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ON THE LOCAL TIMES FOR GAUSSIAN INTEGRATORS

For the Gaussian integrators with values in $\mathbb{R}$ and $\mathbb{R}^2$ the properties of the local time is investigated in terms of the operator which determines the geometry of covariance function. The explicit formula for the modulus of continuity of Gaussian integrators is obtained.

1. Introduction

In this article we study the local times for Gaussian integrators. Such class of processes appeared in the work [1] and was named as integrators since every function from $L_2([0;1])$ can be integrated over the random processes from this class. Also in [1] the extended stochastic integral with respect to integrators is constructed and the Itô formula for it is obtained. It was proved in [2] that the result of acting of a second quantization operator on the smooth function of the diffusion process satisfies the partial stochastic differential equation with the extended stochastic integral over integrator.

Definition 1.1. [1] If for the random process $x(t),\ t\in[0;1]$ there exists $c>0$ such that for any $n\geq 1,\ a_0,\dots,a_{n-1}\in\mathbb{R},\ 0=t_0<t_1<\dots<t_n=1$ the following relation holds

$$E\Big(\sum_{k=0}^{n-1}a_k(x(t_{k+1})-x(t_k))\Big)^2\leq c\sum_{k=0}^{n-1}a_k^2(t_{k+1}-t_k), \tag{1}$$

then the process x is said to be an integrator.

The following statement describes all Gaussian integrators.

Lemma 1.1. *The Gaussian process x is an integrator iff it can be represented as*

$$x(t)=(A1\!\!1_{[0;t]},\xi),\ t\in[0;1] \tag{2}$$

with some continuous linear operator A in $L_2([0;1])$ and a certain Gaussian white noise in the same space.

Proof. Suppose that the Gaussian process x has a representation (2), then

$$E\Big(\sum_{k=0}^{n-1}a_k(x(t_{k+1})-x(t_k))\Big)^2=E\Big(\sum_{k=0}^{n-1}a_k(A1\!\!1_{[t_k;t_{k+1}]},\xi)\Big)^2=$$
$$=\Big\|A\sum_{k=0}^{n-1}a_k1\!\!1_{[t_k;t_{k+1}]}\Big\|^2\leq\|A\|^2\sum_{k=0}^{n-1}a_k^2(t_{k+1}-t_k).$$

Inversely, suppose that x is an integrator. Denote by $\overline{LS\{x\}}$ the closure of the linear span of values of x with respect to square mean norm. $\overline{LS\{x\}}$ is a separable Hilbert space. There exists the subspace H_1 of $L_2([0;1])$ which is isomorphic to $\overline{LS\{x\}}$. Denote

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by $H_1 = j(\overline{LS\{x\}})$, where $j : \overline{LS\{x\}} \to H_1$ is isomorphism. Then $L_2([0;1])$ can be represented as a direct sum $H_1 \oplus H_2$. Suppose that ξ_2 is a Gaussian white noise in H_2, which is independent of x. Put $(h_1, \xi_1) = j^{-1}(h_1)$ and $(h, \xi) = (h_1, \xi_1) + (h_2, \xi_2),\ h_1 \in H_1,\ h_2 \in H_2,\ h \in L_2([0;1])$. The independence of ξ_2 and x implies that ξ is a white noise in $L_2([0;1])$. Define a linear operator $B : L_2([0;1]) \to \overline{LS\{x\}}$ by the rule

$$B\Big(\sum_{k=0}^{n-1} a_k 1\!\!1_{[t_k;t_{k+1}]}\Big) \mapsto \sum_{k=0}^{n-1} a_k(x(t_{k+1}) - x(t_k)).$$

Then for an operator $A = jB$

$$B1\!\!1_{[0;t]} = x(t) = (jx(t), \xi) = (A1\!\!1_{[0;t]}, \xi).$$

Since x satisfies (1) for any step function $f \in L_2[0;1]$ the next inequality holds

$$\|Af\|^2 \le c\|f\|^2. \tag{3}$$

The set of all step functions on $[0;1]$ is dense in $L_2([0;1])$. Consequently (3) ends the proof. □

It is not difficult to check that the Wiener process, the Brownian bridge, the Ornstein–Unlenbeck process are integrators. Let us check whether the fractional Brownian motion $B^\alpha_t,\ \alpha \in (0;1),\ t \in [0;1]$ is an integrator. Recall the definition. A zero mean Gaussian process with the $\mathrm{cov}(B^\alpha_{t_1}, B^\alpha_{t_2}) = \frac{1}{2}(t_1^{2\alpha} + t_2^{2\alpha} - (t_2 - t_1)^{2\alpha})$ is said to be a fractional Brownian motion, where α is a real number in $(0;1)$, called the Hurst index or Hurst parameter associated with the fractional Brownian motion. If B^α_t were integrator, then $c > 0$ would exist such that

$$E(B^\alpha_{t_2} - B^\alpha_{t_1})^2 = (t_2 - t_1)^{2\alpha} \le c(t_2 - t_1).$$

For $\alpha < \frac{1}{2}$

$$\lim_{t_2 - t_1 \to 0} (t_2 - t_1)^{2\alpha - 1} = +\infty. \tag{4}$$

(4) implies that for $\alpha < \frac{1}{2}$ the fractional Brownian motion B^α_t is not an integrator. In the case of $\alpha = \frac{1}{2}$ the process B^α_t as a standard Brownian motion is an integrator. Let us investigate the case $\alpha > \frac{1}{2}$. To do that we will use the following obvious statement.

Lemma 1.2. *The process x is an integrator iff there exists $c > 0$ such that for any continuous two times differentiable function f on $[0;1]$ with $f(0) = f(1) = 0$ the following relation holds*

$$E\Big(\int_0^1 x(t)f'(t)dt\Big)^2 \le c\int_0^1 f^2(t)dt. \tag{5}$$

Let us check that for $\alpha > \frac{1}{2}$ the process B^α_t satisfies the inequality (5). Really,

$$E\Big(\int_0^1 B^\alpha_t f'(t)dt\Big)^2 = -\frac{1}{2}\int_0^1\int_0^1 f'(t_1)f'(t_2)(t_2 - t_1)^{2\alpha}dt_2dt_1 =$$

$$= -2\alpha(2\alpha - 1)\int_0^1 f(t_1)\int_{t_1}^1 f(t_2)(t_2 - t_1)^{2\alpha - 2}dt_2dt_1.$$

By using the Shur test [3] one can see that for $\alpha > \frac{1}{2}$ the integral operator in $L_2([0;1])$ with the kernel $K(t_1, t_2) = (t_2 - t_1)^{2\alpha - 2}1\!\!1_{\{t_2 > t_1\}}$ is bounded. It implies that for $\alpha > \frac{1}{2}$ the fractional Brownian motion B^α_t is an integrator. In the partial case when $A = I + S$ with the identity operator I and the compact operator S the planar integrators were studied in [4–7]. The renormalization for their self-intersection local times was proposed. The main feature of the integrators with $I + S$ representation is $E(x(t_2) - x(t_1))^2 \sim t_2 - t_1,\ t_2 - t_1 \to 0$. It can be verified that for $\alpha > \frac{1}{2}$ the fractional Brownian motion

B_t^α does not have a representation with $A = I + S$. Really, if the operator A in the representation (2) for the process $B_t^\alpha,\ \alpha > \frac{1}{2}$ has the form $I + S$, then the compactness of the operator S implies that

$$\lim_{t_2 - t_1 \to 0} \frac{E(B_{t_2}^\alpha - B_{t_1}^\alpha)^2}{t_2 - t_1} = 1$$

(see [4]). Since for $\alpha > \frac{1}{2}$

$$\frac{E(B_{t_2}^\alpha - B_{t_1}^\alpha)^2}{t_2 - t_1} = (t_2 - t_1)^{2\alpha - 1} \to 0,\ t_2 - t_1 \to 0,$$

then the operator A does not have the form $I + S$. If A is the multiplication operator in $L_2([0;1])$ on the bounded measurable function χ, then the corresponding integrator x can be considered as a certain Wiener process β with the change of time, i.e.

$$x(t) = \beta\Big(\int_0^t \chi(s)^2 ds\Big),\ t \in [0;1].$$

The present paper consists of two parts. In the first part we investigate the geometry of one dimensional integrators. Under some condition on the operator A we will prove the existence of the local time. The existence of the local time for Gaussian processes is connected with the local nondeterminism property introduced by S. Berman in [8]. This property means that the finite sets of increments of the process have uniformly nondegenerate distributions. One can expect that the integrator with a continuously invertible A will have the local nondeterminism property. We will check this statement in the article. Also we present an explicit formula for the modulus of continuity of Gaussian integrators in terms of the operator A. The second part of the paper is devoted to the self-intersection local times for planar integrators. We will check that with positive probability the trajectories of planar integrator have double self-intersections. Also we will prove the existence of self-intersection local time for planar Gaussian integrator on $\Delta_2^\delta = \{0 \le t_1 \le t_2 \le 0,\ t_2 - t_1 \ge \delta\},\ \delta > 0$. Finally we investigate the dependence between the growth of the modulus of continuity of planar integrator and its approximations of double self-intersection local time. We will prove that the smoother process the lesser the number of points of self-intersections.

2. The Geometry of trajectories of Gaussian integrators in one dimension. Local time, local nondeterminism, modulus of continuity

In this section we investigate the connection between the geometry of a covariance function of Gaussian integrator in one dimension and the existence of its local time. To do that we use the following two approaches. The first approach belongs to J. Rosen [9] and is based on the square integrability of the Fourier transform of the occupation measure. The second approach is related to the Berman local nondeterminism property [8]. Suppose that x is one dimensional integrator with the representation (2). Consider its occupation measure μ on $\mathbb{R}$ defined by the formula $\mu(E) = \int_0^1 1\!\!\mathrm{I}_E(x(s))ds$. Denote by λ_n the Lebesgue measure on $\mathbb{R}^n$.

Definition 2.1. If $\mu \ll \lambda_1$, then the Radon-Nikodym derivative $\frac{d\mu}{d\lambda_1}$ is said to be a local time of x on $[0;1]$.

Put $\alpha(u) := \frac{d\mu}{d\lambda_1}(u)$ if it exists. Then

$$\int_0^1 f(x(t))dt = \int_{\mathbb{R}} f(u)\alpha(u)du \tag{6}$$

for all bounded Borel functions f.

Theorem 2.1. *Suppose that the operator A in the representation* (2) *is continuously invertible, then x has a local time α on* $[0;1]$ *and* $\alpha \in L_2(\mathbb{R}, d\lambda_1)$ *a.s.*

In the paper under a continuously invertible operator in the Hilbert space we mean a bijective bounded linear operator in the Hilbert space. In this case the Banach theorem guarantees the existence of continuous A^{-1}.

Proof. To prove the theorem it suffices to check that

$$E\int_{\mathbb{R}} |\widehat{\mu}(z)|^2 dz < +\infty,$$

where as usual

$$\widehat{\mu}(z) = \int_{\mathbb{R}} e^{izv}\mu(dv).$$

It follows from the relation (6) that

$$\begin{aligned} &E\int_{\mathbb{R}} |\widehat{\mu}(z)|^2 dz = \\ &= 2\int_{\mathbb{R}}\int_{\Delta_2} Ee^{iz(x(t_2)-x(t_1))} d\vec{t}dz, \end{aligned} \tag{7}$$

where $d\vec{t} = dt_1dt_2,\ \Delta_2 = \{0 \le t_1 \le t_2 \le 1\}$. The continuous invertibility of A implies that

$$E(x(t_2) - x(t_1))^2 = \|A1\!\!1_{[t_1;t_2]}\|^2 \ge c(t_2 - t_1),\ c > 0. \tag{8}$$

It follows from (8) that (7) less or equal to

$$2\int_{\mathbb{R}}\int_{\Delta_2} e^{-\frac{1}{2}cz^2(t_2-t_1)} d\vec{t}dz < +\infty.$$

□

Further in this section we discuss the local nondeterminism property for one dimensional integrators. Recall the definition. Let $\{y(t),\ t \in J\}$ be $\mathbb{R}$-valued zero mean Gaussian process on an open interval J. Suppose that there exists $d > 0$ such that

1) $E(y(t) - y(s))^2 > 0$, for all $s, t \in J:\ 0 \le |t - s| \le d$;

2) $Ey^2(t) > 0$ for all $t \in J$.

For $m \ge 2,\ t_1, \ldots, t_m \in J,\ t_1 < t_2 < \ldots < t_m$ put

$$V_m = \frac{\mathrm{Var}((y(t_m) - y(t_{m-1})/y(t_1), \ldots, y(t_{m-1}))}{\mathrm{Var}(y(t_m) - y(t_{m-1}))}.$$

Definition 2.2. [8] A Gaussian process y is said to be a locally nondetermined on J if for every $m \ge 2$

$$\lim_{c\to 0}\ \inf_{t_m - t_1 \le c} V_m > 0.$$

Roughly speaking, the property of local nondeterminism means that small increment of the process uniformly is not linearly depended on the increments from the "past". The following statement was proved in [8] and demonstrates that such uniform nondegeneracy of distributions of increments is one of the sufficient conditions for existence and smoothness of the local time for general Gaussian process.

Theorem 2.2. [8] *Let $y(t),\ t \in [0;T]$, be a zero mean Gaussian process satisfying the following three conditions:*

1) $y(0) = 0$ a.s.;

2) y is locally nondetermined on $(0;T)$;

3) there exist positive real numbers γ, δ *and a continuous even function* $b(t)$ *such that* $b(0) = 0,\ b(t) > 0,\ t \in (0; T]$,

$$\lim_{h \to 0} h^{-\gamma} \int_0^h [b(t)]^{-1-2\delta} dt = 0$$

and $M(y(t)-y(s))^2 \geq b^2(t-s)$, *for all* $s, t \in [0; T]$. *Then there exists a version* $\alpha(u, t),\ u \in \mathbb{R},\ t \in [0; T]$ *of the local time of the the process* y *which is jointly continuous in* (u, t) *and which satisfying the Hölder condition in* t *uniformly in* u, *i.e. for every* $\gamma' < \gamma$ *there exist positive and finite random variables* η' *and* η *such that*

$$\sup_u |\alpha(u, t+h) - \alpha(u, t)| \leq \eta' |h|^{\gamma'}$$

for all $s, t, t+h \in [0; T]$ *and all* $|h| < \eta$.

To answer the question does a Gaussian integrator x satisfy conditions 1)–3) of the theorem 2.2 we will need the following reformulation of the notion of local nondeterminism. Denote by $G(e_1, \ldots, e_n)$ the Gram determinant constructed by the vectors $e_1, \ldots, e_n$. Let $g \in C([0; 1], L_2([0; 1])),\ \Delta g(t_i) = g(t_{i+1}) - g(t_i),\ i = 1, \ldots, m-1$. It is not difficult to check that the following statement holds (see [4]).

Lemma 2.1. *The Gaussian process* $y(t) = (g(t), \xi)$, *where* ξ *is a white noise in* $L_2([0; 1])$ *is a locally nondetermined on* J *iff for every* $m \geq 2$

$$\lim_{c \to 0} \inf_{t_m - t_1 \leq c} \frac{G(g(t_1), \Delta g(t_2), \ldots, \Delta g(t_{m-1}))}{\|g(t_1)\|^2 \|\Delta g(t_2)\|^2 \ldots \|\Delta g(t_{m-1})\|^2} > 0.$$

By using the lemma 2.1 let us prove the following statement.

Theorem 2.3. *Suppose that the operator* A *in the representation* (2) *is continuously invertible, then there exists a version* $\alpha(u, t),\ u \in \mathbb{R},\ t \in [0; T]$ *of the local time for the integrator* (2) *which is jointly continuous in* (u, t) *and which satisfying the Hölder condition in* t *uniformly in* u, *i.e. for every* $\gamma' < \frac{1}{2}$ *there exist positive and finite random variables* η' *and* η *such that*

$$\sup_u |\alpha(u, t+h) - \alpha(u, t)| \leq \eta' |h|^{\gamma'}$$

for all $s, t, t+h \in [0; T]$ *and all* $|h| < \eta$.

To prove the theorem 2.3 we will need the following statement which is interesting by itself.

Lemma 2.2. *Suppose that* A *is a continuously invertible operator in the Hilbert space* H. *Then for all* $k \geq 1$ *there exists a positive constant* $c(k)$ *which depends on* k *such that for any* $e_1, \ldots, e_k \in H$ *the following relation holds*

$$G(Ae_1, \ldots, Ae_k) \geq c(k) G(e_1, \ldots, e_k).$$

Proof. For $e_1, \ldots, e_k \in H$ let $f_1, \ldots, f_k$ be an orthonormal system which is obtained from $e_1, \ldots, e_k$ via the Gram-Schmidt orthogonalization procedure. To prove the statement of lemma it suffices to check that

$$\inf G\left(\frac{Af_1}{\|Af_1\|}, \ldots, \frac{Af_k}{\|Af_k\|}\right) > 0,$$

where infimum is taking over all orthonormal systems $(f_1, \ldots, f_k)$. Again by using the Gram-Schmidt orthogonalization procedure consider the orthogonal system $q_1, \ldots, q_k$ which is obtained from $\frac{Af_1}{\|Af_1\|}, \ldots, \frac{Af_k}{\|Af_k\|}$, where

$$q_j = \frac{Af_j}{\|Af_j\|} - \sum_{i=1}^{j-1} \alpha_{ij} \frac{Af_i}{\|Af_i\|}$$

with some α_{ij}.

Let us prove that

$$\inf_{(f_1,\ldots,f_k)} G\Big(\frac{Af_1}{\|Af_1\|},\ldots,\frac{Af_k}{\|Af_k\|}\Big) = \inf_{(f_1,\ldots,f_k)} \prod_{i=1}^{k}\|q_i\|^2 > 0.$$

If it is not so, then there exist the sequence $\{f_1^n,\ldots,f_k^n\}_{n\geq 1},\ j=\overline{1,k}$ such that $\|q_j^n\| \to 0,\ n\to\infty$. The invertibility of the operator A implies that

$$\Big\|\frac{f_j^n}{\|Af_j^n\|} - \sum_{i=1}^{j-1}\alpha_{ij}^n\frac{f_i^n}{\|Af_i^n\|}\Big\| \to 0,\ n\to\infty.$$

But

$$\Big\|\frac{f_j^n}{\|Af_j^n\|} - \sum_{i=1}^{j-1}\alpha_{ij}^n\frac{f_i^n}{\|Af_i^n\|}\Big\| \geq \frac{1}{\|Af_j^n\|} \geq c > 0.$$

The contradiction we get proves the lemma. □

Proof of the theorem 2.3. To prove the lemma let us check that the process x satisfies conditions 1)–3) of the theorem 2.2. It is obvious that $x(0)=0$ a.s. Lemma 2.1 and lemma 2.2 imply that x is locally nondetermined. Let us check that the process x satisfies the condition 3) of the theorem 2.2. Let $b(t)=c\sqrt{t},\ c>0$. Pick $\delta<\frac{1}{2}$ and then γ such that $\gamma<\frac{1}{2}-\delta$. One can see that

$$\lim_{h\to 0} h^{-\gamma}\int_0^h t^{-\frac{1}{2}-\delta}dt =$$

$$= \frac{2}{1-2\delta}\lim_{h\to 0} h^{\frac{1}{2}-\delta-\gamma} = 0.$$

□

It must be mentioned that to discuss the local time of the processes one can use another general definition. Let $\{f_\varepsilon\},\ \varepsilon>0$ be the family of functions which weakly converges to delta-function at the point zero as, $\varepsilon\to 0$.

Definition 2.3. If there exists

$$L_2 - \lim_{\varepsilon\to 0}\int_0^1 f_\varepsilon(z(t))dt, \tag{9}$$

then such limit is said to be a local time of the process z at the point zero.

Let μ be an occupation measure of the process z. Suppose that $\mu \ll \lambda_1$ and $\alpha\in C(\mathbb{R})$ is the Radon-Nikodym derivative. Then it follows from (6) that

$$\int_0^1 f_\varepsilon(z(t))dt = \int_{\mathbb{R}} f_\varepsilon(u)\alpha(u)du \to \alpha(0),\ \varepsilon\to 0.$$

Let us check does the convergence (9) hold for one dimensional Gaussian integrator x with a continuous invertible operator A. To do that we will use the following statement which was proved in [10]. Let (T,Ξ) be a measurable space with the finite measure ν. Consider a centered Gaussian random field ζ on (T,Ξ). Denote by $B(s,t)$ the covariance matrix of the vector $(\zeta(s),\zeta(t))$. Put $I_\varepsilon=\int_T f_\varepsilon(\zeta(t))\nu(dt)$.

Theorem 2.4. [10] *Suppose that*

$$\int_T\int_T \frac{1}{\sqrt{detB(s,t)}}\nu(ds)\nu(dt) < \infty, \tag{10}$$

then the limit of I_ε *exists in* $L_2(\Omega)$, *as* $\varepsilon\to 0$.

Apply the theorem 2.4 to our case. For the process (2) the condition (10) means that

$$\int_0^1\int_0^1 \frac{1}{\sqrt{G(A1_{[0;s]}, A1_{[0;t]})}}dsdt < \infty.$$

Since A is continuously invertible, then

$$\int_0^1\int_0^1 \frac{1}{\sqrt{G(A1_{[0;s]}, A1_{[0;t]})}}dsdt \le 2c\int_0^1\int_0^t \frac{1}{\sqrt{s(t-s)}}dsdt < +\infty,\ c > 0. \tag{11}$$

The estimation (11) implies the following statement.

Lemma 2.3. *Suppose that A in the representation* (2) *is continuously invertible, then the limit of $\int_0^1 f_\varepsilon(x(t))dt$ exists in $L_2(\Omega)$.*

At the end of this section we describe the modulus of continuity for one dimensional Gaussian integrators. Let us recall the following definition.

Definition 2.4. [11] The function $w:\mathbb{R}_+\to\mathbb{R}_+,\ w(0)=0$ is a uniform m-modulus of continuity for the random process $\{y(t),\ t\in[0;1]\}$ if a.s.

$$\overline{\lim_{\delta\to 0}}\sup_{\substack{t,s\in[0;1]\\ |t-s|\le\delta}}\frac{|y(t)-y(s)|}{w(\delta)} < +\infty.$$

To describe a uniform m-modulus of continuity for the process (2) we will need the following statement which is proved in [11]. Suppose that $y(t),\ t\in[0;1]$ is a centered Gaussian process. Denote by

$$d(s,t) = \sqrt{E(y(t)-y(s))^2}.$$

Theorem 2.5. *Suppose that for the Gaussian process y there exists strictly increasing majorant for d, i.e. strictly increasing function ψ with $\psi(0)=0$ such that $d(s,t)\le\psi(|s-t|)$, then the function*

$$w(\delta) = \psi(\delta)\sqrt{\ln\frac{1}{\delta}} + \int_0^\delta \frac{\psi(u)}{u\sqrt{\ln\frac{1}{u}}}du$$

is a uniform m-modulus of continuity for the process y.

To obtain the m-modulus of continuity for Gaussian integrator we introduce the next characteristics of the operator A [3]. Denote by $Q_{a,b}$ the multiplication operator on $1\!\!1_{[a;b]}$ in $L_2([0;1])$. Put

$$\varphi(t) := \sup_{b-a\le t}\|AQ_{a,b}\|.$$

The following example shows that the asymptotic behavior of the function φ can be variously dependent on the structure of the operator A.

Example 2.1. Assume that P_i is a projection on $1\!\!1_{\Delta_i}$, where $\Delta_i = \left[\frac{1}{i+1};\frac{1}{i}\right],\ i=1,2,\ldots$ For decreasing to zero sequence $\{\lambda_i,\ i\ge 1\}$ put $A=\sum_{i=1}^\infty \lambda_i P_i$. Note that A is a compact operator in $L_2([0;1])$. For an arbitrary number $i_*,\ z\in L_2([0;1]),\ [a;b]\subset[0;1]$

$$\|AQ_{a,b}z\|^2 = \|\sum_{i=1}^\infty \lambda_i P_i 1\!\!1_{[a;b]}z\|^2 =$$

$$= \sum_{i=1}^\infty \lambda_i^2 \frac{\left(\int_{\Delta_i\cap[a;b]} z(s)ds\right)^2}{|\Delta_i|} =$$

$$= \sum_{i=1}^{i_*-1} \lambda_i^2 \frac{\left(\int_{\Delta_i \cap [a;b]} z(s)ds\right)^2}{|\Delta_i|} + \sum_{i=i_*}^{\infty} \lambda_i^2 \frac{\left(\int_{\Delta_i \cap [a;b]} z(s)ds\right)^2}{|\Delta_i|} \leq$$

$$\leq (b-a) \max_{i=\overline{1,i_*}} \frac{\lambda_i^2}{|\Delta_i|} \|z\|^2 + \lambda_{i_*}^2 \|z\|^2 \tag{12}$$

(the last estimate is obtained by applying the Cauchy inequality in the first and the second summands). The relation (12) implies that

$$\varphi(t)^2 \leq \inf_{i_*} \left[\lambda_{i_*}^2 + t \max_{i=\overline{1,i_*}} \frac{\lambda_i^2}{|\Delta_i|}\right]. \tag{13}$$

Suppose that $\lambda_i = \frac{1}{i}$. Let us check that in this case

$$\overline{\lim_{t\to 0}} \frac{\varphi(t)}{\sqrt{t}} \leq \sqrt{2}, \tag{14}$$

$$\underline{\lim}_{t\to 0} \frac{\varphi(t)}{\sqrt{t}} \geq 1. \tag{15}$$

Really, it follows from the estimate (13) that

$$\varphi(t) \leq \sqrt{2t}. \tag{16}$$

Suppose that $i_0 = \left[\frac{1}{\sqrt{t}}\right]$. Consider $z = 1\!\!1_{[\frac{1}{i_0+1};\frac{1}{i_0}]}\sqrt{i_0(i_0+1)}$. One can see that the following relation holds

$$\varphi(t) = \sup_{b-a\leq t} \|AQ_{a,b}z\|^2 \geq \frac{1}{i_0^2}. \tag{17}$$

Estimates (16), (17) imply (14), (15). In the case $\lambda_i = \frac{1}{\sqrt{i}}$ we have

$$c_1 \sqrt[4]{t} \leq \varphi(t) \leq c_2 \sqrt[4]{t},\ c_1, c_2 > 0$$

which confirms the different behavior of the function φ comparably to the previous case.

One can see that previously defined distance

$$d(s,t) = \|AQ_{s\wedge t, s\vee t} 1\!\!1_{[s\wedge t; s\vee t]}\| \leq$$

$$\leq \sup_{b-a\leq|t-s|} \|AQ_{a,b}\| \sqrt{|t-s|} = \varphi(|t-s|)\sqrt{|t-s|}.$$

It implies that $\varphi(|t-s|)\sqrt{|t-s|}$ is strictly increasing majorant for $d(s,t)$. Applying the theorem 2.5 we get the following statement

Theorem 2.6. *The function*

$$w(\delta) = \varphi(\delta)\sqrt{\delta \ln \frac{1}{\delta}} + \int_0^\delta \frac{\varphi(u)}{\sqrt{u \ln \frac{1}{u}}} du$$

is a uniform m-modulus of continuity for the Gaussian integrator.

Example 2.2. For the operators from the previous example we have the next estimations. Suppose that $\varphi(t) \leq c\sqrt{t},\ c > 0$. Then $w(\delta) \leq c_1\delta\sqrt{\ln\frac{1}{\delta}},\ c_1 > 0$. If $\varphi(t) \leq c\sqrt[4]{t},\ c > 0$ then $w(\delta) \leq c_1\delta^{\frac{3}{4}}\sqrt{\ln\frac{1}{\delta}},\ c_1 > 0$.

3. Properties of planar Gaussian integrators

The main object of investigation in this section is planar Gaussian integrator

$$x(t) = ((A1\!\!I_{[0;t]}, \xi_1), (A1\!\!I_{[0;t]}, \xi_2)),\ t \in [0;1]. \tag{18}$$

Here A is a continuous linear operator in $L_2([0;1])$, ξ_1, ξ_2 are independent Gaussian white noises in $L_2([0;1])$. One of our main goals is to construct the self-intersection local time for the process (18). To investigate the self-intersection local time we have to know that the process (18) has points of self-intersection. The following statement describes the geometry of trajectories of planar Gaussian integrators.

Theorem 3.1. *Suppose that A in the representation* (18) *is continuously invertible, then with positive probability the trajectories of planar Gaussian integrator x have double self-intersections, i.e. $P\{\exists\ t_1 \neq t_2 :\ x(t_1) = x(t_2)\} > 0$.*

Proof. The Kolmogorov condition implies that every coordinate of the process (18) can be considered as an element of the space $C_0([0;1])$, where $C_0([0;1])$ is the space of all continuous functions on the interval $[0;1]$ equal to zero at the point zero. To prove the theorem it suffices to check that the support of the distribution of one coordinate of the process x coincides with $C_0([0;1])$. Really, if $f \in C_0^1([0;1], \mathbb{R}^2)$ is a curve which has double points, then for sufficiently small $\varepsilon > 0$ the functions g such that $\|g-f\|_{C_0([0;1],\mathbb{R}^2)} < \varepsilon$ also have double points. $C_0^1([0;1], \mathbb{R}^2)$ is the space of all continuously differentiable functions from $[0;1]$ to $\mathbb{R}^2$ which equal to zero at the point zero. Consequently, the process x will have double points with positive probability if $P\{\|x - f\|_{C_0([0;1],\mathbb{R}^2)} < \varepsilon\} > 0$. The independence of coordinates of the process x implies the following relation

$$P\{\|x - f\|_{C_0([0;1],\mathbb{R}^2)} < \varepsilon\} = P\Big\{\sqrt{\|x_1 - f_1\|^2_{C_0([0;1])} + \|x_2 - f_2\|^2_{C_0([0;1])}} < \varepsilon\Big\} \geq$$

$$\geq P\Big\{\|x_1 - f_1\|^2_{C_0([0;1])} < \frac{\varepsilon^2}{2}\Big\} P\Big\{\|x_2 - f_2\|^2_{C_0([0;1])} < \frac{\varepsilon^2}{2}\Big\}.$$

It implies that the process x will have double points with positive probability if for any $h \in C_0([0;1]),\ \sigma > 0$

$$P\Big\{\|x_1 - h\|^2_{C_0([0;1])} < \sigma\Big\} > 0. \tag{19}$$

The relation (19) is equivalent to the condition that the support of the distribution of one coordinate of the process x coincides with $C_0([0;1])$. Further in the proof one coordinate of the initial process x is denoted by the same letter x. Define the mapping $F_0 : L_2([0;1]) \to C_0([0;1])$ by the rule

$$L_2([0;1]) \ni h \overset{F_0}{\mapsto} (1\!\!I_{[0;t]}, h).$$

and the mapping $F : L_2([0;1]) \to C_0([0;1])$ as

$$L_2([0;1]) \ni h \overset{F}{\mapsto} (A1\!\!I_{[0;t]}, h) = (1\!\!I_{[0;t]}, A^*h) = F_0(A^*h). \tag{20}$$

It is known [12] that $F(L_2([0;1]))$ is the set of admissible shifts for the distribution of x. Since the closure of the set of admissible shifts of x coincides with the support of the distribution of x, then it suffices to show that $\overline{F(L_2([0;1]))} = C_0([0;1])$. It is known [12] that $F_0(L_2([0;1]))$ is the set of admissible shifts for the Wiener process and $\overline{F_0(L_2([0;1]))} = C_0([0;1])$. (20) implies that to prove the theorem we have to check that

$$\overline{A^*(L_2([0;1]))} = L_2([0;1])$$

which is the consequence of continuous invertibility of A. □

Since the trajectories of planar Gaussian integrator have double self-intersections one can consider the self-intersection local time for the process x which is formally defined as

$$T_2^x = \int_{\Delta_2} \delta_0(x(t_2) - x(t_1))d\vec{t}, \tag{21}$$

where δ_0 is a delta-function at the point zero. The formal expression (21) registers the moments of time which the process x spends in infinitesimally small neighborhoods of its double self-intersection points. One can try to give a precise definition to (21) by using the following approach. Consider the Gaussian random field defined by formula $X(t_1,t_2) = x(t_2) - x(t_1),\ t_1, t_2 \in \Delta_2$. Define an occupation measure of X as

$$\mu_{\Delta_2}(E) = \int_{\Delta_2} 1\!\!\mathrm{I}_E(X(t_1,t_2))d\vec{t}.$$

Theorem 3.2. *Suppose that A in the representation* (18) *is continuously invertible, then X has a local time on Δ_2 and $\alpha(u, \Delta_2) \in L_2(\mathbb{R}^2, d\lambda_2)$.*

Proof. To prove the lemma let us check that

$$E\int_{\mathbb{R}^2} |\widehat{\mu_{\Delta_2}}(z)|^2 dz < +\infty.$$

Really,

$$E\int_{\mathbb{R}^2} |\widehat{\mu_{\Delta_2^2}}(z)|^2 dz = \int_{\mathbb{R}^2} E\int_{\mathbb{R}^2} e^{i(z,v)}\mu_{\Delta_2}(dv)\int_{\mathbb{R}^2} e^{-i(z,u)}\mu_{\Delta_2}(du)dz =$$

$$= \int_{\mathbb{R}^2}\int_{\Delta_2^2}\prod_{j=1}^{2} Ee^{i(z,x^j(t_2)-x^j(t_1)-x^j(t_4)+x^j(t_3))}d\vec{t}dz =$$

$$= \int_{\mathbb{R}^2}\int_{\Delta_2^2} e^{-\frac{1}{2}\|z\|^2\operatorname{Var}(x^1(t_2)-x^1(t_1)-x^1(t_4)+x^1(t_3))}d\vec{t}dz, \tag{22}$$

where $\Delta_2^2 = \Delta_2 \times \Delta_2$. Denote by π a permutation of $(1, 2, 3, 4)$. Let

$$\Delta_2^2(\pi) = \{0 \le t_{\pi(1)} < t_{\pi(2)} < t_{\pi(3)} < t_{\pi(4)} \le 1\}.$$

Let us check that the integral (22) is finite in the case $\Delta_2^2(\pi)$. The continuous invertibility of the operator A implies the estimate

$$\begin{aligned}&\operatorname{Var}(x^1(t_2) - x^1(t_1) - x^1(t_4) + x^1(t_3)) \ge \\ &c(t_{\pi(2)} - t_{\pi(1)} + t_{\pi(4)} - t_{\pi(3)}),\ c > 0\end{aligned} \tag{23}$$

for $t_1, t_2, t_3, t_4 \in \Delta_2^2(\pi)$. It follows from (23) that (22) is less or equal to

$$\int_{\mathbb{R}^2}\int_{\Delta_2^2(\pi)} e^{-\frac{1}{2}\|z\|^2 c(t_{\pi(2)}-t_{\pi(1)}+t_{\pi(4)}-t_{\pi(3)})}d\vec{t}dz. \tag{24}$$

Changing variables

$$\begin{aligned}t_{\pi(1)} &= s_1\\ t_{\pi(2)} - t_{\pi(1)} &= s_2\\ t_{\pi(3)} - t_{\pi(2)} &= s_3\\ t_{\pi(4)} - t_{\pi(3)} &= s_4\end{aligned}$$

one can see that (24) is bounded by

$$\int_{\mathbb{R}^2}\int_0^1\int_0^1 e^{-\frac{1}{2}\|z\|^2 c(s_2+s_4)}d\vec{s}dz$$

which is finite. By using the same arguments one can check that (22) is finite over each $\Delta_2^2(\pi)$. □

If $\alpha(u,\Delta_2)=\frac{d\mu_{\Delta_2}}{d\lambda_2}(u)$ would be continuous at the point zero, then we could associate T_2^x with $\alpha(0,\Delta_2)$. Unfortunately this is not so even in the case of planar Wiener process [4]. And the reason is to much self-intersections in any neighborhood of the diagonal of Δ_2. But J. Rosen proved in [9] that for any $\delta>0$ the local time $\alpha(u,\Delta_2^\delta)$ of planar Wiener process is Hölder continuous with respect to u. By using the same arguments we will check the Hölder continuity with respect to spatial variable of the local time of planar Gaussian integrator x generated by a continuously invertible operator A.

Theorem 3.3. *Suppose that A in the representation* (18) *is continuously invertible. Then for any compact $K\subset\mathbb{R}^2$ we can choose a version $\alpha(u,\Delta_2^\delta)$ such that a.s.*

$$\sup_{u,v\in K}\frac{|\alpha(u,\Delta_2^\delta)-\alpha(v,\Delta_2^\delta)|}{\|u-v\|^\beta}<+\infty$$

for any $\beta<1$.

Proof. To prove the theorem it suffices to check that there exists $c>0$ such that for any $k\geq 1,\ \gamma<1,\ u,v\in\mathbb{R}^2$ the following estimate holds

$$E(\alpha(u,\Delta_2^\delta)-\alpha(v,\Delta_2^\delta))^k\leq c\|u-v\|^{k\gamma}.$$

Really,

$$E(\alpha(u,\Delta_2^\delta)-\alpha(v,\Delta_2^\delta))^k=$$
$$=E\Big(\frac{1}{2\pi}\int_{\mathbb{R}^2}(e^{-i(u,z)}-e^{-i(v,z)})\widehat{\alpha}(z,\Delta_2^\delta)dz\Big)^k=$$
$$=\frac{1}{(2\pi)^k}E\int_{\mathbb{R}^{2k}}\prod_{j=1}^{k}(e^{-i(u,z_j)}-e^{-i(v,z_j)})\widehat{\alpha}(z_j,\Delta_2^\delta)d\vec{z}. \tag{25}$$

Since

$$\widehat{\alpha}(z,\Delta_2^\delta)=\int_{\mathbb{R}^2}e^{i(z,p)}\alpha(p,\Delta_2^\delta)dp=\int_{\Delta_2^\delta}e^{i(z,x(t)-x(s))}dsdt,$$

then (25) equals

$$\frac{1}{(2\pi)^k}\int_{\mathbb{R}^{2k}}\prod_{j=1}^{k}(e^{-i(u,z_j)}-e^{-i(v,z_j)})\int_{\Delta_2^{\delta k}}Ee^{i\sum_{j=1}^k(z_j,x(t_j)-x(s_j))}d\vec{s}d\vec{t}d\vec{z}, \tag{26}$$

were $\Delta_2^{\delta k}=\Delta_2^\delta\times\ldots\times\Delta_2^\delta$. Note that for any $\gamma<1$

$$|e^{-i(u,z)}-e^{-i(v,z)}|\leq\|z\|^\gamma\|u-v\|^\gamma.$$

It implies that (26) less or equal to

$$\|u-v\|^{k\gamma}\int_{\mathbb{R}^{2k}}\prod_{j=1}^{k}\|z_j\|^\gamma\int_{\Delta_2^{\delta k}}Ee^{i\sum_{j=1}^k(z_j,x(t_j)-x(s_j))}d\vec{s}d\vec{t}d\vec{z}=$$
$$=\|u-v\|^{k\gamma}\int_{\mathbb{R}^{2k}}\prod_{j=1}^{k}\|z_j\|^\gamma\int_{\Delta_2^{\delta k}}\prod_{l=1}^{2}e^{-\frac{1}{2}\operatorname{Var}\left(\sum_{j=1}^k z_j^l(x^l(t_j)-x^l(s_j))\right)}d\vec{s}d\vec{t}d\vec{z}. \tag{27}$$

Put $(\tau_1,\ldots,\tau_{2k})=(s_1,t_1,\ldots,s_k,t_k)$. Assume that $\Delta_2^{\delta k}$ is replaced by $\Delta_2^{\delta k}(\pi)$ for some permutation π of $\{1,\ldots,2k\}$ since (27) is a sum of integrals over such regions. Let us define disjoint intervals

$$R_i=[\tau_{\pi(i)};\tau_{\pi(i+1)}],\ i=\overline{1,2k-1}.$$

Note that

$$x^l(t_j)-x^l(s_j)=\sum_{R_i\subseteq[s_j;t_j]}\widetilde{x}^l(R_i), \tag{28}$$

where $\widetilde{x}^l(R_i) = x^l(\tau_{\pi(i+1)}) - x^l(\tau_{\pi(i)}),\ l = 1, 2$. Denote by

$$\overline{z}_i^l = \sum_{j: R_i \subseteq [s_j; t_j]} z_j^l,\ l = 1, 2. \tag{29}$$

It follows from (28) and (29) that

$$\operatorname{Var}\Big(\sum_{j=1}^{k} z_j^l (x^l(t_j) - x^l(s_j))\Big) = \operatorname{Var}\Big(\sum_{i=1}^{2k-1} \overline{z}_i^l \widetilde{x}^l(R_i)\Big) =$$

$$= \operatorname{Var}\Big(\sum_{i=1}^{2k-1} \overline{z}_i^l (A1\!\!1_{[\tau_{\pi(i)};\tau_{\pi(i+1)})}, \xi_l)\Big) \geq c\Big(\sum_{i=1}^{2k-1} \|\overline{z}_i^l\|^2 (\tau_{\pi(i+1)} - \tau_{\pi(i)})\Big) =$$

$$= c \operatorname{Var}\Big(\sum_{i=1}^{2k-1} \overline{z}_i^l \widetilde{w}^l(R_i)\Big), \tag{30}$$

where $\widetilde{w}^l(R_i) = w^l(\tau_{\pi(i+1)}) - w^l(\tau_{\pi(i)}),\ l = 1, 2$. (30) implies that (27) is less or equal to

$$\|u - v\|^{k\gamma} \int_{\mathbb{R}^{2k}} \int_{\Delta_2^{\delta k}} \prod_{j=1}^{k} \|z_j\|^\gamma e^{-\frac{1}{2} c \operatorname{Var} \sum_{j=1}^{k} (z_j, w(t_j) - w(s_j))} d\vec{s} d\vec{t} d\vec{z}. \tag{31}$$

Further we will need the following lemma which was proved in [9].

Lemma 3.1. [9] *For any $k \geq 1, \gamma < 1$*

$$\int_{\mathbb{R}^{2k}} \int_{\Delta_2^{\delta k}} \prod_{l=1}^{k} \|u_l\|^\gamma e^{-\frac{1}{2} \operatorname{Var} \sum_{j=1}^{k} (u_j, w(t_j) - w(s_j))} d\vec{s} d\vec{t} d\vec{u} < +\infty.$$

The lemma 3.1 implies that there exists $c > 0$ such that (31) is less or equal to $c\|u - v\|^{k\gamma}, \gamma < 1$. Therefore, we proved that there exists $c > 0$ such that for any $k \geq 1, \gamma < 1$

$$E(\alpha(u, \Delta_2^\delta) - \alpha(v, \Delta_2^\delta))^k \leq c\|u - v\|^{k\gamma}. \tag{32}$$

To end the proof of the theorem we need the following modification of the Kolmogorov condition.

Theorem 3.4. *(moments and continuity, Kolmogorov, Loeve, Chentsov)* [14] *Let y be a process on $\mathbb{R}^d$ with values in a complete metric space (S, ρ). Assume for some $c, a, b > 0$ that*

$$E\{\rho(y(s), y(t))\}^a \leq c|s - t|^{d+b},\ s, t \in \mathbb{R}^d.$$

Then y has a continuous version, and the latter is a.s. locally Hölder continuous with exponent γ for any $\gamma \in \Big(0; \frac{b}{a}\Big)$.

The estimation (32) and the theorem 3.4 end the proof of the theorem. □

The following technical lemma will be useful in the calculation of moments of the local time of two dimensional integrators. Denote by $\Delta_n = \{0 \leq s_1 \leq \ldots \leq s_n \leq 1\}$. For $s_1, \ldots, s_n \in \Delta_n, u_1, \ldots, u_n \in \mathbb{R}^2$ let $p_{s_1 \ldots s_n}(u_1, \ldots, u_n)$ be the density of Gaussian vector $(x(s_1), \ldots, x(s_n))$ in $\mathbb{R}^{2n}$. Let us check that the following statement holds.

Lemma 3.2. *Suppose that A in the representation* (18) *is continuously invertible. Then there exist positive constants $c_1(n), c_2$ such that the following relation holds*

$$p_{s_1 \ldots s_n}(u_1, \ldots, u_n) \leq \frac{c_1(n)}{s_1(s_2 - s_1) \ldots (s_n - s_{n-1})} e^{-c_2 \sum_{i=0}^{n-1} \frac{\|u_{i+1} - u_i\|^2}{s_{i+1} - s_i}}.$$

Proof. The independence of the coordinates of our process implies that

$$p_{s_1\ldots s_n}(u_1,\ldots,u_n)=\prod_{l=1}^{2}\widetilde{p}_{s_1\ldots s_n}(u_1^l,\ldots,u_n^l),$$

where

$$\widetilde{p}_{s_1\ldots s_n}(u_1^l,\ldots,u_n^l)=\frac{1}{\sqrt{2\pi^n}}\frac{1}{\sqrt{G(g(s_1),\ldots,g(s_n))}}$$
$$\cdot e^{-\frac{1}{2}(B^{-1}(g(s_1),\ldots,g(s_n))u^l,u^l)}.$$

Here $g(t)=Ag_0(t),\ g_0(t)=1\!\!1_{[0;t]},\ u^l=(u_1^l,\ldots,u_n^l),\ B(e_1,\ldots,e_n)$ is the Gramian matrix constructed by $e_1,\ldots,e_n,\ G(e_1,\ldots,e_n)$ is the Gram determinant. Since A is a continuously invertible operator, then by lemma 2.2 there exists a positive constant $c(n)$ such that the following estimate holds

(33) $G(g(s_1),\ldots,g(s_n))\geq c(n)G(g_0(s_1),\ldots,g_0(s_n))=c(n)s_1(s_2-s_1)\cdot\ldots\cdot(s_n-s_{n-1}).$

Put $c_1(n)=\frac{1}{c(n)}$. Let us estimate

$$(B^{-1}(g(s_1),\ldots,g(s_n))u^l,u^l),\ l=1,2.$$

It was proved in [4,5] that in the case

$$u^l=((h_0,g(s_1)),\ldots,(h_0,g(s_n))),\ h_0\in L_2([0;1])$$

the following relation holds

$$(B^{-1}(g(s_1),\ldots,g(s_n))u^l,u^l)=\|P_{g(s_1)\ldots g(s_n)}h_0\|^2,$$

where $P_{e_1\ldots e_n}$ is a projection on $LS\{e_1,\ldots,e_n\}$. If $h_0\in LS\{g(s_1),\ldots,g(s_n)\}$, then

$$(B^{-1}(g(s_1),\ldots,g(s_n))u^l,u^l)=\|h_0\|^2.$$

One can note that

$$((h_0,g(s_1)),\ldots,(h_0,g(s_n)))=((A^*h_0,g_0(s_1)),\ldots,(A^*h_0,g_0(s_n))).$$

Since

$$\begin{aligned}(A^*h_0,g_0(s_1))&=u_1^l\\(A^*h_0,g_0(s_2))&=u_2^l\\&\vdots\\(A^*h_0,g_0(s_n))&=u_n^l,\end{aligned}$$

then

$$A^*h_0=\sum_{j=0}^{n-1}\frac{g_0(s_{j+1})-g_0(s_j)}{\|g_0(s_{j+1})-g_0(s_j)\|}(u_{j+1}^l-u_j^l)+r,\tag{34}$$

where $r\perp g_0(s_i),\ i=\overline{1,n}$. Let us remark that continuous invertibility of the operator A implies the existence of A^{*-1}. (34) implies that

$$(B^{-1}(g(s_1),\ldots,g(s_n))u^l,u^l)=$$

$$=\|h_0\|^2\Big\|A^{*-1}\Big(\sum_{j=0}^{n-1}\frac{g_0(s_{j+1})-g_0(s_j)}{\|g_0(s_{j+1})-g_0(s_j)\|}(u_{j+1}^l-u_j^l)+r\Big)\Big\|^2.\tag{35}$$

It follows from (33) and (35) that

$$\widetilde{p}_{s_1\ldots s_n}(u^l) \le \frac{\sqrt{c_1(n)}}{\sqrt{s_1(s_2-s_1)\cdot\ldots\cdot(s_n-s_{n-1})}}\cdot$$

$$\cdot\exp\Big\{-\frac{1}{2}\Big\|A^{*-1}\Big(\sum_{j=0}^{n-1}\frac{g_0(s_{j+1})-g_0(s_j)}{\|g_0(s_{j+1})-g_0(s_j)\|}(u^l_{j+1}-u^l_j)+r\Big)\Big\|^2\Big\}\le$$

$$\le \frac{\sqrt{c_1(n)}}{\sqrt{s_1(s_2-s_1)\cdot\ldots\cdot(s_n-s_{n-1})}}\cdot$$

$$\cdot\exp\Big\{-\frac{1}{2}c_2\Big(\Big\|\sum_{j=0}^{n-1}\frac{g_0(s_{j+1})-g_0(s_j)}{\|g_0(s_{j+1})-g_0(s_j)\|}(u^l_{j+1}-u^l_j)\Big\|^2+\|r\|^2\Big)\Big\}\le$$

$$\le \frac{\sqrt{c_1(n)}}{\sqrt{s_1(s_2-s_1)\cdot\ldots\cdot(s_n-s_{n-1})}}e^{-\frac{1}{2}c_2\sum_{j=0}^{n-1}\frac{(u^l_{j+1}-u^l_j)^2}{s_{j+1}-s_j}}.$$

□

Further we discuss the connection between the growth of modulus of continuity for two dimensional Gaussian integrators and the growth of its approximations for the self-intersection local time. Let $f_\varepsilon(z)=\frac{1}{2\pi}e^{-\frac{\|z\|^2}{2\varepsilon}},\ \varepsilon>0,\ z\in\mathbb{R}^2$. Denote by $\Phi(t):=\varphi(t)^2t$.

Lemma 3.3. *Suppose that there exists $\alpha>1$ such that*

$$\Phi(t)\le ct^\alpha,\ c>0, \tag{36}$$

then there exists $c_1>0$ such that

$$E\int_{\Delta_2}f_\varepsilon(x(t_2)-x(t_1))d\vec{t}\ge c_1\varepsilon^{\frac{1}{\alpha}-1}.$$

Let us give an example of the integral operator which satisfies the condition (36) of the lemma 3.3.

Example 3.1. Let A be an integral operator in $L_2([0;1])$ which is acting by the rule

$$Af(u_2)=\int_0^1 1\!\!1_{\{u_2>u_1\}}f(u_1)du_1.$$

Then

$$AQ_{a,b}f(u_2)=\int_0^1 1\!\!1_{\{u_2>u_1\}}1\!\!1_{[a;b]}(u_1)f(u_1)du_1.$$

One can check that the integral operator $AQ_{a,b}$ with the kernel

$$K(u_1,u_2)=1\!\!1_{\{u_2>u_1\}}1\!\!1_{[a;b]}(u_1)$$

is the Hilbert–Schmidt operator with $\|AQ_{a,b}\|_2=\sqrt{2(b-a)}$. Here $\|\cdot\|_2$ is the Hilbert–Schmidt norm. Consequently $\|AQ_{a,b}\|\le\sqrt{2(b-a)}$ and $\Phi(t)\le 2t^2$. Therefore, the condition (36) holds with $c=2,\ \alpha=2$.

Proof of the lemma 3.3.

$$E\int_{\Delta_2}f_\varepsilon(x(t_2)-x(t_1))d\vec{t}=$$

$$=\int_{\Delta_2}\frac{1}{2\pi}\frac{1}{\|AQ_{t_1,t_2}1\!\!1_{[t_1;t_2]}\|^2+\varepsilon}d\vec{t}\ge$$

$$\ge\int_{\Delta_2}\frac{1}{2\pi}\frac{1}{(\sup_{b-a\le t_2-t_1}\|AQ_{a,b}\|)^2(t_2-t_1)+\varepsilon}d\vec{t}=$$

$$= \frac{1}{2\pi} \int_{\Delta_2} \frac{1}{\Phi(t_2 - t_1) + \varepsilon} d\vec{t} \geq \frac{1}{2\pi} \int_{\Delta_2} \frac{1}{c(t_2 - t_1)^\alpha + \varepsilon} d\vec{t}. \tag{37}$$

Changing variables

$$t_1 = u_1, \ \frac{t_2 - t_1}{\varepsilon^{1/\alpha}} = u_2$$

we get that (37) equals

$$\frac{1}{2\pi} \varepsilon^{\frac{1}{\alpha} - 1} \int_{\{0 \leq u_1 \leq \varepsilon^{1/\alpha} u_2 + u_1 \leq 1\}} \frac{1}{c u_2^\alpha + 1} du_1 du_2 \geq c_1 \varepsilon^{\frac{1}{\alpha} - 1}, \ c_1 > 0.$$

□

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