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THE PROBLEM OF PASTING TOGETHER TWO DIFFUSION PROCESSES AND CLASSICAL POTENTIALS

The paper is a survey of some analytical methods for constructing a diffusion process in $\mathbb{R}^d$ that is a result of pasting together two diffusion processes. It is an exposition in written of a lecture that the authors delivered at one of the plenary sessions of the Conference "Stochastic analysis and random dynamics" which held in Lviv, June 14–20, 2009.

Introduction

Consider a stochastic differential equation in $\mathbb{R}^d$ of the form

$$dx(t) = a(t, x(t))dt + b(t, x(t))^{1/2}dw(t), \tag{1}$$

where $(a(t,x))_{\substack{t\geq 0\\ x\in\mathbb{R}^d}}$ and $(b(t,x))_{\substack{t\geq 0\\ x\in\mathbb{R}^d}}$ are given functions with their values in $\mathbb{R}^d$ and $\mathcal{L}^+(\mathbb{R}^d)$, respectively, and $(w(t))_{t\geq 0}$ is a standard Wiener process in $\mathbb{R}^d$. By $\mathcal{L}^+(\mathbb{R}^d)$, we denote the set of all symmetric positive definite linear operators in $\mathbb{R}^d$, and $b(t,x)^{1/2}$ stands for a square root of $b(t,x)$.

This equation is usually treated as a one being intended for describing a dynamical system of the form

$$dx(t) = a(t, x(t))dt \tag{2}$$

in the situation where some random influences on it are to be taken into account.

An alternative point of view on Eq. (1) consists in considering it as a result of perturbing a system of the form

$$dx_0(t) = b(t, x_0(t))^{1/2}dw(t) \tag{3}$$

by a vector field $(a(t,x))_{\substack{t\geq 0\\ x\in\mathbb{R}^d}}$. It turns out that, in the case of the function $(b(t,x))_{\substack{t\geq 0\\ x\in\mathbb{R}^d}}$ being good enough, one can perturb Eq. (3) by such a vector field $(a(t,x))_{\substack{t\geq 0\\ x\in\mathbb{R}^d}}$ which does not generate any dynamical system of the type (2). In such a situation, the former treatment of Eq. (1) becomes, of course, meaningless, and another interpretation for it is to be given.

We discuss below some analytical methods for constructing solutions to Eq. (1) in cases where a given vector field $(a(x))_{x\in\mathbb{R}^d}$ is determined by a generalized function. In Section 1, such a construction is based on the well-known formulae of perturbation. In Sections 2-3, the problem of pasting together two diffusion processes is formulated, and some ways of solving it are discussed. Finally, some examples are considered in Section 3.

1. Diffusion in a medium with membranes

Let a given $\mathcal{L}^+(\mathbb{R}^d)$-valued function $(b(x))_{x\in\mathbb{R}^d}$ be bounded and Hölder continuous, and let the inequality $(b(x)\theta,\theta) \geq c_0|\theta|^2$ with some positive constant c_0 be held for all

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$\theta \in \mathbb{R}^d$ and $x \in \mathbb{R}^d$. Then, as is known, there exists the fundamental solution to the equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\mathrm{tr}(b(x)u''_{xx}),\ t > 0, x \in \mathbb{R}^d. \tag{4}$$

Denote it by $g_0(t,x,y),\ t > 0, x \in \mathbb{R}^d$, and $y \in \mathbb{R}^d$. If $(x_0(t))_{t\geq 0}$ is the solution to the equation

$$dx_0(t) = b(x_0(t))^{1/2}dw(t), \tag{5}$$

then, for any bounded continuous function $(\varphi(x))_{x\in\mathbb{R}^d}$, the function

$$u(t,x) = \mathbb{E}_x\varphi(x_0(t)) = \int_{\mathbb{R}^d}\varphi(y)g_0(t,x,y)dy,\ t > 0, x \in \mathbb{R}^d,$$

satisfies Eq. (4) and the initial condition

$$u(0+,x) = \varphi(x),\ x \in \mathbb{R}^d. \tag{6}$$

Suppose now that S is a closed surface separating $\mathbb{R}^d$ into two open parts: the interior D_i and the exterior D_e. Assume that the surface S belongs to the class $H^{1+\lambda}$ and, moreover, that each point of S has the property of both inner and outer sphericity. Let $\nu(x)$ for $x \in S$ denote the unit outer normal vector to S at x. The vector $N(x) = b(x)\nu(x)$ for $x \in S$ is called conormal. A continuous function $(q(x))_{x\in S}$ with its values in the interval $[-1,1]$ is assumed to be given. Denote, by $(\delta_S(x))_{x\in\mathbb{R}^d}$, the generalized function on $\mathbb{R}^d$, whose action on a test function consists in integrating the latter one over the surface S.

We now show how to construct a solution to the following stochastic differential equation (see [1], [2]):

$$dx(t) = q(x(t))\delta_S(x(t))N(x(t))dt + b(x(t))^{1/2}dw(t). \tag{7}$$

Our construction is based on the well-known formulae of perturbation.

Perturbation formulae

Let $(x(t))_{t\geq 0}$ be a Markov process in $\mathbb{R}^d$ satisfying the stochastic differential equation

$$dx(t) = a(t,x(t))dt + b(t,x(t))^{1/2}dw(t)$$

with the given functions $(b(t,x))_{t\geq 0,\, x\in\mathbb{R}^d}$ and $(a(t,x))_{t\geq 0,\, x\in\mathbb{R}^d}$ as above, and let g denote its transition probability density with respect to the Lebesgue measure in $\mathbb{R}^d$. Suppose that a vector field $(\widetilde{a}(t,x))_{t\geq 0,\, x\in\mathbb{R}^d}$ be now given. In order to construct the transition probability density $\widetilde{g}$ for the Markov process $(\widetilde{x}(t))_{t\geq 0}$, being the solution to the equation

$$d\widetilde{x}(t) = [a(t,\widetilde{x}(t)) + \widetilde{a}(t,\widetilde{x}(t))]dt + b(t,\widetilde{x}(x))^{1/2}dw(t),$$

one should consider the following pair of equations:

$$\widetilde{g}(s,x,t,y) = g(s,x,t,y) + \int_s^t d\tau\int_{\mathbb{R}^d} g(s,x,\tau,z)(\widetilde{g}'_z(\tau,z,t,y),\widetilde{a}(\tau,z))dz,$$

$$\widetilde{g}(s,x,t,y) = g(s,x,t,y) + \int_s^t d\tau\int_{\mathbb{R}^d} \widetilde{g}(s,x,\tau,z)(g'_z(\tau,z,t,y),\widetilde{a}(\tau,z))dz.$$

It is necessary to notice that, by substituting a function of the type $\widetilde{a}(x) = q(x)\delta_S(x)N(x), x \in \mathbb{R}^d$ into these equations, one should define the action of the function δ_S on a function having a jump on S. It is natural to put

$$\langle\delta_S,\varphi\rangle = \frac{1}{2}\int_S[\varphi(x+) + \varphi(x-)]d\sigma_x,$$

where the integral is a surface integral over S.

Returning to Eq. (7), we define, for $k = 0,1,2,\ldots$, the functions Q_k on the set $(0,+\infty)\times S\times S$ putting $Q_0(t,x,y) = (N(x),\nabla_x g_0(t,x,y))$ and, for $k \geq 1$,

$$Q_k(t,x,y) = \int_0^t d\tau\int_S Q_{k-1}(\tau,x,z)Q_0(t-\tau,z,y)q(z)d\sigma_z =$$

$$= \int_0^t d\tau \int_S Q_0(\tau, x, z) Q_{k-1}(t-\tau, z, y) q(z) d\sigma_z.$$

By some estimates that are standard in the theory of single-layer potentials, one can prove that the series

$$\sum_{k=0}^{\infty} Q_k(t, x, y),\ t > 0, x \in S, y \in S,$$

is uniformly (locally in t) convergent. Denote its sum by $R(t, x, y)$ and define a function g of the arguments $(t, x, y) \in (0, +\infty) \times \mathbb{R}^d \times \mathbb{R}^d$ by each of the relations

$$g(t, x, y) = g_0(t, x, y) + \int_0^t d\tau \int_S g_0(\tau, x, z) V(t-\tau, z, y) q(z) d\sigma_z, \tag{8}$$

$$g(t, x, y) = g_0(t, x, y) + \int_0^t d\tau \int_S \widetilde{V}(\tau, x, z) \frac{\partial g_0(t-\tau, z, y)}{\partial N(z)} q(z) d\sigma_z, \tag{9}$$

where the functions V and $\widetilde{V}$ are given by

$$V(t, x, y) = \frac{\partial g_0(t, x, y)}{\partial N(x)} + \int_0^t d\tau \int_S R(\tau, x, z) \frac{\partial g_0(t-\tau, z, y)}{\partial N(z)} q(z) d\sigma_z$$

for $t > 0, x \in S$, and $y \in \mathbb{R}^d$ and

$$\widetilde{V}(t, x, y) = g_0(t, x, y) + \int_0^t d\tau \int_S g_0(\tau, x, z) R(t-\tau, z, y) q(z) d\sigma_z$$

for $t > 0, x \in \mathbb{R}^d$, and $y \in S$ (one can verify that the right-hand sides of (8) and (9) coincide).

The function g turns out to be the transition probability density of a continuous strong Markov process $(x(t))_{t\geq 0}$ in $\mathbb{R}^d$, for which a W-functional $(\eta_t)_{t\geq 0}$ given by

$$\eta_t = \int_0^t \delta_S(x(\tau)) d\tau,\ t \geq 0,$$

is well defined. It can be now verified that the process

$$x(t) - x(0) - \int_0^t q(x(\tau)) N(x(\tau)) d\eta_\tau,\ t \geq 0,$$

is a continuous square integrable martingale, whose square characteristic is given by

$$\int_0^t b(x(\tau)) d\tau,\ t \geq 0.$$

In other words, the process $(x(t))_{t\geq 0}$ is a solution to Eq. (7).

We have thus seen that although there is no dynamical system generated by the vector field $(q(x)\delta_S(x)N(x))_{x\in\mathbb{R}^d}$, however, this field can serve as a drift for some diffusion process.

The process constructed can be interpreted as a one that describes the movement of a diffusing particle in the medium containing a membrane located on S. The membrane does not affect the particle, while it is wandering in $D_e \cup D_i,$. Whenever it is hitting the surface S, it receives a pulse in the conormal direction. To be perceptible, this pulse must be of the "infinite intensity with the coefficient $q(\cdot)$". The extreme cases $q(x) \equiv +1$ and $q(x) \equiv -1$ mean that the points of S are reflecting into the directions $N(x)$ and $-N(x)$, respectively. In the rest of cases, the membrane is penetrable for movements into both directions. There is no membrane if $q(x) \equiv 0$.

For any continuous bounded function $(\varphi(x))_{x\in\mathbb{R}^d}$ with real values, we put

$$u(t, x, \varphi) = \int_{\mathbb{R}^d} g(t, x, y) \varphi(y) dy,\ t > 0, x \in \mathbb{R}^d.$$

It is not difficult to see that this function possesses the following properties:

a$_0$) it is continuous in $(t, x) \in (0, +\infty) \times \mathbb{R}^d$;

b$_0$) it satisfies the equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\mathrm{tr}(b(x)u''_{xx})$$

in the domain $t > 0, x \in \mathbb{R}^d \setminus S$;

c$_0$) it satisfies the initial condition

$$u(0+, x, \varphi) = \varphi(x),\ \ x \in \mathbb{R}^d;$$

d$_0$) it satisfies the boundary condition

$$\frac{1+q(x)}{2}\frac{\partial u}{\partial N(x)}(t, x+, \varphi) - \frac{1-q(x)}{2}\frac{\partial u}{\partial N(x)}(t, x-, \varphi) = 0$$

for $t > 0, x \in S$.

As a matter of fact, a solution to this conjugate problem was constructed above.

2. Case of discontinuous coefficients

Suppose that some elliptic differential operators $\mathcal{L}_i$ and $\mathcal{L}_e$ in the domains $D_i \cup S$ and $D_e \cup S$, respectively, are given. Each of these operators is defined by a pair of functions: its drift vector and its diffusion operator. Denote them by $(a_i(x), b_i(x))_{x\in D_i\cup S}$ and $(a_e(x), b_e(x))_{x\in D_e\cup S}$, respectively. We suppose that each of these pairs can be extended onto the whole space $\mathbb{R}^d$ as bounded Hölder continuous functions satisfying the uniform ellipticity condition. Then the functions

$$a(x) = \begin{cases} a_i(x) \text{ if } x \in D_i \\ a_e(x) \text{ if } x \in D_e, \end{cases} \qquad b(x) = \begin{cases} b_i(x) \text{ if } x \in D_i \\ b_e(x) \text{ if } x \in D_e \end{cases}$$

determine an elliptic operator in $\mathbb{R}^d$, whose coefficients have jumps (in general) at the points of S, and we have two conormal directions: $N_i(x) = b_i(x)\nu(x)$ and $N_e(x) = b_e(x)\nu(x), x \in S$. Therefore, the problem a$_0$)–d$_0$) should be rewritten in this situation as follows:

for a given continuous bounded function $(\varphi(x))_{x\in\mathbb{R}^d}$, a function

$$(u(t, x, \varphi))_{\substack{t>0\\ x\in\mathbb{R}^d}}$$

is looking for such that

a$_1$) it is continuous in $(t, x) \in (0, +\infty) \times \mathbb{R}^d$;

b$_1$) it satisfies the equation

$$\frac{\partial u}{\partial t} = (a_e(x), u'_x(t, x)) + \frac{1}{2}\mathrm{tr}(b_e(x)u''_{xx}(t, x)) \equiv \mathcal{L}_e u(t, x)$$

in the domain $t > 0, x \in D_e$ and the equation

$$\frac{\partial u}{\partial t} = (a_i(x), u'_x(t, x)) + \frac{1}{2}\mathrm{tr}(b_i(x)u''_{xx}(t, x)) \equiv \mathcal{L}_i u(t, x)$$

in the domain $t > 0, x \in D_i$;

c$_1$) it satisfies the initial condition

$$u(0+, x, \varphi) = \varphi(x),\ \ x \in \mathbb{R}^d;$$

d$_1$) it satisfies the “boundary” condition

$$\frac{1+q(x)}{2}\frac{\partial u}{\partial N_e(x)}(t, x+, \varphi) - \frac{1-q(x)}{2}\frac{\partial u}{\partial N_i(x)}(t, x-, \varphi) = 0$$

at the points $t > 0$ and $x \in S$ (as above, q is a given continuous function on S with its values in the interval $[-1, 1]$).

It is natural to look for a solution to this problem in the form

$$u(t, x, \varphi) = \int_{\mathbb{R}^d} g_e(t, x, y)\varphi(y)dy + \int_0^t d\tau \int_S g_e(\tau, x, y)V_e(t-\tau, y, \varphi)d\sigma_y$$

for $t > 0$ and $x \in D_e$ and

$$u(t,x,\varphi) = \int_{\mathbb{R}^d} g_i(t,x,y)\varphi(y)dy + \int_0^t d\tau \int_S g_i(\tau,x,y)V_i(t-\tau,y,\varphi)d\sigma_y$$

for $t > 0$ and $x \in D_i$, where V_e and V_i are some unknown functions, and g_e and g_i are the fundamental solutions for the operators $\mathcal{L}_e$ and $\mathcal{L}_i$, respectively.

The condition d$_1$) and the theorem on the jump of the conormal derivative of a single-layer potential gives us the first integral equation for the functions V_e and V_i. One more equation is a consequence of condition a$_1$). But unlike the previous one, it is an equation of a bad kind. Fortunately, this equation can be transformed by a special procedure into an equivalent equation of a nice kind (see Section 3). Then this pair of equations can be solved by the method of successive approximations. As a result, we obtain a diffusion process in $\mathbb{R}^d$ with its drift vector given by

$$a(x) + \frac{\sigma_e(x)+\sigma_i(x)}{2}\frac{(1+q(x))N_e(x)-(1-q(x))N_i(x)}{(1+q(x))\sigma_e(x)+(1-q(x))\sigma_i(x)}\delta_S(x),\ x \in \mathbb{R}^d,$$

where $\sigma_e(x)^2 = (N_e(x),\nu(x))$ and $\sigma_i(x)^2 = (N_i(x),\nu(x))$ for $x \in S$ and its diffusion operator given by the function $(b(x))_{x\in D_e\cup D_i}$ and by the expression

$$\frac{(1+q(x))\sigma_e(x)b_e(x)+(1-q(x))\sigma_i(x)b_i(x)}{(1+q(x))\sigma_e(x)+(1-q(x))\sigma_i(x)}$$

if $x \in S$.

3. General problem

Problems a$_0$)–d$_0$) and a$_1$)–d$_1$) considered above can be generalized in the following way.

As above, a pair of elliptic differential operators $\mathcal{L}_i$ and $\mathcal{L}_e$ are assumed to be given, as well as a continuous function $(q(x))_{x\in S}$ with its values in the interval $[-1,1]$. In addition, some continuous functions $(r(x))_{x\in S}$ and $(k(x))_{x\in S}$ with non-negative values should be given, which will characterize the property of our membrane to be sticky and absorbing, respectively. Moreover, an elliptic differential operator $\mathcal{L}_0$ on the manifold S should be given that will operate the diffusion along the surface S.

Given the objects listed, the problem is to construct a real-valued function $(u(t,x,\varphi))_{\substack{t>0\\ x\in\mathbb{R}^d}}$ (for any given continuous bounded function $(\varphi(x))_{x\in\mathbb{R}^d}$) such that

a) it is continuous in $(t,x) \in (0,+\infty)\times\mathbb{R}^d$;

b) it satisfies the equation

$$\frac{\partial u}{\partial t} = \mathcal{L}_e u,\ t>0, x\in D_e,$$

and the equation

$$\frac{\partial u}{\partial t} = \mathcal{L}_i u,\ t>0, x\in D_i;$$

c) it satisfies the initial condition

$$u(0+,x,\varphi) = \varphi(x),\ x\in\mathbb{R}^d;$$

d) it satisfies the conjugation condition

$$r(x)\frac{\partial u}{\partial t}(t,x,\varphi) = \frac{1+q(x)}{2}\frac{\partial u}{\partial N_e(x)}(t,x+,\varphi) -$$
$$-\frac{1-q(x)}{2}\frac{\partial u}{\partial N_i(x)}(t,x-,\varphi) + \mathcal{L}_0 u(t,x,\varphi) - k(x)u(t,x,\varphi)$$

for $t>0$ and $x\in S$.

Notice that condition d) is a slightly transformed version of the general Wentzel condition [3] (as we restrict ourselves to considering the continuous processes only, we do not write down the term in the Wentzel condition that corresponds to the possibilities for the process to jump into $D_e \cup D_i$ from the membrane S). It turns out that a solution of problem a)–d) can be constructed in a form similar to that of problem a$_1$)–d$_1$), where the unknown functions V_e and V_i in the corresponding single-layer potentials are determined from solving a certain system of the Volterra integral equations of the second kind (see [4]-[7]).

We give a short description of constructing such a solution of problem a)–d) under the following additional assumptions:

1°) the functions $(a_e(x), b_e(x))_{x\in\mathbb{R}^d}$ and $(a_i(x), b_i(x))_{x\in\mathbb{R}^d}$ are bounded and Hölder continuous with the exponent $\alpha \in (0,1)$;

2°) the matrices $(b_e(x))_{x\in\mathbb{R}^d}$ and $(b_i(x))_{x\in\mathbb{R}^d}$ are symmetric and uniformly positive definite;

3°) the surface S is a $(d-1)$-dimensional hypersurface of the class $H^{2+\alpha}$;

4°) the functions $(r(x))_{x\in S}, (q(x))_{x\in S}, (k(x))_{x\in S}$ and the coefficients of the operator $\mathcal{L}_0$ are bounded and Hölder continuous functions with the exponent α;

5°) the delaying coefficient $(r(x))_{x\in S}$ satisfies the condition $\inf_{x\in S} r(x) > 0$, and the operator $\mathcal{L}_0$ is a uniformly elliptic operator in the tangent variables;

6°) the initial function $(\varphi(x))_{x\in\mathbb{R}^d}$ is a one of the class $H^{2+\alpha}(\mathbb{R}^d)$, and it satisfies the corresponding consistency condition.

If conditions 1°)–6°) are fulfilled, then problem a)–d) has a unique classical solution $u(t,x) = u(t,x,\varphi),\ t > 0, x \in \mathbb{R}^d$, the restrictions of which on the regions $t > 0, x \in D_e$ and $t > 0, x \in D_i$ belong to the classes

$$H^{1+\frac{\alpha}{2},2+\alpha}([0,T]\times\overline{D}_e) \quad \text{and} \quad H^{1+\frac{\alpha}{2},2+\alpha}([0,T]\times\overline{D}_i),$$

respectively.

In order to prove this assertion, one should write down the solution desired as the sum of two potentials: the Poisson potential

$$u_{0e}(t,x,\varphi) = \int_{\mathbb{R}^d} g_e(t,x,y)\varphi(y)dy$$

in the region $t > 0, x \in D_e$ or

$$u_{0i}(t,x,\varphi) = \int_{\mathbb{R}^d} g_i(t,x,y)\varphi(y)dy$$

in the region $t > 0, x \in D_i$ and the single-layer potential

$$u_{1e}(t,x,\varphi) = \int_0^t d\tau \int_S g_e(t-\tau,x,y)V_e(\tau,y,\varphi)d\sigma_y$$

in the region $t > 0, x \in D_e$ or

$$u_{1i}(t,x,\varphi) = \int_0^t d\tau \int_S g_i(t-\tau,x,y)V_i(\tau,y,\varphi)d\sigma_y$$

in the region $t > 0, x \in D_i$, where V_e and V_i are unknown functions.

Suppose *a priori* that V_e and V_i are functions from the class

$$\overset{\frac{\alpha}{2},\alpha}{\underset{0}{H}}([0,T]\times S).$$

In order to find them, we make use of conditions a) and d). Consider first condition d). Setting $r(x) \equiv 1$, we write down this condition in the form

d′) $\mathcal{L}'u(t,x,\varphi) \equiv \frac{\partial u}{\partial t}(t,x,\varphi) - \mathcal{L}_0 u(t,x,\varphi) + k(x)u(t,x,\varphi) = \theta_0(t,x,\varphi)$

for $t > 0, x \in S$, where

$$\theta_0(t,x,\varphi) = \frac{1+q(x)}{2}\frac{\partial u(t,x+,\varphi)}{\partial N_e(x)} - \frac{1-q(x)}{2}\frac{\partial u(t,x-,\varphi)}{\partial N_i(x)}.$$

The derivatives

$$\frac{\partial u(t,x+,\varphi)}{\partial N_e(x)} \quad \text{and} \quad \frac{\partial u(t,x-,\varphi)}{\partial N_i(x)}$$

can be found out from the well-known theorem on the jump of the conormal derivative of a single-layer potential. As a result, we get the following formula for the function θ_0:

$$\theta_0(t,x,\varphi)=\frac{1+q(x)}{2}\frac{\partial u_{0e}(t,x,\varphi)}{\partial N_e(x)}-\frac{1-q(x)}{2}\frac{\partial u_{0i}(t,x,\varphi)}{\partial N_i(x)}+$$
$$+\frac{1+q(x)}{2}\int_0^t d\tau\int_S\frac{\partial g_e(t-\tau,x,y)}{\partial N_e(x)}V_e(\tau,y,\varphi)d\sigma_y-$$
$$-\frac{1-q(x)}{2}\int_0^t d\tau\int_S\frac{\partial g_i(t-\tau,x,y)}{\partial N_i(x)}(t-\tau,x,y)V_i(\tau,y,\varphi)d\sigma_y-$$
$$-\frac{1+q(x)}{2}V_e(t,x,\varphi)-\frac{1-q(x)}{2}V_i(t,x,\varphi).$$

Condition d$'$) can be considered as an autonomous parabolic equation in the region $t>0, x\in S$. Conditions 1°)–6°), the *a priori* assumptions on the functions V_e and V_i, and the well-known properties of the potentials in the formula for θ_0 allow us to assert that the coefficients of equation d$'$) belong to the class $H^\alpha(S)$, and its right-hand side is a function of the class $H^{\frac{\alpha}{2},\alpha}([0,T],S)$. Then, as it follows from conditions 1°)–6°), there exists a fundamental solution $\Gamma(t,x,y), t>0, x\in S, y\in S$, for the operator $\mathcal{L}'$ (see [5], [6]). This allows us to conclude that the classical solution to equation d$'$) satisfying the initial condition

$$u(0+,x,\varphi)=\varphi(x),\ \ x\in S,$$

exists, and it is unique. In addition, $u\in H^{1+\frac{\alpha}{2},2+\alpha}([0,T]\times S)$ and its norm satisfies the inequality

$$\|u\|_{H^{1+\frac{\alpha}{2},2+\alpha}([0,T]\times S)}\le \text{const}\Big(\|\varphi\|_{H^{2+\alpha}(S)}+\|\theta_0\|_{H^{\frac{\alpha}{2},\alpha}([0,T]\times S)}\Big).$$

Moreover, this solution can be written in the form

$$u(t,x,\varphi)=\int_S\Gamma(t,x,y)\varphi(y)d\sigma_y+\int_0^t d\tau\int_S\Gamma(t-\tau,x,y)\theta_0(\tau,y,\varphi)d\sigma_y,\ \ t>0, x\in S.$$

We now have two different expressions for the function $(u(t,x,\varphi))_{\substack{t>0\\ x\in S}}$: the previous one and that which can be obtained from the representations of u as a sum of the potentials restricted to S. Taking into account condition a) and the formula for θ_0, we arrive at the following system of integral equations for the functions V_e and V_i:

$$\int_0^t d\tau\int_S g_i(t-\tau,x,y)V_i(\tau,y,\varphi)d\sigma_y+\int_0^t d\tau\int_S[K_i(t-\tau,x,y)V_i(\tau,y,\varphi)+$$
$$+K_e(t-\tau,x,y)V_e(\tau,y,\varphi)]d\sigma_y=f_i(t,x,\varphi),\ \ t>0, x\in S;$$
$$\int_0^t d\tau\int_S g_e(t-\tau,x,y)V_e(\tau,y,\varphi)d\sigma_y+\int_0^t d\tau\int_S[K_i(t-\tau,x,y)V_i(\tau,y,\varphi)+$$
$$+K_e(t-\tau,x,y)V_e(\tau,y,\varphi)]d\sigma_y=f_e(t,x,\varphi),\ \ t>0, x\in S,$$

where

$$K_i(t-\tau,x,y)=\Gamma(t-\tau,x,y)\frac{1-q(y)}{2}+\int_\tau^t ds\int_S\Gamma(t-s,x,z)\frac{1-q(z)}{2}\frac{\partial g_i(s-\tau,z,y)}{\partial N_i(z)}d\sigma_z,$$
$$K_e(t-\tau,x,y)=\Gamma(t-\tau,x,y)\frac{1+q(y)}{2}-\int_\tau^t ds\int_S\Gamma(t-s,x,z)\frac{1+q(z)}{2}\frac{\partial g_e(s-\tau,z,y)}{\partial N_e(z)}d\sigma_z,$$
$$f_i(t,x,\varphi)=f_0(t,x,\varphi)-u_{0i}(t,x,\varphi),$$
$$f_e(t,x,\varphi)=f_0(t,x,\varphi)-u_{0e}(t,x,\varphi),$$
$$f_0(t,x,\varphi)=\int_S\Gamma(t,x,y)\varphi(y)d\sigma_y+\int_0^t d\tau\int_S\Gamma(t-\tau,x,y)\times$$
$$\times\left[\frac{1+q(y)}{2}\frac{\partial u_{0e}(\tau,y,\varphi)}{\partial N_e(y)}-\frac{1-q(y)}{2}\frac{\partial u_{0i}(\tau,y,\varphi)}{\partial N_i(y)}\right]d\sigma_y.$$

The functions f_e and f_i belong to the class $\underset{0}{H}^{1+\frac{\alpha}{2},2+\alpha}([0,T]\times S)$, and the kernels K_i and K_e satisfy the inequalities

$$|K_i(t-\tau,x,y)| \le C(t-\tau)^{-\frac{d-1}{2}}\exp\left\{-c\frac{|y-x|^2}{t-\tau}\right\},$$

$$|K_e(t-\tau,x,y)| \le C(t-\tau)^{-\frac{d-1}{2}}\exp\left\{-c\frac{|y-x|^2}{t-\tau}\right\},$$

in any region of the form $0\le\tau<t\le T, x\in S, y\in S$.

As seen, each equation of the system is a Volterra integral equation of the first kind. In order to regularize them, we introduce integro-differential operators $\mathcal{E}_i$ and $\mathcal{E}_e$ for the first and the second equation, respectively. They are analogous to those given in [8], [9] for the solution of the first boundary-value problem. In particular, if $S=\mathbb{R}^{d-1}$, then the operator $\mathcal{E}_e$ acts on a function $(f(t,x'))_{\substack{t>0\\ x'\in\mathbb{R}^{d-1}}}$ according to the formula

$$\mathcal{E}_e(t,x')f=\sqrt{\frac{2}{\pi}}\left\{\frac{\partial}{\partial t}\int_0^t(t-\tau)^{-\frac{1}{2}}d\tau\int_{\mathbb{R}^{d-1}}h_e(\widetilde{t}-\tau,x',y')f(\tau,y')dy'\right\}\bigg|_{\widetilde{t}=t},$$

where $h_e(t,x',y'), t>0, x'\in\mathbb{R}^{d-1}, y'\in\mathbb{R}^{d-1}$, is the fundamental solution associated with a uniformly parabolic operator that is the trace of the main part of the operator $\frac{\partial}{\partial t}-\mathcal{L}_e$ on the region $\sum=(0,\infty)\times\mathbb{R}^{d-1}$ in the global inner coordinates of this surface. The operator $\mathcal{E}_i$ is defined by analogy. In the case of a general manifold $\sum=(0,\infty)\times S$, one should make use of the atlas of a $(d-1)$-dimensional manifold S generated by a partition of unity. We would like to emphasize that the operators $\mathcal{E}_e$ and $\mathcal{E}_i$ are constructed by a local procedure with the use of fundamental solutions corresponding to the operators that are the traces of the main parts of the operators $\mathcal{L}_e$ and $\mathcal{L}_i$ on $\sum$ in local inner coordinates.

After the operators $\mathcal{E}_i$ and $\mathcal{E}_e$ have been applied to both sides of the corresponding equations, the system is transformed into an equivalent one of Volterra integral equations of the second kind. They can be written in the form

$$V_i(t,x,\varphi)+\int_0^t d\tau\int_S[R_{1i}(t-\tau,x,y)V_i(\tau,y,\varphi)+R_{1e}(t-\tau,x,y)V_e(\tau,y,\varphi)]d\sigma_y=$$
$$=f_{1i}(t,x,\varphi), t>0, x\in S,$$

$$V_e(t,x,\varphi)+\int_0^t d\tau\int_S[R_{2i}(t-\tau,x,y)V_i(\tau,y,\varphi)+R_{2e}(t-\tau,x,y)V_e(\tau,y,\varphi)]d\sigma_y=$$
$$=f_{2e}(t,x,\varphi), t>0, x\in S,$$

where

$$f_{1i}(t,x,\varphi)=(b_i(x)\nu(x),\nu(x))^{1/2}\mathcal{E}_i(t,x)f_i,$$

$$f_{1e}(t,x,\varphi)=(b_e(x)\nu(x),\nu(x))^{1/2}\mathcal{E}_e(t,x)f_e,$$

and the kernels $R_{1i},R_{1e},R_{2i},R_{2e}$ for $0\le\tau<t\le T, x\in S$, and $y\in S$ are estimated by a function of the form

$$C(t-\tau)^{-\frac{d+1-\alpha}{2}}\exp\left\{-c\frac{|y-x|^2}{t-\tau}\right\}.$$

It is also proved that the functions $\mathcal{E}_if_i,\mathcal{E}_ef_e$ and f_{1i},f_{1e} are elements from the spaces

$$\underset{0}{H}^{\frac{1+\alpha}{2},1+\alpha}([0,T]\times S)$$

and $\underset{0}{H}^{\frac{\alpha}{2},\alpha}([0,T]\times S)$, respectively.

We can now solve this new system of equations for the functions V_i and V_e by the method of successive approximations and establish that they belong to the class

$\underset{0}{H}^{\frac{\alpha}{2},\alpha}([0,T]\times S)$, and their norms satisfy the inequalities

$$\|V_i\|_{\underset{0}{H}^{\frac{\alpha}{2},\alpha}([0,T]\times S)} \leq C\|\varphi\|_{H^{2+\alpha}(\mathbb{R}^d)},$$
$$\|V_e\|_{\underset{0}{H}^{\frac{\alpha}{2},\alpha}([0,T]\times S)} \leq C\|\varphi\|_{H^{2+\alpha}(\mathbb{R}^d)}.$$

We have just constructed a solution to problem a)–d). As for other properties of the solution declared above, they can be easily established by observing that the restrictions of this solution onto each of the regions $(t > 0, x \in D_i)$ and $(t > 0, x \in D_e)$ can be treated as a solution of the first boundary-value problem for the corresponding parabolic equation.

If the initial function φ does not satisfy the consistency condition, then the solution to problem a)–d) is a bounded continuous function in the region $t \geq 0, x \in \mathbb{R}^d$, and its restrictions onto the regions $(t > 0, x \in D_i)$ and $(t > 0, x \in D_e)$ belong to the classes $\mathbb{C}^{1,2}((0,\infty)\times D_i)$ and $\mathbb{C}^{1,2}((0,\infty)\times D_e)$, respectively.

The integral representation of the solution to problem a)–d) given above is used then for constructing the process desired. It turns out to be a diffusion process in the Kolmogorov's sense: its local characteristics (drift vector and diffusion operator) exist in an ordinary sense as ordinary (though discontinuous) functions (they are not generalized functions).

Remark. Some probabilistic methods for constructing various classes of generalized diffusion processes were proposed by L. Zaitseva (1999–2003), and the results were given in her PhD thesis, Kyiv, 2004 (see [10]).

We conclude this section with the following aspects of the problem discussed above.

1) The absorbing coefficient can be taken into account by the arguments based on the Feynman–Kac formula. For example, if $(x(t))_{t\geq 0}$ is a continuous Markov process constructed in Section 1, then, for any continuous bounded function $(k(x))_{x\in S}$ with non-negative values, the function

$$u(t,x) = \mathbb{E}_x\varphi(x(t))\exp\left\{-\int_0^t k(x(\tau))d\eta_\tau\right\},\ t\geq 0, x\in\mathbb{R}^d,$$

satisfies conditions a$_0$)–c$_0$) and condition

d$_0'$) $\dfrac{1+q(x)}{2}\dfrac{\partial u}{\partial N(x)}(t,x+) - \dfrac{1-q(x)}{2}\dfrac{\partial u}{\partial N(x)}(t,x-) - k(x)u(t,x) = 0$

valid for $t > 0$ and $x \in S$.

2) The property of the points of S to be sticky can be reached by a random change of time in the process, for which $r(x) \equiv 0$. For example, if $(x(t))_{t\geq 0}$ is the process of Section 1 and a given function $(r(x))_{x\in S}$ with positive values is continuous and bounded, then we put, for $t \geq 0$,

$$\zeta_t = \inf\{s:\ s + \int_0^s r(x(\tau))d\eta_\tau \geq t\},\ \widehat{x}(t) = x(\zeta_t).$$

It can be proved that, for any continuous bounded function $(\varphi(x))_{x\in\mathbb{R}^d}$, the function

$$\widehat{u}(t,x) = \mathbb{E}_x\varphi(\widehat{x}(t)),\ t\geq 0, x\in\mathbb{R}^d,$$

satisfies conditions a$_0$)–c$_0$) and condition

d$_0''$) $r(x)\dfrac{\partial\widehat{u}}{\partial t}(t,x) = \dfrac{1+q(x)}{2}\dfrac{\partial\widehat{u}}{\partial N(x)}(t,x+) - \dfrac{1-q(x)}{2}\dfrac{\partial\widehat{u}}{\partial N(x)}(t,x-)$

valid for $t > 0$ and $x \in S$.

It is interesting to notice that the process $(\widehat{x}(t))_{t\geq 0}$ is a diffusion one in Kolmogorov's sense. This circumstance was observed by B. Kopytko and Zh. Tsapovska in 1998 (see [4]). Moreover, the process $(\widehat{x}(t))_{t\geq 0}$ turns out to have the following property:

the process

$$\left(\widehat{x}(t)-\widehat{x}(0)-\int_0^t \frac{q(\widehat{x}(\tau))}{r(\widehat{x}(\tau))}1\!\!\mathrm{I}_S(\widehat{x}(\tau))N(\widehat{x}(\tau))d\tau\right)_{t\geq 0}$$

is a square integrable martingale, whose square characteristic is given by

$$\int_0^t b(\widehat{x}(\tau))1\!\!\mathrm{I}_{\mathbb{R}^d\setminus S}(\widehat{x}(\tau))d\tau,\ t\geq 0.$$

In other words, the process $(\widehat{x}(\tau))_{t\geq 0}$ solves the martingale problem with the coefficients $\widehat{a}(x)=\frac{q(x)}{r(x)}1\!\!\mathrm{I}_S(x)N(x)$ and $\widehat{b}(x)=b(x)1\!\!\mathrm{I}_{\mathbb{R}^d\setminus S}(x), x\in\mathbb{R}^d$. This problem has, clearly, more than one solution.

It is curious to notice that there is a very simple way to explain why, under the random change of time described above, the diffusion process with the coefficients $(a(x)=q(x)\delta_S(x)N(x),\ b(x))_{x\in\mathbb{R}^d}$ is transformed into the process with the coefficients $(\widehat{a}(x),\widehat{b}(x))_{x\in\mathbb{R}^d}$. Namely, we have, for $x\in\mathbb{R}^d$,

$$\widehat{a}(x)=\frac{q(x)\delta_S(x)N(x)}{1+r(x)\delta_S(x)}=1\!\!\mathrm{I}_S(x)\frac{q(x)N(x)}{r(x)},$$

$$\widehat{b}(x)=\frac{b(x)}{1+r(x)\delta_S(x)}=1\!\!\mathrm{I}_{\mathbb{R}^d\setminus S}(x)b(x).$$

Of course, this arithmetic cannot serve as a rigorous proof of the result; a quite rigorous one is given by O. Aryasova and M. Portenko in [2].

4. Examples

1.The process of the first example is a result of pasting together two diffusion process on a real line $\mathbb{R}^1$ (see [11]). Let the numbers $b_1>0, b_2>0$, and $q\in[-1,1]$ be fixed.

We put

$$d=(1-q)\sqrt{b_1}+(1+q)\sqrt{b_2};\ q_1=d^{-1}(1-q)\sqrt{b_1};$$
$$q_2=d^{-1}(1+q)\sqrt{b_2},\ c=2(\sqrt{b_1}+\sqrt{b_2})^{-1};$$
$$D_1=\{x\in\mathbb{R}^1: x<0\},\ D_2=\{x\in\mathbb{R}^1: x>0\}.$$

For $t>0, x\in D_i\cup\{0\}$, and $y\in D_j$, where $i=1,2$ and $j=1,2$, we define the function g by setting

$$g(t,x,y)=(2\pi b_j t)^{-1/2}\Bigg[\exp\left\{-\frac{1}{2t}\left(\frac{y}{\sqrt{b_j}}-\frac{x}{\sqrt{b_i}}\right)^2\right\}+$$
$$+(q_2-q_1)\mathrm{sign}y\exp\left\{-\frac{1}{2t}\left(\frac{|y|}{\sqrt{b_j}}+\frac{|x|}{\sqrt{b_i}}\right)^2\right\}\Bigg].$$

For $y=0$, we set

$$g(t,x,0)=\frac{c}{2}[\sqrt{b_1}g(t,x,0-)+\sqrt{b_2}g(t,x,0+)], t>0, x\in\mathbb{R}^1.$$

It is not a difficult exercise to verify that there exists a continuous Markov process $(x(t))_{t\geq 0}$ in $\mathbb{R}^1$ such that its transition probability density is given by the function g. Moreover, $(x(t))_{t\geq 0}$ is a (generalized) diffusion process with its drift coefficient $(a(x))_{x\in\mathbb{R}^1}$ given by

$$a(x)=\widetilde{q}\cdot\widetilde{\delta}(x),\ x\in\mathbb{R}^1$$

and its diffusion coefficient $(b(x))_{x\in\mathbb{R}^1}$ given by

$$b(x)=\sum_{i=1}^2 b_i 1\!\!\mathrm{I}_{D_i}(x)+1\!\!\mathrm{I}_{\{0\}}(x)(q_1b_1+q_2b_2), x\in\mathbb{R}^1,$$

where $\widetilde{q}=c^{-1}[q_2\sqrt{b_2}-q_1\sqrt{b_1}]$ and $\widetilde{\delta}$ is the non-symmetric (in general) Dirac function: $\langle\widetilde{\delta},\varphi\rangle=\frac{c}{2}[\sqrt{b_1}\varphi(0-)+\sqrt{b_2}\varphi(0+)]$.

A W-functional $(\eta_t)_{t\ge 0}$ of the process $(x(t))_{t\ge 0}$ can be defined for which

$$\mathbb{E}_x\eta_t = c\int_0^t \exp\left\{-\frac{x^2}{2b_i\tau}\right\}\frac{d\tau}{\sqrt{2\pi\tau}}, t\ge 0,\ x\in D_i\cup\{0\}, i=1,2.$$

The functional $(\eta_t)_{t\ge 0}$ is a non-symmetric (in general) local time of the process $(x(t))_{t\ge 0}$ at the point $x=0$. It can be written in the form

$$\eta_t = \int_0^t \widetilde{\delta}(x(s))ds, t\ge 0.$$

Then the process $(x(t)-x(0)-\widetilde{q}\eta_t)_{t\ge 0}$ is a square integrable martingale with its characteristic given by

$$\int_0^t b(x(\tau))d\tau,\ t\ge 0.$$

For a fixed number $r>0$, we now put

$$\zeta_t = \inf\{s : s + r\eta_s \ge t\},\ \widehat{x}(t) = x(\zeta_t),\ t\ge 0.$$

Then the process $(\widehat{x}(t))_{t\ge 0}$ is a continuous strong Markov process in $\mathbb{R}^1$, for which

$$\mathbb{E}_x\varphi(\widehat{x}(t)) = \varphi(0)h(t,x) + \int_{\mathbb{R}^1}\varphi(y)\widehat{g}(t,x,y)dy, \quad t>0, x\in\mathbb{R}^1,$$

where

$$h(t,x) = \frac{2}{\sqrt{2\pi t}}\int_0^\infty \exp\left\{-\frac{2\theta}{cr} - \frac{1}{2t}\left(\theta + \frac{|x|}{\sqrt{b_i}}\right)^2\right\}d\theta,\ x\in D_i\cup\{0\}, i=1,2,$$

$$\widehat{g}(t,x,y) = \frac{1}{\sqrt{2\pi b_j t}}\left[\exp\left\{-\frac{1}{2t}\left(\frac{y}{\sqrt{b_j}} - \frac{x}{\sqrt{b_i}}\right)^2\right\} - \exp\left\{-\frac{1}{2t}\left(\frac{|y|}{\sqrt{b_j}} + \frac{|x|}{\sqrt{b_i}}\right)^2\right\}\right] +$$

$$+\frac{4}{cdr}(1+q\operatorname{sign}y)\int_0^\infty \exp\left\{-\frac{2\theta}{cr} - \frac{1}{2t}\left(\theta + \frac{|y|}{\sqrt{b_j}} + \frac{|x|}{\sqrt{b_i}}\right)^2\right\}\frac{d\theta}{\sqrt{2\pi t}}$$

for $t>0, x\in D_i\cup\{0\}$, and $y\in D_j$ $(i=1,2$ and $j=1,2)$. For $y=0$, we set

$$\widehat{g}(t,x,0) = \frac{c}{2}[\widehat{g}(t,x,0-)\sqrt{b_1} + \widehat{g}(t,x,0+)\sqrt{b_2}].$$

Denote, by $\widehat{P}$, the transition probability for the process $(\widehat{x}(t))_{t\ge 0}$.

Then a simple calculation shows that, for $t\ge 0$, $x\in\mathbb{R}^1$, and any real-valued twice continuously differentiable function $(\varphi(x))_{x\in\mathbb{R}^1}$ that is bounded together with its derivatives, the relation

$$\int_{\mathbb{R}^1}\varphi(y)\widehat{P}(t,x,dy) = \varphi(x) + \varphi'(0)\frac{\widetilde{q}}{r}\int_0^t h(s,x)ds + \frac{1}{2}\int_0^t ds\int_{\mathbb{R}^1}\varphi''(y)b(y)g(s,x,y)dy$$

is true.

Of course, this relation can be extended to some class of unbounded functions. In particular, one can easily conclude from this relation that the following equalities are fulfilled:

(i) $\sup\limits_{x\in\mathbb{R}^1}\int_{\mathbb{R}^1}(y-x)^4\widehat{P}(t,x,dy) = O(t^2)$ as $t\downarrow 0$;

(ii) $\lim\limits_{t\downarrow 0} t^{-1}\int_{\mathbb{R}^1}(y-x)\widehat{P}(t,x,dy) = \widehat{a}(x)$ for all $x\in\mathbb{R}^1$, where $\widehat{a}(x) = \frac{\widetilde{q}}{r}1\!\!I_{\{0\}}(x), x\in\mathbb{R}^1$;

(iii) $\lim\limits_{t\downarrow 0} t^{-1}\int_{\mathbb{R}^1}(y-x)^2\widehat{P}(t,x,dy) = \widehat{b}(x)$ for all $x\in\mathbb{R}^1$, where $\widehat{b}(x) = \sum\limits_{i=1}^2 b_i 1\!\!I_{D_i}(x), x\in\mathbb{R}^1$.

These equalities show that the transition probability $\widehat{P}$ defines a Markov process that is a diffusion one in Kolmogorov's sense, and the coefficients $(\widehat{a}(x),\widehat{b}(x))_{x\in\mathbb{R}^1}$ do not determine the process uniquely. The corresponding martingale problem also turns out to be ill-posed.

2. Let $(x(t))_{t\ge 0}$ be a skew Brownian motion in $\mathbb{R}^1$ that is determined by a fixed parameter $q\in[-1,1]$. This is a continuous Markov process in $\mathbb{R}^1$, whose transition

probability density is given by

$$G_0(s,x,t,y)=(2\pi(t-s))^{-1/2}\left[\exp\left\{-\frac{(y-x)^2}{2(t-s)}\right\}+q\,\mathrm{sign}y\exp\left\{-\frac{(|y|+|x|)^2}{2(t-s)}\right\}\right]$$

for $0\le s<t, x\in\mathbb{R}^1$, and $y\in\mathbb{R}^1$ (we believe that $\mathrm{sign}0=0$).

Denote, by $(\eta_t^s)_{0\le s\le t}$, the symmetric local time for the process $(x(t))_{t\ge 0}$ at the point $x=0$. As above, it can be written in the form

$$\eta_t^s=\int_s^t\delta(x(\tau))d\tau,\ 0\le s\le t,$$

where $(\delta(x))_{x\in\mathbb{R}^1}$ is an ordinary symmetric Dirac δ-function.

Then the functional $(w_t^s)_{0\le s\le t}$ defined by the relation

$$w_t^s=x(t)-x(s)-q\eta_t^s$$

is a square integrable martingale with its characteristic given by $(t-s)_{t\ge s}$.

For any measurable locally bounded function $(\varphi(t))_{t\ge 0}$ with real values, we put

$$\mathcal{E}_t^s(\varphi)=\exp\left\{\int_s^t\varphi(\tau)dw_\tau^s-\frac{1}{2}\int_s^t\varphi(\tau)^2d\tau\right\},0\le s\le t,$$

and denote, by G_φ, the function of the arguments (s,x,t,y) for $0\le s<t, x\in\mathbb{R}^1$, and $y\in\mathbb{R}^1$ defined by the relation

$$\mathbb{E}_{s,x}\mathcal{E}_t^s(\varphi)f(x(t))=\int_{\mathbb{R}^1}G_\varphi(s,x,t,y)f(y)dy$$

valid for all $0\le s\le t, x\in\mathbb{R}^1$, and a measurable bounded real-valued function $(f(x))_{x\in\mathbb{R}^1}$. Then the function G_φ has the representations

$$G_\varphi(s,x,t,y)=g_\varphi(s,x,t,y)+q\int_s^tV_\varphi(s,x,\tau)\frac{\partial g_\varphi(\tau,z,t,y)}{\partial z}\bigg|_{z=0}d\tau,$$

$$G_\varphi(s,x,t,y)=g_\varphi(s,x,t,y)+q\int_s^tg_\varphi(s,x,\tau,0)\widetilde{V}_\varphi(\tau,t,y)d\tau,$$

where

$$g_\varphi(s,x,t,y)=(2\pi(t-s))^{-1/2}\exp\left\{-\frac{1}{2(t-s)}(y-x-\Phi(t)+\Phi(s))^2\right\},$$

$\Phi(t)=\int_0^t\varphi(\tau)d\tau$, and the kernels V_φ and $\widetilde{V}_\varphi$ are defined by the relations

$$V_\varphi(s,x,t)=g_\varphi(s,x,t,0)+\int_s^tg_\varphi(s,x,\tau,0)R_\varphi(\tau,t)d\tau$$

and

$$\widetilde{V}_\varphi(s,t,y)=\frac{\partial g_\varphi(s,x,t,y)}{\partial x}\bigg|_{x=0}+\int_s^tR_\varphi(s,\tau)\frac{\partial g_\varphi(\tau,x,t,y)}{\partial x}\bigg|_{x=0}d\tau.$$

The kernel $(R_\varphi(s,t))_{0\le s<t}$ in these formulae is given by the sum

$$R_\varphi(s,t)=\sum_{n=1}^\infty(-q)^nK_\varphi^{(n)}(s,t),$$

where

$$K_\varphi^{(1)}(s,t)=\frac{\Phi(t)-\Phi(s)}{\sqrt{2\pi(t-s)^3}}\exp\left\{-\frac{(\Phi(t)-\Phi(s))^2}{2(t-s)}\right\}$$

and, for $n\ge 2$,

$$K_\varphi^{(n)}(s,t)=\int_s^tK_\varphi^{(1)}(s,\tau)K_\varphi^{(n-1)}(\tau,t)d\tau=\int_s^tK_\varphi^{(n-1)}(s,\tau)K_\varphi^{(1)}(\tau,t)d\tau.$$

Another pair of representations for G_φ is given by the equalities

$$G_\varphi(s,x,t,y)=G_0(s,x,t,y)+\int_s^t\varphi(\tau)d\tau\int_{\mathbb{R}^1}G_0(s,x,\tau,z)\frac{\partial G_\varphi(\tau,z,t,y)}{\partial z}dz,$$

$$G_\varphi(s,x,t,y) = G_0(s,x,t,y) + \int_s^t \varphi(\tau)d\tau \int_{\mathbb{R}^1} G_\varphi(s,x,\tau,z)\frac{\partial G_0(\tau,z,t,y)}{\partial z}dz.$$

3. Let $(r_0(t))_{t\geq 0}$ be the radial part of a two-dimensional Wiener process. It is a continuous Markov process in $\mathbb{R}_+ = [0,+\infty)$, whose transition probability density h_0 is given by

$$h_0(t,\rho,r) = \frac{r}{t}\exp\left\{-\frac{\rho^2+r^2}{2t}\right\}I_0\left(\frac{\rho r}{t}\right)$$

for $t>0, \rho\in\mathbb{R}_+, r\in\mathbb{R}_+$, where I_0 is the so-called modified Bessel function

$$I_0(z) = \sum_{n=0}^\infty \left(\frac{z}{2}\right)^{2n} \Big/ (n!)^2.$$

The process $(r_0(t))_{t\geq 0}$ can be considered as a diffusion one with its drift coefficient $a_0(r) = (2r)^{-1}$ and its diffusion coefficient $b_0(r)\equiv 1$.

Fix two parameters: $R>0$ and $q\in[-1,1]$. A (generalized) diffusion process $(r(t))_{t\geq 0}$ in $\mathbb{R}_+$ exists such that its drift $(a(r))_{r\in\mathbb{R}_+}$ is given by $a(r) = a_0(r) + q\delta_R(r)$, and its diffusion coefficient $(b(r))_{r\geq 0}$ coincides with that of the process $(r_0(t))_{t\geq 0}$. Denote, by h, the transition probability density of the process $(r(t))_{t\geq 0}$. Then the following two representations for the function h are valid:

$$h(t,\rho,r) = h_0(t,\rho,r) + q\int_0^t V(\tau,\rho)Q(t-\tau,R,r)d\tau,$$

$$h(t,\rho,r) = h_0(t,\rho,r) + q\int_0^t h_0(\tau,\rho,R)\widetilde{V}(t-\tau,r)d\tau,$$

where

$$Q(t,\rho,r) = \frac{\partial h_0(t,\rho,r)}{\partial \rho}$$

for $t>0, \rho>0$, and $r>0$, and the functions V and $\widetilde{V}$ are the solutions to the renewal equations (respectively)

$$V(t,\rho) = h_0(t,\rho,R) + q\int_0^t V(\tau,\rho)H_R(t-\tau)d\tau$$

and

$$\widetilde{V}(t,r) = Q(t,R,r) + q\int_0^t \widetilde{V}(\tau,r)H_R(t-\tau)d\tau$$

with the kernel $(H_R(t))_{t\geq 0}$ given by

$$H_R(t) = Q(t,R,R) = \frac{R^2}{t}\exp\left\{-\frac{R^2}{t}\right\}\left[I_1\left(\frac{R^2}{t}\right) - I_0\left(\frac{R^2}{t}\right)\right]$$

(here, $I_1(z) = I_0'(z) = \sum\limits_{n=0}^\infty \left(\frac{z}{2}\right)^{2n+1}\Big/ n!(n+1)!$).

The values of the function $(H_R(t))_{t>0}$ are negative, and

$$\int_0^\infty H_R(t)dt = -1.$$

One can easily write down the solutions to the equations for V and $\widetilde{V}$. Namely, if we put $H_R^{(1)}(t) = H_R(t)$ and, for $n\geq 2$,

$$H_R^{(n)}(t) = \int_0^t H_R^{(n-1)}(\tau)H_R(t-\tau)d\tau,\ t\geq 0,$$

then the series

$$G_R(t) = \sum_{n=1}^\infty q^n H_R^{(n)}(t)$$

is convergent locally uniformly in $t > 0$. The functions V and $\widetilde{V}$ can be now represented as follows:

$$V(t,\rho) = h_0(t,\rho,R) + \int_0^t h_0(\tau,\rho,R)G_R(t-\tau)d\tau,$$

$$\widetilde{V(t,r)} = Q(t,R,r) + \int_0^t Q(\tau,R,r)G_R(t-\tau)d\tau.$$

Substituting these expressions into the formulae for the function h, we obtain some explicit representations for it.

Notice that the process $(r_0(t))_{t\geq 0}$ is the radial part of a two-dimensional Wiener process. Its circular part is given by $\left(\theta\left(\int\limits_0^t r_0(s)^{-2}ds\right)\right)_{t\geq 0}$, where $(\theta(t))_{t\geq 0}$ is a Brownian motion on a circle of unit radius that does not depend on the process $(r_0(t))_{t\geq 0}$. Therefore, $\left(r(t),\theta\left(\int\limits_0^t r(s)^{-2}ds\right)\right)_{t\geq 0}$ is a two-dimensional Wiener process with a membrane on the circle of radius R. This process was constructed in [12].

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