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ON ORNSHTEIN-UHLENBECK'S MEASURE OF A HILBERT BALL IN THE SPACE OF CONTINUOUS FUNCTIONS

An explicit formula for the characteristic function of the L_2-norm of a path of the Ornshtein-Uhlenbeck process is established and some application of the result is given.

1. Introduction

In 1944, R. H. Cameron and W. R. Martin obtained an explicit formula for the characteristic function of the L_2-norm of a path of the Wiener process (see [1]). The aim of this paper is to give an analogous formula in the case of the Ornshtein-Uhlenbeck process. Besides, we give some application of our result.

2. Cameron-Martin's result

Let $(w(t))_{t\geq 0}$ be a standard Wiener process on a real line $\mathbb{R}$ for which $w(0) = 0$. The correlation function of this process is given by

$$\mathbb{E}w(s)w(t) = s \wedge t, \quad s \geq 0,\ t \geq 0.$$

Denote by $L_2[0,1]$ the Hilbert space of all measurable square integrable functions with real values defined on the interval $[0,1]$. It is well known that the functions

$$\left(\sqrt{2}\sin\left(\left(k-\frac{1}{2}\right)\pi t\right)\right)_{t\in[0,1]}, \quad k = 1, 2, \dots,$$

form an orthonormal basis in $L_2[0,1]$, and the following relation

$$\int_0^1 (s\wedge t)\sin\left(\left(k-\frac{1}{2}\right)\pi t\right)\,dt = \frac{4}{(2k-1)^2\pi^2}\sin\left(\left(k-\frac{1}{2}\right)\pi s\right)$$

is valid for all $s \in [0,1]$ and $k = 1, 2, \dots$. This implies that the Fourier coefficients of the Wiener process on this basis

$$\eta_k = \int_0^1 w(t)\sqrt{2}\sin\left(\left(k-\frac{1}{2}\right)\pi t\right)\,dt, \quad k = 1, 2, \dots,$$

form a sequence of indepedent normal random variables with $\mathbb{E}\eta_k = 0$ and $\mathbb{E}\eta_k^2 = 4/(2k-1)^2\pi^2$. Taking into account Parseval's identity, we arrive at the relation

$$\int_0^1 w^2(t)\,dt = \sum_{k=1}^{\infty}\eta_k^2 \tag{1}$$

valid almost surely (one can easily verify that the series on the right hand side of (1) is convergent almost surely).

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A very simple calculation shows that the following formula

$$\mathbb{E}\exp\left\{\theta\eta_k^2\right\} = \left(1 - \frac{8\theta}{(2k-1)^2\pi^2}\right)^{-\frac{1}{2}}$$

holds true for any real number θ satisfying the inequality $\theta < (2k-1)^2\pi^2/8$. Therefore, if $\theta < \pi^2/8$, then the relation

$$\mathbb{E}\exp\left\{\theta\int_0^1 w^2(t)\,dt\right\} = \left(\prod_{k=1}^{\infty}\left(1 - \frac{8\theta}{(2k-1)^2\pi^2}\right)\right)^{-\frac{1}{2}}$$

holds true. Now, making use of the formula

$$\cos z = \prod_{k=1}^{\infty}\left(1 - \frac{4z^2}{(2k-1)^2\pi^2}\right)$$

valid for an arbitrary complex number z other than any number of the set

$$\left\{\left(k + \frac{1}{2}\right)\pi : \, k = 0, \pm1, \pm2, \dots\right\},$$

we arrive at the equality

$$\mathbb{E}\exp\left\{\theta\int_0^1 w^2(s)\,ds\right\} = \frac{1}{\sqrt{\cos\sqrt{2\theta}}}, \quad \theta < \frac{\pi^2}{8}. \tag{2}$$

This is the Cameron-Martin formula (see [1] and also [2], Chapter II, §12).

Remark 2.1. Denote by $C[0,1]$ the set of all real-valued continuous functions defined on the interval $[0,1]$, and for $r > 0$ we put

$$B_r = \left\{x(\cdot) \in C[0,1] : \, x(0) = 0, \, \int_0^1 x^2(s)\,ds < r\right\}.$$

This is a Hilbert ball of radius $\sqrt{r}$ in the space of continuous functions starting from the origin. For $r > 0$, denote by $F(r)$ the value of the Wiener measure on the set B_r. Then the formula (2) can be rewritten in the following form

$$\int_0^{\infty} e^{-\theta r}\,dF(r) = \frac{1}{\sqrt{\cosh\sqrt{2\theta}}}, \quad \theta \geq 0. \tag{3}$$

3. The Ornshtein-Uhlenbeck process

For a fixed parameter $\rho > 0$, we put

$$x(t) = e^{-\rho t}\int_0^t e^{\rho s}\,dw(s), \quad t \geq 0, \tag{4}$$

where the integral on the right hand side of this equality is the Wiener integral

$$\int_0^t e^{\rho s}\,dw(s) = e^{\rho t}w(t) - \rho\int_0^t e^{\rho s}w(s)\,ds, \quad t \geq 0.$$

The stochastic process defined by (4) is called the Ornshtein-Uhlenbeck process starting from the origin ($x(0) = 0$).

Denote by $(K(s,t))_{s\geq 0,\, t\geq 0}$ the correlation function of this process. It can be easily calculated, namely,

$$K(s,t) = \frac{1}{\rho}\exp\{-\rho(s\vee t)\}\sinh(\rho(s\wedge t)), \quad s\geq 0,\, t\geq 0.$$

For $k=1,2,\dots$, denote by μ_k the unique root of the equation $\tan\mu = -\mu/\rho$ on the interval $\left(\left(k-\frac{1}{2}\right)\pi, k\pi\right)$ and put

$$\lambda_k = \frac{1}{\rho^2+\mu_k^2}, \quad \varphi_k(t) = \frac{1}{\varkappa_k}\sin(\mu_k t), \quad t\in[0,1],$$

where

$$\varkappa_k^2 = \int_0^1 \sin^2(\mu_k t)\,dt = \frac{1}{2}\frac{\rho^2+\rho+\mu_k^2}{\rho^2+\mu_k^2}.$$

The following relations

$$\int_0^1 K(s,t)\varphi_k(t)\,dt = \lambda_k\varphi_k(s), \quad s\in[0,1], \quad k=1,2,\dots,$$

can be established by very simple calculations. Moreover, one can verify that the functions $(\varphi_k(t))_{t\in[0,1]}$, $k=1,2,\dots$, form an orthonormal basis in the space $L_2[0,1]$ (the property of this system to be complete is a simple consequence of the theorem proved in Section 55, Chapter V of [4]).

Now, as in Section 2, we can assert that the Fourier coefficients

$$\xi_k = \int_0^1 x(t)\varphi_k(t)\,dt, \quad k=1,2,\dots,$$

of the Ornshtein-Uhlenbeck process on this basis form a sequence of independent normally distributed random variables for which $\mathbb{E}\xi_k = 0$, $\mathbb{E}\xi_k^2 = \lambda_k$. Again, making use of Parseval's identity, we get

$$\int_0^1 x^2(s)\,ds = \sum_{k=1}^{\infty}\xi_k^2, \tag{5}$$

where, as above, the series on the right hand side is convergent almost surely. Since the formula

$$\mathbb{E}\exp\left\{\theta\xi_k^2\right\} = (1-2\theta\lambda_k)^{-1/2} = \left(1-\frac{2\theta}{\rho^2+\mu_k^2}\right)^{-1/2}$$

holds true for any real number $\theta < 1/(2\lambda_k) = (\rho^2+\mu_k^2)/2$, the equality (5) implies the following relation

$$\mathbb{E}\exp\left\{\theta\int_0^1 x^2(s)\,ds\right\} = \left[\prod_{k=1}^{\infty}\left(1-\frac{2\theta}{\rho^2+\mu_k^2}\right)\right]^{-1/2} \tag{6}$$

valid for all $\theta < (\rho^2+\mu_1^2)/2$.

Note that μ_k for $k\geq 1$ is a function of ρ, hence, the right hand side of (6) is a function of $\rho>0$ and $\theta < (\rho^2+\mu_1^2)/2$; we denote it by $\Phi(\rho,\theta)$. An explicit formula will be found out for this function in the next section.

4. Calculating the function Φ

Taking the logarithmic derivative of Φ in the argument θ, we get

$$\frac{\partial}{\partial\theta}\ln\Phi(\rho,\theta)=\sum_{k=1}^{\infty}\frac{1}{\rho^2+\mu_k^2-2\theta}. \tag{7}$$

In what follows, we are trying to represent the series (7) as the sum of terms being equal to the products of the Fourier coefficients (in the basis $(\varphi_k)_{k\geq 1}$) of two appropriate functions.

We note, first of all, that an antiderivative of the function $\Big(\sinh(t\sqrt{\rho^2-2\theta})\sin\mu_k t\Big)_{t\in\mathbb{R}}$ can be easily calculated. This fact and the relation $\rho\sin\mu_k+\mu_k\cos\mu_k=0$ (see the definition of the number μ_k) allow us to write down the following equality

$$\int_0^1\frac{\sinh(t\sqrt{\rho^2-2\theta})}{V(\rho,\theta)}\varphi_k(t)\,dt=\frac{\sin\mu_k}{\varkappa_k(\rho^2+\mu_k^2-2\theta)} \tag{8}$$

valid for $\rho>0$, $\theta<\rho^2/2$ and $k=1,2,\ldots$, where we put

$$V(\rho,\theta)=\sqrt{\rho^2-2\theta}\cosh\sqrt{\rho^2-2\theta}+\rho\sinh\sqrt{\rho^2-2\theta}.$$

It thus remains to find out such a function $(h(\rho,t))_{t\in[0,1]}$ that satisfies the relation

$$\int_0^1 h(\rho,t)\varphi_k(t)\,dt=\frac{\varkappa_k}{\sin\mu_k}$$

for all $\rho>0$ and $k=1,2,\ldots$. Rewrite this relation in the form

$$\int_0^1 h(\rho,t)\sin\mu_k t\,dt=\frac{\varkappa_k^2}{\sin\mu_k}=\frac{\varkappa_k^2\sin\mu_k}{\mu_k^2}(\rho^2+\mu_k^2),$$

where the equality $\sin^2\mu_k=\mu_k^2/(\rho^2+\mu_k^2)$ has just been used. Substituting into the right hand side of this relation instead of $\varkappa_k$ its expression in terms of ρ and μ_k (see above), we arrive at the conclusion that the function h must satisfy the condition ($\rho>0$, $k=1,2,\ldots$)

$$\int_0^1 h(\rho,t)\sin\mu_k t\,dt=\frac{1}{2}\sin\mu_k+\frac{1}{2}\rho(\rho+1)\frac{\sin\mu_k}{\mu_k^2}. \tag{9}$$

It is clear that the function h should be equal to the sum of two functions: h_1 and h_2 for which

$$\int_0^1 h_1(\rho,t)\sin\mu_k t\,dt=\frac{1}{2}\sin\mu_k,$$

$$\int_0^1 h_2(\rho,t)\sin\mu_k t\,dt=\frac{1}{2}\rho(\rho+1)\frac{\sin\mu_k}{\mu_k^2}$$

for all $\rho>0$ and $k=1,2,\ldots$. The first of these relations means that h_1 coincides (within the multiplicator $1/2$) with the Dirac δ-function $(\delta_1(t))_{t\in[0,1]}$ concentrated at the point

$t = 1$. In order to find out the function h_2, let us calculate the integral (using the relation $\rho \sin \mu_k + \mu_k \cos \mu_k = 0$)

$$\int_0^1 t \sin \mu_k t dt = -\frac{1}{\mu_k} \cos \mu_k + \frac{\sin \mu_k}{\mu_k^2} = (\rho + 1)\frac{\sin \mu_k}{\mu_k^2}.$$

It follows from this that $h_2(\rho, t) = \rho t/2$ for $t \in [0, 1]$ and $\rho > 0$. We have thus found out the function h satisfying the equality (9) for all $\rho > 0$ and $k = 1, 2, \dots$, namely

$$h(\rho, t) = \frac{1}{2}\delta_1(t) + \frac{1}{2}\rho t, \quad t \in [0, 1].$$

Note that the function δ_1 does not belong to $L_2[0, 1]$. Nevertheless, it is possible to write down the Parseval identity for it and the function

$$\left(\frac{\sinh(t\sqrt{\rho^2 - 2\theta})}{V(\rho, \theta)}\right)_{t\in[0,1]}$$

because the Fourier coefficients of the latter function form an absolutely convergent series as the formula (8) shows (we remind that $\mu_k \in \left(\left(k - \frac{1}{2}\right)\pi, k\pi\right)$). So we get

$$\sum_{k=1}^{\infty} \frac{1}{\rho^2 + \mu_k^2 - 2\theta} = \int_0^1 \frac{\sinh(t\sqrt{\rho^2 - 2\theta})}{V(\rho, \theta)} \left[\frac{1}{2}\delta_1(t) + \frac{1}{2}\rho t\right] dt. \tag{10}$$

Calculating the integral here, we arrive at the formula

$$\frac{\partial}{\partial \theta} \ln \Phi(\rho, \theta) = \frac{1}{2V(\rho, \theta)} \left[\frac{\rho^2 - \rho - 2\theta}{\rho^2 - 2\theta} \sinh \sqrt{\rho^2 - 2\theta} + \frac{\rho}{\sqrt{\rho^2 - 2\theta}} \cosh \sqrt{\rho^2 - 2\theta}\right].$$

Note that the expression on the right hand side of this formula coincides with the partial derivative in the argument θ of the function

$$\ln \left[e^{\rho/2} \left(\cosh \sqrt{\rho^2 - 2\theta} + \frac{\rho}{\sqrt{\rho^2 - 2\theta}} \sinh \sqrt{\rho^2 - 2\theta}\right)^{-1/2}\right],$$

and since the value of this function at the point $\theta = 0$ is equal to zero, we can write down the final formula

$$\Phi(\rho, \theta) = e^{\rho/2} \left(\cosh \sqrt{\rho^2 - 2\theta} + \frac{\rho}{\sqrt{\rho^2 - 2\theta}} \sinh \sqrt{\rho^2 - 2\theta}\right)^{-1/2} \tag{11}$$

that holds true for all $\rho > 0$ and $\theta < \rho^2/2$.

The arguments that led us to (11) remain applicable also in the case $\theta \geq \rho^2/2$, but $\theta < (\rho^2 + \mu_1^2)/2$; the corresponding formula can be written as follows

$$\Phi(\rho, \theta) = e^{\rho/2} \left(\cos \sqrt{2\theta - \rho^2} + \frac{\rho}{\sqrt{2\theta - \rho^2}} \sin \sqrt{2\theta - \rho^2}\right)^{-1/2}. \tag{12}$$

The Ornshtein-Uhlenbeck process generates on the space $C[0, 1]$ a probabilistic measure that is called the Ornshtein-Uhlenbeck measure. If we denote by $F_\rho(r)$ the value of this measure on the Hilbert ball B_r for $r > 0$ (see above), then we can write down the following formula

$$\mathbb{E}\exp\left\{-\theta\int\limits_0^1 x^2(s)\,ds\right\} = \int\limits_0^\infty e^{-\theta r}\,dF_\rho(r) = \tag{13}$$

$$= e^{\rho/2}\left(\cosh\sqrt{\rho^2+2\theta} + \frac{\rho}{\sqrt{\rho^2+2\theta}}\sinh\sqrt{\rho^2+2\theta}\right)^{-1/2}$$

valid for all $\theta \geq 0$.

Note that the Wiener measure is a limitting one for the Ornshtein-Uhlenbeck measures, as $\rho \to 0+$. It is easily seen that when passing to the limit, as $\rho \to 0+$, in formulae (12) and (13), we get the formulae (2) and (3) respectively.

5. An application

The Ornshtein-Uhlenbeck process defined above by (4) is such a solution to the stochastic differential equation

$$dx(t) = -\rho x(t)\,dt + dw(t)$$

that satisfies the initial condition $x(0) = 0$. We put

$$\mathcal{E}(\rho) = \exp\left\{-\rho\int_0^1 w(s)\,dw(s) - \frac{\rho^2}{2}\int\limits_0^1 w^2(s)\,ds\right\},$$

where the first integral on the right hand side is the Itô stochastic integral.

There are problems (see, for example, [3]), where the necessity to have a formula for $\mathbb{E}\mathcal{E}^2(\rho)$ arises. We will show that the formula (11) can be very useful in such a situation.

Note that

$$\mathcal{E}^2(\rho) = \mathcal{E}(2\rho)\exp\left\{\rho^2\int\limits_0^1 w^2(s)\,ds\right\}, \quad \rho > 0.$$

Girsanov's theorem now implies the equality

$$\mathbb{E}\mathcal{E}^2(\rho) = \Phi(2\rho, \rho^2).$$

Taking into account (11), we arrive at the formula

$$\mathbb{E}\mathcal{E}^2(\rho) = e^\rho\left(\cosh\rho\sqrt{2} + \sqrt{2}\sinh\rho\sqrt{2}\right)^{-1/2}.$$

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