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A. YU. PILIPENKO AND YU. E. PRYKHODKO

LIMIT BEHAVIOR OF A SIMPLE RANDOM WALK WITH NON-INTEGRABLE JUMP FROM A BARRIER

Consider a Markov chain on $\mathbb{Z}_+$ with reflecting barrier at 0 such that jumps of the chain outside of 0 are i.i.d. with mean zero and finite variance. It is assumed that the jump from 0 has a distribution that belongs to the domain of attraction of non-negative stable law. It is proved that under natural scaling of a space and a time a limit of this scaled Markov chain is a Brownian motion with some Wentzell's boundary condition at 0.

Introduction

Consider a homogeneous Markov chain $(X(k),\ k \geqslant 0)$ on $\mathbb{Z}$ with transition probabilities

$$p_{i,j} = \mathrm{P}(\varepsilon = j - i),\ |i| > 0,$$
$$p_{0,j} = \mathrm{P}(\xi = j),\ j \in \mathbb{Z},$$

where $\mathrm{E}\varepsilon = 0, \mathrm{D}\varepsilon = \sigma^2 < \infty$, ξ is a random variable with values in $\mathbb{Z}$.

We study a limit behavior of a sequence of processes

$$X_n = \Big(X_n(t) = \tfrac{1}{\sqrt{n}} X\big([nt]\big),\ t \geqslant 0\Big),\ n \in \mathbb{N}.$$

If the random variable ξ is integrable (see [1], [2], [3]), then $\{X_n\}$ converges weakly to σW_γ, where W_γ is a skew Brownian motion, i.e., a continuous Markov process with transition density

$$p_t(x,y) = \varphi_t(x-y) + \gamma\,\mathrm{sign}(y)\,\varphi_t(|x|+|y|),\ x,y \in \mathbb{R},$$

where $\varphi_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$ is a density of $N(0,t)$.

For example, if $\mathrm{P}(\varepsilon = \pm 1) = 1/2$, then [1] parameter $\gamma \in [0,1]$ of a limit process W_γ equals

$$\gamma = \frac{\mathrm{E}\xi}{\mathrm{E}|\xi|}.$$

We study a limit of a sequence $\{X_n\}$ when ξ is non-integrable positive random variable that belongs to a domain of attraction of a positive stable law with $\alpha \in (0,1)$. A limit process X is Markov, but discontinuous. Naturally, it should behave like a Wiener process outside of 0 and the only points of discontinuity may be moments of hitting 0. More precise, X will be a Wiener process with Wentzell's boundary condition where a jump measure $\mu(du) = u^{-(\alpha+1)}du$ has infinite variation. In this case the limit process a.s. has infinitely many jumps in any neighbourhood of an instant when it hits 0.

Complete description of possible boundary conditions for diffusions was done by Wentzell [4, 5]. Construction and investigation of the corresponding processes with non-local boundary condition was done by different methods, including semigroup theory,

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stochastic differential equations, martingale problem, etc., see for ex. [6, 7, 8, 9, 10, 11, 12, 13, 14, 15].

Our proof of the main result is based on the Skorokhod representation theorem, continuous mapping theorem, and properties of some reflection problem with possibly jump-type reflection.

1. Main results

Assume that a random variable ξ belongs to a domain of attraction of α-stable law U_α on $[0,\infty)$, $\alpha\in(0,1)$. That is, there exists a sequence $\{a_n\}$ such that

$$\lim_{n\to\infty}\frac{\sum_{k=1}^n \xi_k}{a_n}\overset{d}{=}U_\alpha, \tag{1}$$

where $\{\xi_k,\ k\geqslant 1\}$ are i.i.d. copies of ξ,

$$\mathrm{E}e^{-\lambda U_\alpha}=e^{-\lambda^\alpha},\ \lambda\geqslant 0.$$

It is known [16] that (1) yields a weak convergence of distributions of processes

$$\frac{\sum_{k=1}^{[nt]}\xi_k}{a_n}\Rightarrow U_\alpha(t),\ n\to\infty,\ \text{in } \mathcal{D}([0,\infty)), \tag{2}$$

where $U_\alpha(\cdot)$ is α-stable process, i.e. a cadlag process with homogeneous independent increments such that $U_\alpha(0)=0$ and

$$\mathrm{E}e^{-\lambda U_\alpha(t)}=e^{-\lambda^\alpha t},\ t\geqslant 0,\ \lambda\geqslant 0.$$

Recall that Levy-Khinchine measure of U_α equals

$$\mu(du)=cu^{-(\alpha+1)}du,$$

where $c=\frac{\alpha}{\Gamma(1-\alpha)}$.

Theorem 1.1. *Let $\{X(k),k\geqslant 0\}$ be a homogeneous Markov chain on $\mathbb{Z}_+$ such that*

$$p_{ij}=\mathrm{P}(\varepsilon=j-i),\ i\geqslant 1,$$

$$p_{0j}=\mathrm{P}(\xi=j),$$

where a random variable ε takes values in a set $\{-1,0,1,2,\dots\}$,

$$\mathrm{E}\varepsilon=0,\mathrm{D}\varepsilon=\sigma^2\in(0,\infty),$$

ξ takes values in $\mathbb{N}$, and ξ belongs to a domain of attraction of α-stable law U_α on $[0,\infty)$, $\alpha\in(0,1)$.

Let a process $\{X_n(t),t\geqslant 0\}$ has the same distribution as $\{X([nt])/\sqrt{n},t\geqslant 0\}$ given $X(0)=x_n$.

If $x_n/\sqrt{n\sigma^2}\to x_0, n\to\infty$, then sequences $\{X_n\}$ converges weakly in $\mathcal{D}([0,\infty))$ to the process

$$X_\infty(t):=\sigma\widetilde{W}_\alpha(t):=\sigma\left(x_0+W(t)+U_\alpha\big(U_\alpha^{(-1)}(M(t))\big)\right),\ t\geqslant 0,$$

where $(W(t))$ is a standard Wiener process, $M(t)=-\min_{s\in[0,t]}\big((x_0+W(s))\wedge 0\big)$, $t\geqslant 0$, $(U_\alpha(t))$ is a α-stable process, $U_\alpha^{(-1)}(t)=\inf\{s\geqslant 0\colon U_\alpha(s)\geqslant t\}$, $t\geqslant 0$, and processes $(W(t))$ and $(U_\alpha(t))$ are independent.

Remark 1.1. If we take a random variable $\widetilde{\xi}=m\xi$, $m\in\mathbb{N}$, instead of ξ, then the limit process for the sequence $\{X_n\}$ will be the same.

Remark 1.2. It follows from [17], §3 that $\widetilde{W}_\alpha$ satisfies the following martingale problem. For any $f \in \mathrm{C}^2([0,\infty))$ with a compact support such that

$$\int_0^\infty (f(u) - f(0))u^{-(\alpha+1)}du = 0,$$

a process

$$f(\widetilde{W}_\alpha(t)) - \tfrac{\sigma^2}{2}\int_0^t f''(\widetilde{W}_\alpha(s))ds$$

is a martingale. Moreover, using reasoning of [17] it can be shown that $\widetilde{W}_\alpha$ is a Markov (and even strong Markov) process. Hence $\widetilde{W}_\alpha$ is a diffusion with generator $A = \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}$, where a core of A is $C_0^2([0,\infty)) \cap \{g \ : \ \int_0^\infty (g(u) - g(0))u^{-(\alpha+1)}du = 0\}$. Existence of a diffusion with such generator follows from [4], where all possible generators of diffusions on an interval were described.

Remark 1.3. A non-negative random variable η belongs to the domain of attraction of α-stable law on $(0,\infty)$ if and only if $\mathrm{P}(\eta > x) \sim x^{-\alpha}\ell(x), \ x \to +\infty$, where ℓ is a slowly varying function [18, § XIII.6].

Next result is a generalization of Theorem 1.1 to the case when X is a Makov chain on $\{-m, -m+1, \dots, -1\} \cup \mathbb{Z}_+$ and a symmetry of a Markov chain is violated not only at 0 but on $\{-m, -m+1, \dots, 0\}$.

Theorem 1.2. *Let X be a Markov chain on $\{-m, -m+1, \dots, -1\} \cup \mathbb{Z}_+$ such that all states are connected and*

$$p_{i,j} = \mathrm{P}(\varepsilon = j - i), \ i \geqslant 1,$$

where ε is from Theorem 1.1.
Put $\tau = \inf\{k \geqslant 1 : \ X(k) \geqslant 1\}$. Assume that a distribution of $X(\tau)$ given $X(0) = 0$, belongs to a domain of attraction of a positive α-stable law with $\alpha \in (0,1)$. Then the distributions of $\{X_n, n \geqslant 1\}$ constructed in Theorem 1.1 converge weakly in $\mathcal{D}([0,\infty))$ to the distribution of the process $\sigma\widetilde{W}_\alpha(t)$.

Remark 1.4. Denote $\bar{F}_i(x) = \mathrm{P}(X(1) \geqslant x/X(0) = i)$. Condition of Theorem 1.2 on domain of attraction of α-stable law is satisfied if exists $k \geqslant 1$ and $i_1, ..., i_k \in \{1, \dots, m\}$ such that

$$\bar{F}_i(x) = x^{-\alpha}l_i(x), x \to \infty, \quad \text{for } i \in \{i_1, ..., i_k\},$$

and for other i

$$\bar{F}_i(x) = o(x^{-(\alpha+\delta)}), x \to \infty,$$

where δ is a fixed positive number.

Remark 1.5. We conjecture that the limit in Theorem 1.2 is $\sigma\widetilde{W}_\alpha(t)$ if we replace condition $\varepsilon \geqslant -1$ by boundedness of ε from below.

By $p_t^\alpha(x,y)$ denote a transition density of a process $\widetilde{W}_\alpha$ constructed in Theorem 1.1 (it is not difficult to show that this density exists). Let $p_t^{(0)}(x,y)$ be a transition density of a Brownian motion killed at zero: $p_t^{(0)}(x,y) = (\varphi_t(x-y) - \varphi_t(x+y))1\!\!1_{xy>0}$, where $\varphi_t(x) = \frac{1}{\sqrt{2\pi t}}e^{-x^2/2t}$. Theorem 1.2 and the same reasoning as in a proof of the Andre reflection principle yields the following result.

Theorem 1.3. *Assume that $X(k), k \geqslant 0$ is a Markov chain on $\mathbb{Z}$ such that*
1) $p_{i,i\pm1} = 1/2, \ |i| \geqslant 1$,
2) $p_{0,j} = \beta\mathrm{P}(\xi = j), \ p_{0,-j} = (1-\beta)\mathrm{P}(\xi = j), \ j \geqslant 1$,
where $\beta \in [0,1]$, and random variable ξ belongs to a domain of attraction of a positive α-stable law with $\alpha \in (0,1)$.

Then the distributions of the sequence $\{X_n(t) = \frac{1}{\sqrt{n}}X\big([nt]\big), t \geqslant 0\}_{n\geqslant 1}$ *converge weakly in* $\mathcal{D}([0,\infty))$ *to the distribution of a Markov process* $\widetilde{W}_{\alpha,\beta}(t)$ *with a transition density*

$$\widetilde{p}_t^{\alpha,\beta}(x,y) = p_t^{(0)}(x,y) + \tfrac{1+(2\beta-1)\,\mathrm{sign}(y)}{2}\left(p_t^{\alpha}(|x|,|y|) - p_t^{(0)}(|x|,|y|)\right),\ x,y \in \mathbb{R}.$$

Remark 1.6. By analogy with a skew Brownian motion, the process $\widetilde{W}_{\alpha,\beta}(t)$ can be called "the skew Brownian motion with α-stable boundary condition at 0".

2. Proof of Theorem 1.1

2.1. **Generalization of the Skorokhod reflecting problem.** We need the following generalization of Skorokhod's reflecting problem (see for ex. [17]).

Let $f \in \mathcal{D}([0,\infty))$ be increasing function, $f(0) = 0$, and $w \in \mathrm{C}([0,\infty)),\ w(0) \geqslant 0$.

Consider an equation with respect to a pair of unknown functions (X, L):

$$X(t) = w(t) + f(L(t)),\ t \geqslant 0, \tag{3}$$

where $X(t) \geqslant 0$ for $t \geqslant 0$, a function L is continuous and non-decreasing, $L(0) = 0$, and

$$\int_0^t \mathbb{1}_{X(s)=0}\,dL(s) = L(t),\ t \geqslant 0.$$

This system will be called $W(w,f)$ reflecting problem.

Theorem 2.1 ([17]). *Assume that* $\lim_{x\to\infty} f(x) = \infty$. *Then there exists a unique solution to the problem* (3). *Moreover*

$$L(t) = f^{(-1)}(-\min_{s\in[0,t]}(w(s)\wedge 0)),\ t \geqslant 0,$$

i.e.

$$X(t) = w(t) + f\big(f^{(-1)}(-\min_{s\in[0,t]}(w(s)\wedge 0))\big),\ t \geqslant 0,$$

where $f^{(-1)}(x) = \inf\{y \geqslant 0\colon f(y) \geqslant x\},\ x \geqslant 0.$

Remark 2.1. Reflection problem for $f(x) = x$ was originally introduced by A.V.Skorokhod in [19]. If f is arbitrary continuous increasing function, $f(0) = 0$, then $f\big(f^{(-1)}(x)\big) = x,\ x \geqslant 0$, and X (but not L) coincides with the solution of the Skorokhod reflection problem with $f(x) = x$.

Remark 2.2. Let $\widetilde{f}(x) = f(Cx)$, where $C > 0$, and $(\widetilde{X}, \widetilde{L})$ be the solution of $W(w,\widetilde{f})$. Then $\widetilde{X}(t) = X(t),\ \widetilde{L}(t) = L(t)/C$.

2.2. **Construction of a copy of Markov chain X as a solution of reflection problem.** In this section we associate a distribution of X with a solution of a reflection problem for some random walk. We will assume for simplicity that all $x_n = 0$. The proof of the general case leaves the same, but we would need always add some initial terms.

Let $(S(k), k \geqslant 0)$ be a random walk on $\mathbb{Z}$ with jumps that have the same distribution as ε. Extend S for all $t \geqslant 0$ continuously by linearity:

$$S(t) = S([t]) + (t - [t])\big(S([t]+1) - S([t])\big),\ t \geqslant 0.$$

Consider a reflecting problem $W(S,F)$:

$$\widetilde{X}(t) = S(t) + F(L(t)),\ t \geqslant 0, \tag{4}$$

where

$$F(x) = x + \sum_{k=1}^{[x]}(\xi_k - 1),\ x \geqslant 0,$$

$\{\xi_k,\ k \geqslant 1\}$ are i.i.d.r.v. with the same distribution as ξ and are independent of S.

It follows from Theorem 2.1 that there exists a unique solution $\widetilde{X}$ of (4), and

$$L(t) = F^{(-1)}(-\min_{s\in[0,t]}(S(s)\wedge 0)),\ t \geqslant 0,$$

where $F^{(-1)}(x) = \inf\{y \geqslant 0\colon F(y) \geqslant x\},\ x \geqslant 0$.

It is easy to see that $L(\cdot)$ is continuous and non-decreasing piece-wise linear process such that

$$\text{either } L(t) = L(k), \quad \text{or } L(t) = L(k) + (t-k) \text{ for all } t \in [k, k+1].$$

In particular,

$$L(k+1) - L(k) \in \{0, 1\},\ k \in \mathbb{N}.$$

By $(t_i, i \geqslant 1)$ denote points of growth of the function $L([t]),\ t \geqslant 0$. Note that t_i may take only integer values, $\widetilde{X}(t)$ jumps only when $t = t_i$ for some i. This means that $\widetilde{X}(t) = 0, t \in [t_i - 1, t_i)$, $S_{t_i} - S_{t_i - 1} = -1$, $\widetilde{X}(t_i) = \xi_i - 1$, and

$$L(k) = i \text{ for } k = \overline{t_i, t_{i+1} - 1}. \tag{5}$$

Put

$$\bar{X}(k) := 1 + \widetilde{X}(k),\ k = \overline{0, t_1{-}1},$$

$$\bar{X}(t_1) := 0.$$

Further define $\bar{X}$ by a recursion

$$\bar{X}(k+i) := 1 + \widetilde{X}(k),\ k = \overline{t_i{+}1, t_{i+1}{-}1},$$

$$\text{and } \bar{X}(t_{i+1} + i) := 0.$$

Taking into account (5), a construction of X can be described as follows

$$\bar{X}(k + L(k)) = 1 + \widetilde{X}(k),\ k \geqslant 0,$$

$$\bar{X}(t_i + L(t_i) - 1) = 0,\ i \geqslant 1.$$

It can be easily verified that distributions of the sequence $(\bar{X}(k), k \in \mathbb{Z}_+)$ and Markov chain $(X(k), k \in \mathbb{Z}_+)$ coincide.

To prove Theorem 1.1 it suffices to verify that $\bar{X}_n \Rightarrow X_\infty$ as $n \to \infty$, where $\bar{X}_n(t) = \frac{1}{\sqrt{n}}\bar{X}([nt]),\ t \geqslant 0,\ n \in \mathbb{N}$.

Put

$$\widetilde{X}_n(t) = \frac{1}{\sqrt{n}}\widetilde{X}(nt),$$

$$\lambda(t) := t + \sum_{k\leqslant t} \mathbb{1}_{\bar{X}(k)=0}.$$

Then

$$\widetilde{X}([t]) = \bar{X}(\lambda([t])) - 1,\ t \geqslant 0,$$

and

$$\widetilde{X}_n\left(\frac{[nt]}{n}\right) = \frac{\widetilde{X}([nt])}{\sqrt{n}} = \frac{\bar{X}(\lambda([nt])) - 1}{\sqrt{n}} = \bar{X}_n\left(\frac{\lambda([nt])}{n}\right) - \frac{1}{\sqrt{n}}. \tag{6}$$

Now the proof of the Theorem is as follows. First we prove that $\widetilde{X}_n \Rightarrow X_\infty$ as $n \to \infty$ in $D([0,\infty))$. Secondly we show that for any $T > 0$

$$\sup_{t\in[0,T]} |\frac{\lambda([nt])}{n} - t| \to 0,\ n \to \infty, \tag{7}$$

in probability, and then complete the proof with some minor details.

2.3. **Passage to the limit.** It follows form (4) that

$$\frac{1}{\sqrt{n}}\widetilde{X}(nt) = \frac{1}{\sqrt{n}}S(nt) + \frac{1}{\sqrt{n}}F(L(nt)),\ t \geqslant 0,\ n \in \mathbb{N}. \tag{8}$$

It is well known that distributions of a sequence

$$\big(S_n(t) = \tfrac{1}{\sqrt{n}}S(nt),\ t \geqslant 0\big),\ n \in \mathbb{N},$$

converge weakly in $\mathrm{C}([0,\infty))$ to a distribution of a Wiener process (Donsker's theorem, see for ex. [20])

$$S_n(t) \Rightarrow \sigma W(t),\ n \to \infty,\ \text{in } \mathrm{C}([0,\infty)). \tag{9}$$

Consider a limit behavior of the second term in (8).

Recall that,

$$F(x) = x + \sum_{n=1}^{[x]}(\xi_k - 1),\ x \geqslant 0,$$

where a distribution of random variables ξ_k belongs to a domain of attraction of the distribution U_α. Then, see (2), there exists a sequence $\{a_n\}$ such that

$$\frac{F(nt)}{a_n} \Rightarrow U_\alpha(t),\ n \to \infty,$$

in $\mathcal{D}([0,\infty))$.

Take $k(n) := \inf\{k\colon a_k \geqslant \sqrt{n}\}$. Then $a_n \to \infty$ and $\frac{a_{n+1}}{a_n} \to 1$ as $n \to \infty$ (see [18, §VIII.3, Lemma 3]). So

$$\frac{a_{k(n)}}{\sqrt{n}} \to 1,\ n \to \infty. \tag{10}$$

Hence

$$\frac{F(k(n)t)}{a_{k(n)}} \Rightarrow U_\alpha(t),\ n \to \infty,\ \text{in } \mathcal{D}([0,\infty)).$$

Therefore (10) yields

$$F_n(t) := \frac{1}{\sqrt{n}}F(k(n)t) \Rightarrow U_\alpha(t),\ n \to \infty,\ \text{in } \mathcal{D}([0,\infty)). \tag{11}$$

Taking into account Remark 2.2, the process $\widetilde{X}_n(t) = \frac{1}{\sqrt{n}}\widetilde{X}(nt)$ from reflection problem (8) coincides with a solution of the reflection problem

$$\widetilde{X}_n(t) = S_n(t) + F_n(L_n(t)),\ t \geqslant 0,$$

and

$$L_n(t) = L(nt)/k(n). \tag{12}$$

We need the following result on a continuity of a solution of reflection problem $W(w,f)$ on functions f and w.

Proposition 2.1. *Let $\{w_n, n \geqslant 0\} \subset \mathrm{C}([0,\infty))$, $\{f_n, n \geqslant 0\} \subset \mathcal{D}([0,\infty))$ be increasing functions such that $f_n(0) = 0$, $lim_{x\to\infty} f_n(x) = +\infty$, and (X_n, L_n) be solutions of $W(w_n, f_n)$, $n \geqslant 0$.*
Assume that
1) $f_n \to f_0$, $n \to \infty$ in $\mathcal{D}([0,\infty))$;
2) $w_n \to w_0$, $n \to \infty$ in $\mathrm{C}([0,\infty))$;
3) if $\hat{t}$ is a point of discontinuity of a function f_0, then equation (with respect to the variable $t \geq 0$)

$$-\min_{s\in[0,t]}(w_0(s) \wedge 0) = f_0(\hat{t} - 0), \tag{13}$$

has at most one solution.

Then

$$X_n \to X_0,\ n \to \infty,\ \textit{in } \mathcal{D}([0,\infty)),$$

and

$$L_n \to L_0,\ n \to \infty,\ \textit{in } \mathrm{C}([0,\infty)).$$

The proof follows from [17, Corollaries 1 and 2].

Let us continue the proof of Theorem 1.1.

By Skorokhod's representation theorem [21, §I.6], (9) and (11), there exists a single probability space that contains copies $\hat{S}_n$, $\hat{W}$, $\hat{F}_n$, $\hat{U}_\alpha$ of processes S_n, W, F_n and U_α, such that convergence with probability 1 holds

$$\hat{S}_n \to \sigma\hat{W} \text{ as } n \to \infty \text{ in } \mathrm{C}([0,\infty)) \text{ a.s.},$$

and

$$\hat{F}_n \to \hat{U}_\alpha \text{ as } n \to \infty \text{ in } \mathcal{D}([0,\infty)) \text{ a.s.}$$

Note that S_n and F_n are independent, so $\hat{S}_n$ and $\hat{F}_n$ are independent.

Let us apply Proposition 2.1 to the process $\sigma\hat{W}$ as w_0 and $\hat{U}_\alpha$ as f_0.

Equation (13) takes a form

$$-\sigma \min_{s\in[0,t]} (\hat{W}(s) \wedge 0) = \hat{U}_\alpha(t_k - 0),$$

where $\hat{W}$ is a Wiener process, $\{t_k,\ k \geqslant 1\}$ are points of discontinuity of $\hat{U}_\alpha(\cdot)$.

Since $\hat{W}$ and $\hat{U}_\alpha$ are independent, the last equation has at most one solution with probability 1.

Thus we have the following convergence on some probability space

$$\hat{X}_n \to \hat{X}_\infty \text{ as } n \to \infty \text{ in } \mathcal{D}([0,\infty)) \text{ a.s.,} \tag{14}$$

where $\hat{X}_n,\ \hat{X}_\infty$ are solutions of the reflection problems

$$\hat{X}_n(t) = \hat{S}_n(t) + \hat{F}_n(\hat{L}_n(t)),\ t \geqslant 0,$$
$$\hat{X}_\infty(t) = \hat{W}(t) + \hat{U}_\alpha(\hat{L}_\infty(t)),\ t \geqslant 0.$$

From Theorem 2.1,

$$\hat{X}_\infty(t) = \sigma\hat{W}(t) + \hat{U}_\alpha\big(\hat{U}_\alpha^{(-1)}(-\sigma \min_{s\in[0.t]} (\hat{W}(s) \wedge 0)\big),\ t \geqslant 0.$$

It can be easily seen that the processes $\big(\hat{U}_\alpha\big(\hat{U}_\alpha^{(-1)}(\sigma t)\big), t \geqslant 0\big)$ and $\big(\sigma\hat{U}_\alpha\big(\hat{U}_\alpha^{(-1)}(t)\big), t \geqslant 0\big)$ have the same distribution. Since $\hat{U}_\alpha$ and $\hat{W}$ are independent, the distribution of $\hat{X}_\infty(t), t \geq 0$, coincides with the distribution of

$$\sigma\left(\hat{W}(t) + \hat{U}_\alpha\big(\hat{U}_\alpha^{(-1)}(-\min_{s\in[0.t]} (\hat{W}(s) \wedge 0))\big)\right),\ t \geqslant 0.$$

Observe that

$$|\hat{X}_n\left(\frac{[nt]}{n}\right) - \hat{X}_n(t)| \leq \frac{1}{\sqrt{n}}.$$

So (14) yields the convergence

$$\hat{X}_n\left(\frac{[nt]}{n}\right) \to \hat{X}_\infty(t) \text{ as } n \to \infty, \quad \text{in } \mathcal{D}([0,\infty)) \text{ a.s.}$$

Hence, see (6),

$$\bar{X}_n\left(\frac{\lambda([nt])}{n}\right) \Rightarrow \hat{X}_\infty(t),\ n \to \infty, \quad \text{in } \mathcal{D}([0,\infty)). \tag{15}$$

The following statement can be easily proved from the definition of convergence in a Skorokhod space, cf. [20, § 14].

Proposition 2.2. *Let $\{f_n,\ n \geqslant 0\} \subset D([0,\infty))$. Assume that a sequence of non-negative non-decreasing cadlag functions $\{\lambda_n,\ n \geqslant 1\}$ is such that*

$$\forall T > 0 : \lim_{n\to\infty} \sup_{t\in[0,T]} |\lambda_n(t) - t| \to 0,$$

$$f_n(\lambda_n(t)) \to f(t), n \to \infty, \quad \text{in } D([0,\infty)).$$

Let $\{t_k^{(n)}\}$ be the set of jumps of λ_n, $u_k^{(n)} := \lambda_n(t_k^{(n)}-), v_k^{(n)} := \lambda_n(t_k^{(n)})$.
If for any $T > 0$

$$\sup_k \sup_{s\in[u_k^{(n)}, v_k^{(n)})\cap[0,T]} |f_n(s) - f_n(u_k^{(n)})| \to 0, n \to \infty, \tag{16}$$

then

$$f_n \to f, n \to \infty, \quad \text{in } D([0,\infty)).$$

Taking into account (15) and Proposition 2.2, to prove Theorem 1.1 it suffices to verify (7).

We have

$$\sup_{t\in[0,T]} \left| \frac{\lambda([nt])}{n} - t \right| \leqslant \frac{1}{n} \sum_{k\leqslant nT} 1\!\!1_{\bar{X}(k)=0} + \frac{1}{n}.$$

Observe that $\sum_{k\leqslant nT} 1\!\!1_{\bar{X}(k)=0} \leqslant L(nT)$ and a sequence $L_n(T) = L(nT)/k(n)$ is weakly convergent by Proposition 2.1. Since $n \sim a_n^\alpha \ell(a_n), n \to \infty$, [18, § XIII.6], where ℓ is a slowly varying function, and $\frac{a_{k(n)}}{\sqrt{n}} \to 1,\ n \to \infty$ (see (10)), we have $k(n)/n \to 0, n \to \infty$.

Therefore $L(nT)/n$ converges to 0 weakly and hence in probability. We have verified (7). Theorem 1.1 is proved.

3. Proof of Theorem 1.2

We will assume that all $x_n = 0$. The general case is considered similarly.

Let a random variable ξ have a distribution $X(\tau)$ given $X(0) = 0$:

$$\mathrm{P}(\xi = j) = \mathrm{P}(X(\tau) = j/X(0) = 0).$$

By τ_k denote the lengths of the excursion that starts at the instant of kth visit of X to the set $\{-m, -m+1, \ldots, 0\}$ and finishes at the instant of exit from this set.

Consider a process $\widetilde{X}$ constructed in §2.2. A copy of a process X can be constructed from a process $\widetilde{X}$ inserting excursions of lengths $\{\widetilde{\tau}_k\}$, where $\{\widetilde{\tau}_k\}$ are copies of $\{\tau_k\}$. This excursions describe behavior of X inside the set $\{-m, -m+1, \ldots, 0\}$. Denote the constructed copy of X by $\bar{X}$.

Remark 3.1. Random variables $\{\widetilde{\tau}_k\}$ are i.i.d., but they may depend on $\widetilde{X}$.

Similarly to reasoning of previous section we have representation (6) with

$$\lambda(t) := t + \sum_{k\leqslant t} 1\!\!1_{\bar{X}(k)\leqslant 0}.$$

Observe that $\sum_{k\leqslant nT} 1\!\!1_{\bar{X}(k)\leqslant 0} \leqslant \sum\limits_{k=1}^{L(nT)} \widetilde{\tau}_k$. Since all states of initial Markov chain are connected, the average time spent in the set $\{-m, -m+1, \ldots, 0\}$ until the exit from it is finite:

$$M := \mathrm{E}\widetilde{\tau}_k < \infty.$$

By the strong law of large numbers

$$\sum_{k=1}^{n} \widetilde{\tau}_k/n \to M,\ n \to \infty,\ \text{a.s.}$$

Since $L(nT)/n \to 0,\ n \to \infty$, in probability,

$$\sum_{k=1}^{L(nT)} \widetilde{\tau}_k/n \to 0, n \to \infty,$$

in probability.

Condition (16) of Proposition 2.2 is satisfied, the corresponding supremum does not exceed $(m+1)/\sqrt{n}$. The rest of the proof is the same as in Theorem 1.1.

References

1. J. M. Harrison and L. A. Shepp, *On skew Brownian motion*, Ann. Probab. **9(2)** (1981), 309–313.
2. R. A. Minlos and E. A. Zhizhina, *Limit diffusion process for a non-homogeneous random walk on a one-dimensional lattice*, Uspekhi Matem. Nauk. **52:2(314)** (1997), 87–100.
3. A. Yu. Pilipenko and Yu. E. Pryhodko, *Limit behavior of symmetric random walks with a membrane*, Teor. Imovir. Mat. Stat. **85** (2011), 84–94. English translation in: Theor. Probability and Math. Statist. **85** (2012), 93–105.
4. A. D. Wentzell, *Semigroups of operators that correspond to a generalized differential operator of second order* (in Russian), Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 269–272.
5. A. D. Wentzell, *On boundary conditions for multi-dimensional diffusion processes*, Theory Probab. Appl. **4** (1959), 164–177.
6. W. Feller, *Generalized second order differential operators and their lateral conditions*, Illinois J. Math. **1** (1957), 459–504.
7. T. Ueno, *The diffusion satisfying Wentzell's boundary condition and the Markov process on the boundary. I, II*, Proc. Japan Acad. **36** (1960), 533–538, 625–629.
8. V. M. Surenkov, *Boundary conditions and an ergodic theorem for processes with independent increments* (Russian. English summary), Teor. Veroyatnost. i Primenen. **24** (1979), no. 1, 52–61.
9. S. V. Anulova, *On stochastic differential equations with boundary conditions in a half-plane*, Izv. AN SSSR Ser. Mat. **45** (1981), no. 3, 491–508.
10. B. Grigelionis and R. Mikulevicius, *On weak convergence to random processes with boundary conditions*, Lecture Notes in Math. **972** (1982), 260–275.
11. Y. Ishikawa, *A remark on the existence of a diffusion process with nonlocal boundary conditions*, J. Math. Soc. Japan **42** (1990), no. 1, 171–184.
12. P. L. Gurevich, *On the existence of a Feller semigroup with atomic measure in a nonlocal boundary condition*, (in Russian. Russian summary) Tr. Mat. Inst. Steklova **260** (2008), Teor. Funkts. i Nelinein. Uravn. v Chastn. Proizvodn., 164–179.
13. S. Watanabe, *Ito's theory of excursion point processes and its developments*, Stochastic Process. Appl. **120** (2010), no. 5, 653–677.
14. B. I. Kopytko and R. V. Shevchuk, *On pasting together two inhomogeneous diffusion processes on a line with the general Feller-Wentzell conjugation condition*, Theory of Stochastic Processes **17(33)** (2011), no. 2, 55–70.
15. P. P. Kononchuk and B. I. Kopitko, *Semigroups of operators that describe a Feller process on the line, which is the result of gluing together two diffusion processes* (in Ukrainian), Teor. Imovir. Mat. Stat. **84** (2011), 84–93. English translation in: Theory Probab. Math. Statist., **84** (2012), 87–97.
16. A. V. Skorokhod, *Limit theorems for stochastic processes with independent increments* (in Russian), Teor. Veroyatnost. i Primenen. **2** (1957), no. 2, 145–177.
17. A. Yu. Pilipenko, *On Skorokhod-type reflection map for equations with a possibility of jump exit from a boundary*, Ukrainian mathematical journal **63** (2011), no. 9, 1241–1256.
18. W. Feller, *An introduction to probability theory and its applications. Vol. II*, John Wiley and Sons, Inc., New York – London – Sydney, 1966.
19. A. V. Skorokhod, *Stochastic equations for diffusion processes with boundaries. I*, Teor. Veroyatn. Primen. **6** (1961), no. 3, 287–298.
20. P. Billingsley, *Convergence of Probability Measures*, New York: John Wiley & Sons, Inc., 1999
21. A. V. Skorokhod, *Studies in the Theory of Random Processes* (in Russian), Kiev University, Kiev, 1961. English translation in: Addison-Wesley Publ. Co., Inc., Reading. Mass., 1965.

Institute of Mathematics, National Academy of Sciences of Ukraine, Tereshchenkivska str. 3, 01601, Kiev, Ukraine
E-mail address: apilip@imath.kiev.ua

NTUU "KPI", 37, Prospect Peremohy, 03056, Kyiv-56, Ukraine
E-mail address: prykhodko@matan.kpi.ua