), no. 1, 2014, pp. 91–99

V. I. SENIN

SOJOURN MEASURES OF RANDOM WALKS ON DETERMINISTIC SEQUENCES

We prove that for some class of random walks $\{Z(n),\, n \geq 0\}$, the random sequence $x_{Z(n)}$ almost surely inherits the property of a deterministic sequence x_n to be uniformly distributed.

1. Introduction

The main object of this article is a random walk on indexes of a deterministic sequence. More specifically, let X be a separable metric space, $\mathcal{F}$ be a Borel σ-algebra on this space, and $\{x_n\}$ be a sequence of points from this space. In the article we will investigate a uniform distribution of the sequence $\{x_{Z(n)},\, n \geq 0\}$, where Z is a random walk on $\mathbb{Z}_+$ with a reflecting screen at zero.

Asymptotic properties of random walks on deterministic sequences are investigated, for example, in the papers [1] and [2], where authors consider a measure-preserving automorphism S of probability space $(M, \mathcal{M}, \mu)$, and a random walk along the orbits of S, that satisfy the following condition: a particle at $x \in M$ jumps to Sx or to $S^{-1}x$. Denote by $\xi_x(n)$ the position at time n of the particle, which starts at x. In [1], [2] the question of the existence of an invariant measure, that is absolutely continuous w.r.t. μ, is investigated for such random walks, along with the question of the convergence for the limiting distribution of $\xi_x(n)$ to such a measure. These questions are considered for diophantine rotations of the torus $\mathbb{T}^d$ in the article [1], and for Anosov diffeomorphisms in the paper [2].

In the present article conditions are given, sufficient for a random walk $\{Z(n), n \geq 0\}$ to have the following property: for any sequence $\{x_n\}$ which is uniformly distributed with some measure μ, almost all trajectories of the corresponding random sequence $\{x_Z(n)\}$ have the same property.

Our approach is based on the decomposition of the time axis $(0, \infty)$ into the random intervals $(\theta_{j-1}, \theta_j], j \geq 0$, where $\theta_{-1} = 0$ and $\theta_j = \sup\{n : Z(n) \leq j\}$ is the moment of the last visit of the state j $(j \geq 0)$, with a subsequent use of the methods of renewal theory. Despite random variables $\{\theta_j, j \geq 0\}$ not being stopping times, such an approach becomes possible thanks to the general results from the paper [4].

2. Main result

Definition 1. *Let $\{x_n, n \geq 0\}$ be a sequence of points from the space X. It is said that this sequence is uniformly distributed with measure μ (u.d. with measure μ), if for any $A \in \mathcal{F}$ such that $\mu(\partial A) = 0$ one has the convergence of sojourn measures μ_N of the set A*

$$\lim_{N\to\infty} \mu_N(A) := \lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} \chi_{x_n \in A} = \mu(A).$$

2010 *Mathematics Subject Classification.* 60B10, 60G50.
Key words and phrases. Uniform distribution, random walk.

Note that being uniformly distributed with measure μ means the weak convergence of sojourn measures μ_N of the sequence $\{x_n, n \geq 0\}$ to μ.

Another definition of this property for a sequence in a compact metric space one can find in [3], chapter 3, § 1.

Define homogeneous Markov chain $\{Z(n), n \geq 0\}$ with phase space $\mathbb{Z}_+$, that starts from zero and has the following transition probabilities:

$p_{j,i} = p_{i-j}$, if $i > 0$; $p_{j,0} = \sum\limits_{n \geq j} p_{-n}$; here $p_n \geq 0$ for integer n; $\sum\limits_{n \in \mathbb{Z}} p_n = 1$. Assume also that the following condition holds:

(A1) For integer $n > 1 : p_n = 0$.

Note that one can interpret Markov chain Z as a random walk with a reflecting screen at zero.

The main result of the article is formulated in the following Theorem.

Theorem 1. *Let the following conditions hold:*

1) $\sum\limits_{n \in \mathbb{Z}} n p_n > 0$;

2) $\sum\limits_{n \in \mathbb{Z}} n^2 p_n < \infty$.

Then for any sequence $\{x_n, n \geq 0\}$, which is uniformly distributed with some probability measure μ, almost surely the sequence $\{x_{Z(n)}, n \geq 0\}$ has the same property.

Below we make several remarks; before doing that, let us recall that θ_j denotes the moment of the last visit of the state j $(j \geq 0)$.

Remark 1. *Recall that the random walk in the theorem starts from zero. After minor changes in the proof of the theorem, one can obtain the similar result for the random walk, which starts from any non-negative state.*

Remark 2. *The first condition of the theorem is necessary because Z has to be transient. Otherwise, the decomposition of the time axis into the random intervals $(\theta_{j-1}, \theta_j], j \geq 0$ does not make any sense, because in this case even $\theta_0 = \infty$ almost surely.*

Remark 3. *The second condition of the theorem is imposed in order to guarantee, that the random variables $\{\theta_j - \theta_{j-1},\ j \geq 1\}$ have finite expectations. This is a crucial property for us, which allows a subsequent use of the methods of the renewal theory. Note that the proof of the fact that $\{\theta_j - \theta_{j-1},\ j \geq 1\}$ has finite expectations is nontrivial and is based on the article* [6].

Remark 4. *In the paper it is proved, that the condition (A1) guarantees the independence of the random variables $\{\theta_j - \theta_{j-1},\ j \geq 1\}$. This condition is an analogue of the following property in continuous time: the process is left-continuous and upper-semicontinuous. The reason for us to impose this condition is that, without it, the random variables $\{\theta_j - \theta_{j-1},\ j \geq 1\}$ have no means to be independent. Seemingly, their dependence can be controlled in terms of properly chosen Markov chain, but then the application of the renewal theory methods becomes more complicated. Hence in this paper, for the sake of simplicity, we restrict ourselves to the chains Z that satisfy condition (A1).*

3. Proof of Theorem 1

To make the structure of the proof more transparent, we omit the proofs of auxiliary Lemmas here. These proofs are given in Section 4.

Lemma 1. *Suppose that the following condition holds true for a sequence of random elements $\{\zeta_n\}$: for each $A \in \mathcal{F}$, such that $\mu(\partial A) = 0$*

$$\lim_{N \to \infty} \frac{1}{N+1} \sum_{n=0}^{N} \chi_{\zeta_n \in A} = \mu(A) \quad a.s.$$

Then the sequence $\{\zeta_n\}$ is uniformly distributed with measure μ almost surely.

By Lemma 1, it is enough to prove that for any A from $\mathcal{F}$, such that $\mu(\partial A) = 0$

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} \chi_{Z(n)\in W} = \mu(A) \quad \text{a.s.,} \tag{1}$$

where $W := \{n \geq 0 : x_n \in A\}$.

We conduct the proof of the statement (1) for such sets in several stages. The first is contained in the following Lemma.

Lemma 2. *Let W be a subset of $\mathbb{Z}_+$, such that $\lim_{N\to\infty} \frac{1}{N+1} \sum_{m=0}^{N} \chi_{m\in W} = a$. Then*

$$\lim_{N\to\infty} \mathbf{E} \frac{1}{N+1} \sum_{m=0}^{N} \chi_{Z(m)\in W} = a.$$

Further, let $\{\xi_i, i \geq 1\}$ be a sequence of i.i.d. random variables and $\mathbf{P}\{\xi_1 = j\} = p_j, j = 1, 0, -1, -2, \ldots$. Let $\hat{Z}$ be a random process, given by the relations

$$\hat{Z}(0) = 0 \quad a.s., \text{ for each natural } i : \hat{Z}(i) = (\hat{Z}(i-1) + \xi_i) \vee 0 \quad a.s. \tag{2}$$

Clearly, then the distributions of the random processes Z and $\hat{Z}$ are the same, hence in what follows we identify Z with $\hat{Z}$.

Recall that $\theta_{-1} = 0$, $\theta_j = \sup\{n : Z(n) \leq j\}$ is the moment of the last visit of the state j $(j \geq 0)$. Decompose the time axis $(0, \infty)$ into the random intervals $(\theta_{j-1}, \theta_j]$, $j \geq 0$.

Lemma 3. *Consider a random sequence $\{\xi_n, n \geq 1\}$. Then the segments of the sequence $\{\xi_{\theta_j+1}, \ldots, \xi_{\theta_{j+1}}\}$, are independent and identically distributed. Particulary, the random variables $\{\theta_1 - \theta_0, \theta_2 - \theta_1, \ldots\}$ are i.i.d.*

In Lemma 2 above, the set W and the constant a are not specified. Now, let $W = \{n \geq 0 : x_n \in A\}$, $a = \mu(A)$. Define the random variables σ_j in the following way

$$\sigma_0 := \sum_{k=0}^{\theta_0} \chi_{Z(k)\in W}, \quad \sigma_j := \sum_{k=\theta_{j-1}+1}^{\theta_j} \chi_{Z(k)\in W}, \quad j \geq 1.$$

Lemma 4. *Suppose that there exists $\lim_{N\to\infty} \mathbf{E} \frac{1}{N+1} \sum_{m=0}^{N} \chi_{Z(m)\in W} = a$, and the random variables $\sigma_j, j \geq 1$ are independent. Then*

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{m=0}^{N} \chi_{Z(m)\in W} = a \quad a.s.$$

Note that for each natural j and non-negative $n : \{\sigma_j = n\} \in \sigma\{\xi_{\theta_{j-1}+1}, \ldots, \xi_{\theta_j}\}$, hence the random variables σ_j, $j \geq 1$ are independent. Therefore the conditions of Lemma 4 hold, so one can deduce (1). The theorem is proved.

4. Proofs of Lemmas 1 – 4

The proof of Lemma 1. Let $\{\alpha_n, n \geq 1\}$ be a dense set in X and $\{r_k, k \geq 1\}$ be a dense set in $(0, \infty)$, such that for any positive integer n and k

$$\mu(\partial B(\alpha_n, r_k)) = 0.$$

Let $\mathcal{A}$ be the class of finite intersections of the balls $\{B(\alpha_n, r_k)), n, k \in \mathbb{N}\}$. Then for any set A from $\mathcal{A}$

$$\lim_{n\to\infty} \mu_n(A) = \mu(A),$$

which implies the weak convergence of measures μ_n to measure μ, see [5], chapter 1, § 2, corollary 1. $\square$

The proof of Lemma 2. We represent $\mathbf{E}\frac{1}{N}\sum\limits_{m=0}^{N}\chi_{Z(m)\in W}$ as a sum $\sum\limits_{n=0}^{N}v_{n,N}c_n$, where $c_n = \frac{1}{n+1}\sum\limits_{k=0}^{n}\chi_{k\in W}$, $v_{n,N}$ are some numbers, which will be identified below.

Write the following chain of transformations

$$\mathbf{E}\frac{1}{N+1}\sum_{m=0}^{N}\chi_{Z(m)\in W} =$$

$$\frac{1}{N+1}\sum_{m=0}^{N}\mathbf{P}\{Z(m)\in W\} = \frac{1}{N+1}\sum_{m=0}^{N}\sum_{n=0}^{m}\mathbf{P}\{Z(m)=n\}\chi_{n\in W} =$$

$$\frac{1}{N+1}\sum_{n=0}^{N}\chi_{n\in W}\sum_{m=n}^{N}\mathbf{P}\{Z(m)=n\} = \sum_{n=0}^{N}w_{n,N}\chi_{n\in W},$$

where

$$w_{n,N} = \frac{1}{N+1}\sum_{m=n}^{N}\mathbf{P}\{Z(m)=n\}.$$

Denote $a_n = \chi_{n\in W}$. Then

$$c_n = \frac{1}{n+1}(a_0+...+a_n)\to a,\ n\to\infty.$$

It is sufficient to prove that

$$w_{0,N}a_0+...+w_{N,N}a_N\to a,\ N\to\infty.$$

To do that, express $w_{0,N}a_0+...+w_{N,N}a_N$ using the numbers $c_0, c_1, ..., c_N$ and apply Toeplitz theorem.

$$w_{0,N}a_0+...+w_{N,N}a_N = w_{0,N}c_0 + w_{1,N}(2c_1-c_0)+...+w_{N,N}((N+1)c_N - Nc_{N-1}) =$$
$$c_0(w_{0,N}-w_{1,N})+...+c_{N-1}(Nw_{N-1,N}-Nw_{N,N})+c_N((N+1)w_{N,N}).$$

Further, denote

$$v_{0,N} = w_{0,N}-w_{1,N},$$
$$v_{1,N} = 2(w_{1,N}-w_{2,N}),$$
$$...$$
$$v_{N-1,N} = N(w_{N-1,N}-w_{N,N}),$$
$$v_{N,N} = (N+1)w_{N,N}.$$

Now, to prove that $\sum\limits_{n=0}^{N}v_{n,N}c_n\to a$, $N\to\infty$, let us verify the conditions of Toeplitz theorem:

(1) $\sum\limits_{n=0}^{N}v_{n,N}\to 1,\ N\to\infty$;
(2) for any non-negative $n: v_{n,N}\to 0,\ N\to\infty$;
(3) for any integer n, N, such that $0\le n\le N: v_{n,N}\ge 0$.

Note that the third property has the following interpretation: the expectation of the number of the visits to the state n during the first N steps of the random walk decreases as n increases.

The first property holds true, because

$$\sum_{n=0}^{N} v_{n,N} = (w_{0,N} - w_{1,N}) + ... + (N(w_{N-1,N} - w_{N,N})) + (N+1)w_{N,N} = \sum_{n=0}^{N} w_{n,N} =$$
$$\frac{1}{N+1}\sum_{n=0}^{N}\sum_{m=n}^{N}\mathbf{P}\{Z(m)=n\} = \frac{1}{N+1}\sum_{m=0}^{N}\sum_{n=0}^{m}\mathbf{P}\{Z(m)=n\} = \frac{1}{N+1}\sum_{n=0}^{N} 1 = 1.$$

Note that $\mathbf{E}\xi_k > 0$, hence Z is transient. Thus $\mathbf{P}\{Z(N) = m\} \to 0, N \to \infty$, so

$$w_{n,N} = \frac{1}{N+1}\sum_{m=n}^{N}\mathbf{P}\{Z(m)=n\} \to 0, \quad N \to \infty,$$

which implies the second property.

Let $1 \le k \le N$. Then the following inequality holds true

$$\sum_{m=0}^{N}\mathbf{P}\{Z(m)=0\} \ge \sum_{m=k}^{N}\mathbf{P}\{Z(m)=k\}. \tag{3}$$

We prove the inequality (3) by induction on N. For $N = 1$ the required inequality is obvious. Note, that for $m \ge n \ge 1$:

$\mathbf{P}\{Z(m) = n\} = \sum_{i=n-1}^{m-1}\mathbf{P}\{Z(m) = n,\, Z(m-1) = i\} = \sum_{i=n-1}^{m-1}\mathbf{P}\{Z(m-1) = i\}p_{n-i}.$

Hence

$$\sum_{m=k}^{N}\mathbf{P}\{Z(m)=k\} = \sum_{m=k}^{N}\sum_{i=k-1}^{m-1}\mathbf{P}\{Z(m-1)=i\}p_{k-i} =$$
$$\sum_{i=k-1}^{N-1}\sum_{m=i+1}^{N}\mathbf{P}\{Z(m-1)=i\}p_{k-i} \le \text{(the induction assumption)} \le$$
$$\sum_{i=k-1}^{N-1}\sum_{m=0}^{N-1}\mathbf{P}\{Z(m)=0\}p_{k-i} = \left(\sum_{i=k-1}^{N-1}p_{k-i}\right)\left(\sum_{m=0}^{N-1}\mathbf{P}\{Z(m)=0\}\right) \le$$
$$\sum_{m=0}^{N-1}\mathbf{P}\{Z(m)=0\} \le \sum_{m=0}^{N}\mathbf{P}\{Z(m)=0\},$$

as required.

Thus $v_{0,N} \ge 0$.

Now let us prove the third property. Let us show that if $1 \le n \le N$, then $v_{n,N} \ge 0$ using induction on N.

For $N = 1$ this is obvious. Suppose the statement is proved for $N = 1, ..., M-1\,(M \ge 2)$. Now let us prove it for $N = M$. Take $M \ge n+2$. Then

$v_{n,M} = w_{n,M} - w_{n+1,M} = \sum_{m=n}^{M}\mathbf{P}\{Z(m) = n\} - \sum_{m=n+1}^{M}\mathbf{P}\{Z(m) = n+1\} =$

$$\sum_{m=n}^{M}\sum_{k=n-1}^{m-1}\mathbf{P}\{Z(m)=n,\ Z(m-1)=k\}-$$
$$\sum_{m=n+1}^{M}\sum_{k=n}^{m-1}\mathbf{P}\{Z(m)=n+1,\ Z(m-1)=k\}=$$

$$\sum_{m=n}^{M}\sum_{k=n-1}^{m-1}\mathbf{P}\{Z(m-1)=k\}p_{n-k}-\sum_{m=n+1}^{M}\sum_{k=n}^{m-1}\mathbf{P}\{Z(m-1)=k\}p_{n+1-k}=$$
$$\sum_{k=n-1}^{M-1}p_{n-k}\sum_{m=k+1}^{M}\mathbf{P}\{Z(m-1)=k\}-\sum_{k=n}^{M-1}p_{n+1-k}\sum_{m=k+1}^{M}\mathbf{P}\{Z(m-1)=k\}=$$
$$\sum_{i=n-1}^{M-1}p_{n-i}\sum_{m=i+1}^{M}\mathbf{P}\{Z(m-1)=i\}-\sum_{i=n-1}^{M-2}p_{n-i}\sum_{m=i+2}^{M}\mathbf{P}\{Z(m-1)=i+1\}=$$
$$\sum_{i=n-1}^{M-1}p_{n-i}\sum_{m=i+1}^{M}\mathbf{P}\{Z(m-1)=i\}-\sum_{i=n-1}^{M-2}p_{n-i}\sum_{m=i+2}^{M}\mathbf{P}\{Z(m-1)=i+1\}\geq$$
$$\sum_{i=n-1}^{M-2}p_{n-i}\left(\sum_{m=i+1}^{M}\mathbf{P}\{Z(m-1)=i\}-\sum_{m=i+2}^{M}\mathbf{P}\{Z(m-1)=i+1\}\right)=$$
$$\sum_{i=n-1}^{M-2}p_{n-i}\left(\sum_{m=i}^{M-1}\mathbf{P}\{Z(m)=i\}-\sum_{m=i+1}^{M-1}\mathbf{P}\{Z(m)=i+1\}\right)\geq 0,$$

by the induction assumption.

For $M=n+1$ and for $M=n$ the third property is obvious. □

The proof of Lemma 3. The proof is based on the Corollary 2 from the article [4]. For the reader convenience we formulate this statement below.

Proposition 1. *Suppose that $\{\xi_k\}_{k\geq 0}$ is a sequence of i.i.d. random variables and that the sequence of random events $\{F_n\}_{n\geq 0}$:*

(F1) stationary (i.e. the sequence of random variables χ_{F_n} is stationary),

$\mathbf{P}\{F_n\}>0$,

(F2) $\chi_{F_n}=g(\xi_{n+1},\xi_{n+2},...)$ for some measurable function g.

Also suppose that

$$F_n\cap F_{n+m}=E'_{n,n+m}\cap F_{n+m},\quad n\geq 0, m>0, \tag{4}$$

For some array of events $\{E'_{n,n+m}\}_{n\geq 0,m>0}$, with each $E'_{n,n+m}\in\sigma_{n+1,n+m}$ (here $\sigma_{n+1,n+m}$ is the σ-algebra generated by the random variables ξ_k, $n+1\leq k\leq n+m$), and such that, for each fixed m, the sequence $\{E'_{n,n+m}\}_{n\geq 0}$, is stationary.

Put $\tau_0=\min\{n\geq 0:\chi_{F_n}=1\}$, $\tau_{k+1}=\min\{n>\tau_k:\chi_{F_n}=1\}$, $k\geq 0$. Suppose, that $\tau_k<\infty$ a.s. for all k. Then the segments of the sequence $\{\xi_{\tau_k+1},...,\xi_{\tau_{k+1}}\}$, are independent and identically distributed. In particular, the random variables $\{\tau_1-\tau_0,\tau_2-\tau_1,...\}$ are i.i.d.

Denote $F_n:=\{\forall m\geq 1:\quad \xi_{n+1}+...+\xi_{n+m}\geq 1\}$, $E'_{n,n+m}:=\{\forall m': m\geq m'\geq 1: \xi_{n+1}+...+\xi_{n+m'}\geq 1\}$, and $g(x_{n+1},x_{n+2,...}):=\chi_{\{\forall m\geq 1\,:\,x_{n+1}+...+x_{n+m}\geq 1\}}$. Recall that Z is given by (2).

Put

$$\widetilde{Z}(0):=0;\quad \widetilde{Z}(n)=\xi_1+...+\xi_n,\quad n\geq 1.$$

Then the inequality $\mathbf{P}\{F_n\}>0$ follows from the transience of the random walk $\widetilde{Z}$. It is easy to see that all other conditions of Proposition 1 hold true. Herein τ_k coincides with θ_k. The lemma is proved. □

The proof of Lemma 4. Recall, that $\theta_j := \sup\{n : Z(n) \le j\}$, and that

$$\sigma_0 := \sum_{k=0}^{\theta_0} \chi_{Z(k)\in W}, \quad \sigma_j := \sum_{k=\theta_{j-1}+1}^{\theta_j} \chi_{Z(k)\in W}, \quad j \ge 1.$$

First of all, note that the random variables $\theta_j, j \ge 0$ has finite expectations. Indeed, let $\widetilde{\theta}_j := \sup\{n : \widetilde{Z}(n) \le j\}$. Condition 2) of Theorem 1 is necessary and sufficient for the random variables $\widetilde{\theta}_j$, $j \ge 1$ to have finite expectations; see [6], Theorem 5.1. Further, for any non-negative $n : \widetilde{Z}(n) \le Z(n)$, hence if $\widetilde{Z}(n) \ge j$, then $Z(n) \ge j$. Thus $\widetilde{\theta}_j \ge \theta_j$, and $\mathbf{E}\theta_j < \infty$, as required.

Let $K > 0$ such that $K > \mathbf{E}(\theta_1 - \theta_0)$. First of all, prove that

$$\mathbf{E}\frac{1}{\theta_n + 1}\sum_{m=0}^{\theta_n} \chi_{Z(m)\in W} \to a, \quad n \to \infty. \tag{5}$$

Indeed, denote $A_n := [n\mathbf{E}(\theta_1 - \theta_0)]$ (here $[x]$ is the integer part of number x). Then from the strong law of large numbers, $\frac{\theta_n+1}{A_n+1} \to 1$, $n \to \infty$, a.s.

Note that

$$\mathbf{E}\frac{1}{A_n + 1}\sum_{m=0}^{A_n} \chi_{Z(m)\in W} \to a, \quad n \to \infty.$$

Further,

$$0 \le \Big|\frac{1}{\theta_n + 1}\sum_{m=0}^{\theta_n} \chi_{Z(m)\in W} - \frac{1}{A_n + 1}\sum_{m=0}^{A_n} \chi_{Z(m)\in W}\Big| \le$$

$$\frac{|\theta_n - A_n|}{(A_n+1)(\theta_n+1)} \sum_{m=0}^{\min\{\theta_n, A_n\}} \chi_{Z(m)\in W} + \frac{|\theta_n - A_n|}{\max\{\theta_n+1, A_n+1\}} \le$$

$$\frac{2|\theta_n - A_n|}{\max\{\theta_n+1, A_n+1\}} \le 2\frac{|\theta_n - A_n|}{A_n+1} \to 0, \quad n \to \infty \quad \text{a.s.}, \tag{6}$$

hence

$$\Big|\frac{1}{\theta_n + 1}\sum_{m=0}^{\theta_n} \chi_{Z(m)\in W} - \frac{1}{A_n + 1}\sum_{m=0}^{A_n} \chi_{Z(m)\in W}\Big| \to 0, \; n \to \infty, \quad \text{a.s.}$$

Since

$$\Big|\frac{1}{\theta_n + 1}\sum_{m=0}^{\theta_n} \chi_{Z(m)\in W} - \frac{1}{A_n + 1}\sum_{m=0}^{A_n} \chi_{Z(m)\in W}\Big| \le$$

$$\Big|\frac{1}{\theta_n + 1}\sum_{m=0}^{\theta_n} \chi_{Z(m)\in W}\Big| + \Big|\frac{1}{A_n + 1}\sum_{m=0}^{A_n} \chi_{Z(m)\in W}\Big| \le 2,$$

then the dominated convergence theorem implies (5). From (5) we have

$$\mathbf{E}\frac{1}{\theta_n + 1}\sum_{m=\theta_0}^{\theta_n} \chi_{Z(m)\in W} \to a, \quad n \to \infty. \tag{7}$$

Let us prove that

$$\mathbf{E}\frac{1}{n + 1}\sum_{m=\theta_0}^{\theta_n} \chi_{Z(m)\in W} \to a\mathbf{E}(\theta_1 - \theta_0), \quad n \to \infty. \tag{8}$$

Note that $\mathbf{E}(\theta_1-\theta_0)=\lim_{n\to\infty}\mathbf{E}\frac{\theta_n+1}{n+1}$, $n\to\infty$ a.s. It follows from this statement and (7) that

$$\limsup_{n\to\infty}\left|\mathbf{E}\frac{1}{n+1}\sum_{m=\theta_0}^{\theta_n}\chi_{Z(m)\in W}-a\mathbf{E}(\theta_1-\theta_0)\right|=$$
$$\limsup_{n\to\infty}\left|\mathbf{E}\frac{\theta_n+1}{n+1}\frac{1}{\theta_n+1}\sum_{m=\theta_0}^{\theta_n}\chi_{Z(m)\in W}-\mathbf{E}\frac{\theta_n+1}{n+1}\mathbf{E}\frac{1}{\theta_n+1}\sum_{m=\theta_0}^{\theta_n}\chi_{Z(m)\in W}\right|=$$
$$\limsup_{n\to\infty}\left|\mathbf{E}\left(\frac{\theta_n+1}{n+1}-\mathbf{E}\frac{\theta_n+1}{n+1}\right)\frac{1}{\theta_n+1}\sum_{m=\theta_0}^{\theta_n}\chi_{Z(m)\in W}\right|\le$$
$$\limsup_{n\to\infty}\mathbf{E}\left|\frac{\theta_n+1}{n+1}-\mathbf{E}\frac{\theta_n+1}{n+1}\right|=0.$$

So (8) is proved.

Note that $\sum_{m=\theta_0}^{\theta_n}\chi_{Z(m)\in W}=\sum_{m=1}^{n}\sigma_m$.

Now we use the following analogue of the strong law of large numbers for the random variables σ_m, that, generally speaking, are not identically distributed, but still are independent.

Proposition 2. $\frac{1}{n+1}\sum_{m=1}^{n}(\sigma_m-\mathbf{E}\sigma_m)\to 0, n\to\infty,$ *a.s.*

Proof. Denote $\sigma'_m:=\sigma_m\chi_{\{\sigma_m\le m\}}$. Then using the fact that for each positive integer m, $0\le\sigma_m\le\theta_m-\theta_{m-1}$ a.s., and that the random variables $\theta_j-\theta_{j-1}$, $j\ge 1$ are independent, one can deduce that

$$\frac{1}{n+1}\sum_{m=1}^{n}(\sigma'_m-\mathbf{E}\sigma'_m)\to 0,\quad n\to\infty,\qquad \text{a.s.} \tag{9}$$

The proof of (9) is similar to that of the classical Khinchine strong law of large numbers (see [7], chapter 4, § 3, Theorem 3).

Further $\mathbf{E}\sigma_m-\mathbf{E}\sigma'_m=\mathbf{E}\sigma_m\chi_{\{\sigma_m>m\}}\le\mathbf{E}\theta_1\chi_{\{\theta_1>m\}}\to 0$, $m\to\infty$ by the dominated convergence theorem. Hence,

$$\frac{1}{n+1}\sum_{m=1}^{n}(\mathbf{E}\sigma'_m-\mathbf{E}\sigma_m)\to 0,\quad n\to\infty. \tag{10}$$

Further

$$\sum_{m\ge1}\mathbf{P}\{\sigma_m>m\}\le\sum_{m\ge1}\mathbf{P}\{\theta_m>m\}=\sum_{m\ge1}\mathbf{P}\{\theta_1>m\}=\sum_{m\ge1}\sum_{l\ge m}\mathbf{P}\{\theta_1=l+1\}=$$
$$\sum_{l\ge1}\sum_{m=1}^{l}\mathbf{P}\{\theta_1=l+1\}=\sum_{l\ge1}l\mathbf{P}\{\theta_1=l+1\}\le\mathbf{E}\theta_1<\infty.$$

Thus, from the Borel-Cantelli lemma, a.s. there exists $M=M(\omega)$ such that for each $m\ge M:\sigma_m=\sigma'_m$.

Hence the sequence $\{\sum_{m=1}^{n}\sigma_m-\sigma'_m\}_{n\ge1}$ is bounded almost surely, and therefore

$$\frac{1}{n+1}\sum_{m=1}^{n}\sigma_m-\sigma'_m\to 0,\quad n\to\infty. \tag{11}$$

Hence (9), (10) and (11) imply that almost surely

$$\frac{1}{n+1}\sum_{m=1}^{n}(\sigma_m-\mathbf{E}\sigma_m)=$$

$$\frac{1}{n+1}\left(\sum_{m=1}^{n}(\sigma_m-\sigma'_m)+\sum_{m=1}^{n}(\sigma'_m-\mathbf{E}\sigma'_m)+\sum_{m=1}^{n}(\mathbf{E}\sigma'_m-\mathbf{E}\sigma_m)\right)\to 0,\quad n\to\infty,$$

which completes the proof. □

So, from Proposition 2 $\frac{1}{n+1}\sum\limits_{m=1}^{n}(\sigma_m-\mathbf{E}\sigma_m)\to 0, n\to\infty$, hence (7) implies that a.s. $\frac{1}{n+1}\sum\limits_{m=1}^{n}\sigma_m\to a\mathbf{E}(\theta_1-\theta_0)$, $n\to\infty$. Thus $\frac{1}{\theta_n+1}\sum\limits_{m=0}^{\theta_n}\chi_{Z(m)\in W}=\frac{n+1}{\theta_n+1}\frac{1}{n+1}\sum\limits_{m=0}^{n}\sigma_m\to a, n\to\infty$. Further, we proved, that a.s.

$$|\frac{1}{\theta_n+1}\sum_{m=0}^{\theta_n}\chi_{Z(m)\in W}-\frac{1}{A_n+1}\sum_{m=0}^{A_n}\chi_{Z(m)\in W}|\to 0,\quad n\to\infty,$$

and thus,

$$\frac{1}{A_n+1}\sum_{m=0}^{A_n}\chi_{Z(m)\in W}\to a,\quad n\to\infty.$$

Let M_N be the biggest non-negative number, such that $A_{M_N}\le N$. Then $|N-A_{M_N}|\le\mathbf{E}(\theta_1-\theta_0)+1$. Further, repeating the transformations (6), we have that a.s.

$$\left|\frac{1}{N+1}\sum_{k=0}^{N}\chi_{Z(k)\in W}-\frac{1}{A_{M_N}+1}\sum_{k=0}^{A_{M_N}}\chi_{Z(k)\in W}\right|\le$$

$$2\frac{N-A_{M_N}}{N+1}\le 2\frac{\mathbf{E}(\theta_1-\theta_0)+1}{N+1}\to 0,\quad n\to\infty,$$

thus a.s. $\frac{1}{N+1}\sum\limits_{k=0}^{N}\chi_{Z(k)\in W}\to a$, $N\to\infty$, which completes the proof. □

References

1. V. Kaloshin and Y. Sinai, *Non-symmetric Simple Random Walks along Orbits of Ergodic Automorphisms*, Amer. Math. Soc. Transl. **198** (2000), no. 2, 109–115.
2. V. Kaloshin and Y. Sinai, *Non-symmetric Simple Random Walks along Orbits of Anosov Diffeomorphisms*, Proc. of Steklov Math. Inst. **228** (2000), 224–233.
3. L. Kuipers and H. Niederreiter, *Uniform distribution of sequences*, Wiley-Interscience, New-York, 1974.
4. S. Foss and S. Zachary, *Stochastic sequences with a regenerative structure that may depend both on the future and on the past* (Submitted). http://arxiv.org/abs/1212.1475
5. P. Billingslley, *Convergence of probability measures*, John Wiley, New-York, 1999.
6. K. Sato and T. Watanabe, *Moments of last exit times for Lévy processes*, Ann. Inst. H. Poincaré Probab. Statist. **40** (2004), no. 2, 207–225.
7. A. Shiryaev, *Probability*, Springer-Verlag, New-York, 1996.

Faculty of Mechanics and Mathematics, Taras Shevchenko National University of Kyiv, building 4-e, Academician Glushkov prospectus, Kyiv, Ukraine, 03127

E-mail address: seninvi@ukr.net