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ANOTHER VIEW ON THE LOCAL TIME OF SELF-INTERSECTIONS FOR A FUNCTION OF THE WIENER PROCESS

The article is devoted to the local time of self-intersections for the process $F(w)$, where $F : \mathbb{R}^2 \to \mathbb{R}^2$ is smooth function, and w is the standard planar Brownian motion. We present the local time of self-intersections for the process $F(w)$ in terms of a manifold.

Introduction

Let us consider the local time of self-intersections for a random process $\{\xi(t),\ t \in [0,1]\}$ formally defined as

$$T_0^{\xi} = \int_0^1 \int_0^{s_2} \delta_0(\xi(s_2) - \xi(s_1)) ds_1 ds_2, \tag{1}$$

where δ_0 is the delta-function concentrated at 0. Expression (1) can be understood as a limit in square mean of the random variables

$$T_{\varepsilon}^{\xi} = \int_0^1 \int_0^{s_2} f_{\varepsilon}(\xi(s_2) - \xi(s_1)) ds_1 ds_2, \tag{2}$$

where $f_{\varepsilon}(x) = \frac{1}{2\pi\varepsilon} e^{-\frac{\|x\|^2}{2\varepsilon}},\ x \in \mathbb{R}^2$.

It is known [1-3] that, for a planar Wiener process w, as well as for a planar diffusion process Y described by the stochastic differential equation

$$\begin{cases} dY(s) = a(Y(s))ds + B(Y(s))dw(s), \\ Y(0) = y_0 \end{cases}$$

with Lipschitz coefficients a and B, such a limit does not exist. That is why, instead of (2), one can consider

$$\widetilde{T}_{\varepsilon}^{\xi} = T_{\varepsilon}^{\xi} - ET_{\varepsilon}^{\xi}.$$

It was proved in [1] that, for the planar Wiener process, a limit in square mean of $\widetilde{T}_{\varepsilon}^{\xi}$ exists. It is known [2, 3] that

$$ET_{\varepsilon}^{w} \sim \frac{1}{2\pi} \ln \frac{1}{\varepsilon},\ \varepsilon \to 0+, \tag{3}$$

$$ET_{\varepsilon}^{Y} \sim \frac{1}{2\pi} \ln \frac{1}{\varepsilon} E \int_0^1 \frac{1}{|\det B(Y(s))|} ds,\ \varepsilon \to 0+ . \tag{4}$$

Using (3) and (4), the hypothesis can be put forward that the constant of renormalization for the process $F(w)$ is equivalent to

$$\frac{1}{2\pi} \ln \frac{1}{\varepsilon} E \int_0^1 \frac{1}{|\det F'(w(s))|} ds,\ \varepsilon \to 0+ .$$

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In this paper, we prove that the local time of self-intersections for the process $F(w)$ does not exist. We also define the local time of self-intersections in terms of a manifold and prove the existence of the limiting expectation on a rectangle.

1. Behavior of the limiting expectation

Let w be a Wiener process in $\mathbb{R}^2$, and let $C^2(\mathbb{R}^2,\mathbb{R}^2)$ be the space of twice continuously differentiable functions from $\mathbb{R}^2$. Here, we investigate a random process $F(w)$, where

1) $F \in C^2(\mathbb{R}^2,\mathbb{R}^2)$,

2) there exists $C_1, C_2 > 0: \ C_1 \le |\det F'| \le C_2$.

Consider

$$T_\varepsilon^{F(w)} = \int_0^1 \int_0^{s_2} f_\varepsilon(F(w(s_2)) - F(w(s_1)))ds_1 ds_2.$$

The following theorem holds.

Theorem 1.1. $ET_\varepsilon^{F(w)} \to +\infty,\ \varepsilon \to 0+.$

Proof. Since $F \in C^2(\mathbb{R}^2,\mathbb{R}^2)$, there exists $L > 0$ for any $R > 0$ such that, for $\|u_1\| \vee \|u_2\| \le R$,

$$\|F(u_1) - F(u_2)\| \le L\|u_1 - u_2\|. \tag{5}$$

Inequality (5) yields

$$ET_\varepsilon^{F(w)} \ge \int_0^1 \int_0^{s_2} \int_{\mathbb{R}^{2\times 2}} \frac{1}{2\pi\varepsilon} e^{-\frac{L^2\|x_2-x_1\|^2}{2\varepsilon}} 1\!\!1_{\{\|x_1\|\vee\|x_2\|\le R\}} \cdot$$

$$\cdot \frac{1}{2\pi s_1} e^{-\frac{\|x_1\|^2}{2s_1}} \frac{1}{2\pi(s_2-s_1)} e^{-\frac{\|x_2-x_1\|^2}{2(s_2-s_1)}} dx_1dx_2ds_1ds_2.$$

Applying Fatou's lemma, we have

$$\varliminf_{\varepsilon\to 0+} ET_\varepsilon^{F(w)} \ge \frac{1}{L^2} \int_0^1 \int_0^{s_2} \int_{\{x_1:\|x_1\|\le R\}} \frac{1}{2\pi(s_2-s_1)} \frac{1}{2\pi s_1} e^{-\frac{\|x_1\|^2}{2s_1}} dx_1ds_1ds_2.$$

Hence,

$$\varliminf_{\varepsilon\to 0+} ET_\varepsilon^{F(w)} \ge \frac{1}{L^2} \int_0^1 \int_0^{s_2} \frac{1}{2\pi(s_2-s_1)} \big(1 - e^{-\frac{R^2}{2}}\big)ds_1ds_2 = +\infty.$$

The theorem is proved. □

2. Local time of self-intersections in terms of a manifold

Theorem 1.1 implies that the random variable $\{T_\varepsilon^{F(w)}\}_{\varepsilon>0}$ does not converge in square mean. One can check that, for a planar Wiener process, there exists

$$L_2 - \lim_{\varepsilon\to 0+} \int_t^1 \int_0^t f_\varepsilon(w(s_2) - w(s_1))ds_1ds_2,$$

where the limit is the local time of self-intersections for the planar Wiener process on the rectangle $[0,t]\times[t,1]$, $t \in (0,1)$. This is a motivation to consider the local time of self-intersections for the process $F(w)$ on this rectangle. We introduce a new definition for the local time of self-intersections for the process $F(w)$ in terms of a manifold and prove the existence of the limiting expectation on the same rectangle for it. There is some connection between the "new" and "old" definitions of the local time of self-intersections for the process $F(w)$ which will be explained further. Let us define the local time of self-intersections for the process $F(w)$ in terms of a manifold. For $M = \{(u,v): F(u) = F(v)\}$, where σ is the surface measure on M, we write

$$\int_t^1 \int_0^t \int_M \delta_0(w(s_1) - u)\delta_0(w(s_2) - v)\sigma(du,dv)ds_1ds_2 \tag{6}$$

instead of

$$\int_t^1 \int_0^t \delta_0(F(w(s_2)) - F(w(s_1)))ds_1 ds_2.$$

Let us define expression (6) in a more precise way. Consider $G : \mathbb{R}^4 \to \mathbb{R}^2$ such that

$$G(u, v) = F(v) - F(u).$$

By definition of M, for any $(u_0, v_0) \in M$, we have

1) $G(u_0, v_0) = 0$,
2) $G \in C^1(\mathbb{R}^2, \mathbb{R}^2)$,
3) $\det G'_v(u_0, v_0) \neq 0$.

Then, by the implicit function theorem, there exist an open ball $B(u_0, r) \subset \mathbb{R}^2$ and a unique function $h \in C^1(B(u_0, r), \mathbb{R}^2)$ such that

1) $h(u_0) = v_0$,
2) $\forall u \in B(u_0, r) : G(u, h(u)) = 0$,
3) $\forall u \in B(u_0, r)$:

$$h'(u) = (G'_v(u, h(u)))^{-1} \cdot G'_u(u, h(u)) =$$
$$= (F'_v(h(u)))^{-1} \cdot F'_u(h(u))$$

and, consequently,

$$\int_{M'} \varphi(u, v)\sigma(du, dv) =$$
$$= \int_{B(u_0,r)} \varphi(u, h(u))\rho(u, h(u))du,$$

where $M' := ((u, v) : u \in B(u_0, r), v = h(u))$, φ is an arbitrary continuous finite function on $\mathbb{R}^4$ with bounded support, and ρ is calculated in usual way [4]. It follows from [4] that $\rho > 0$. We note that the closed manifold M can be covered with a countable number of balls which satisfy conditions 1)-3) of the implicit function theorem. Consequently, the integral over the manifold M can be defined as follows:

$$\int_M \varphi(u, v)\sigma(du, dv) = \int_{\mathbb{R}^2} \sum_{v:F(v)=F(u)} \varphi(u, v)\rho(u, v)du. \tag{7}$$

For $s_1 \in [0, t], s_2 \in [t, 1]$, $t \in (0; 1)$ $u, v \in M$, we denote the expression $Ef_\varepsilon(w(s_1) - u)f_\varepsilon(w(s_2) - v)$ by $g_\varepsilon(s_1, s_2, u, v)$. One can check that

$$g_\varepsilon(s_1, s_2, u, v) = \frac{1}{(2\pi)^2} \frac{1}{(s_1 + \varepsilon)(s_2 + \varepsilon) - s_1^2} \cdot$$
$$\cdot \exp\{-\frac{(s_2 - s_1)\|u\|^2 + s_1\|v - u\|^2 + \varepsilon(\|u\|^2 + \|v\|^2)}{2((s_1 + \varepsilon)(s_2 + \varepsilon) - s_1^2)}\}.$$

It is obvious that

$$\lim_{\varepsilon \to 0+} g_\varepsilon(s_1, s_2, u, v) = \frac{1}{2\pi s_1} e^{-\frac{\|u\|^2}{2s_1}} \frac{1}{2\pi(s_2 - s_1)} e^{-\frac{\|v-u\|^2}{2(s_2-s_1)}}.$$

According to (7), it is natural to suppose that the limit of

$$\int_t^1 \int_0^t \int_M g_\varepsilon(s_1, s_2, u, v)\sigma(du, dv)ds_1 ds_2$$

is equal to

$$\int_t^1 \int_0^t \int_{\mathbb{R}^2} \frac{1}{2\pi s_1} e^{-\frac{\|u\|^2}{2s_1}} \sum_{v:F(v)=F(u)} \frac{1}{2\pi(s_2 - s_1)} e^{-\frac{\|v-u\|^2}{2(s_2-s_1)}} \rho(u, v)du ds_1 ds_2$$

as $\varepsilon \to 0+$.

On the other hand, using a change of variables, one can check that, for a nonnegative function $f:\mathbb{R}^2\to\mathbb{R}^2$, the following equality holds:

$$\int_{\mathbb{R}^2} f(F(x),x)dx=\int_{\mathbb{R}^2}\sum_{x:F(x)=y} f(y,x)\cdot\frac{1}{|\det F'(x)|}dy.$$

This yields

$$Ef_\varepsilon(F(w(s_2))-F(w(s_1)))=\int_{\mathbb{R}^{2\times 2}} f_\varepsilon(x_2-x_1)\cdot$$
$$\cdot\sum_{y_1,y_2:F(y_1)=x_1,F(y_2)=x_2}\frac{1}{2\pi s_1}e^{-\frac{\|y_1\|^2}{2s_1}}\frac{1}{2\pi(s_2-s_1)}e^{-\frac{\|y_2-y_1\|^2}{2(s_2-s_1)}}\cdot$$
$$\cdot\frac{1}{|\det F'(y_1)|}\cdot\frac{1}{|\det F'(y_2)|}dx_1dx_2.$$

We denote

$$q(x_1,x_2):=\sum_{y_1,y_2:F(y_1)=x_1,F(y_2)=x_2}\frac{1}{2\pi s_1}e^{-\frac{\|y_1\|^2}{2s_1}}\frac{1}{2\pi(s_2-s_1)}e^{-\frac{\|y_2-y_1\|^2}{2(s_2-s_1)}}\cdot$$
$$\cdot\frac{1}{|\det F'(y_1)|}\cdot\frac{1}{|\det F'(y_2)|}.$$

In the case where $q\in C_b(\mathbb{R}^{2\times 2})$,

$$\int_{\mathbb{R}^{2\times 2}} f_\varepsilon(x_2-x_1)q(x_1,x_2)dx\longrightarrow$$
$$\longrightarrow\int_{\mathbb{R}^2} q(x_1,x_1)dx_1$$

as $\varepsilon\to 0+$, where

$$\int_{\mathbb{R}^2} q(x,x)dx=\int_{\mathbb{R}^2}\frac{1}{2\pi s_1}e^{-\frac{\|y_1\|^2}{2s_1}}\sum_{y_2:F(y_2)=F(y_1)}\frac{1}{2\pi(s_2-s_1)}e^{-\frac{\|y_2-y_1\|^2}{2(s_2-s_1)}}\cdot$$
$$\cdot\frac{1}{|\det F'(y_2)|}dy_1.$$

Hence, we can expect that

$$\lim_{\varepsilon\to 0+}E\int_t^1\int_0^t\int_M f_\varepsilon(w(s_1)-u)f_\varepsilon(w(s_2)-v)\frac{1}{\rho(u,v)}\cdot$$
$$\cdot\frac{1}{|\det F'(v)|}\sigma(du,dv)ds_1ds_2=$$
$$=\lim_{\varepsilon\to 0+}E\int_t^1\int_0^t f_\varepsilon(F(w(s_2))-F(w(s_1)))ds_1ds_2.$$

In what follows, we use the following expression for an approximation of the local time of self-intersections for the process $F(w)$:

$$\mathcal{T}_\varepsilon=\int_t^1\int_0^t\int_M f_\varepsilon(w(s_1)-u)f_\varepsilon(w(s_2)-v)\gamma(u,v)\sigma(du,dv)ds_1ds_2.$$

Here,

$$\gamma(u,v)=\frac{1}{\rho(u,v)}\cdot\frac{1}{|\det F'(v)|}.$$

Theorem 2.1. *Suppose that, for a random variable η with density $p(x)=\frac{c}{1+\|x\|^4}$, the random variable $F(\eta)$ has a bounded density on $\mathbb{R}^2$. Then there exists a finite limit of $E\mathcal{T}_\varepsilon$ as $\varepsilon\to 0+$.*

Proof. We note that $g_\varepsilon(s_1, s_2, u, v)$ can be rewritten as

$$g_\varepsilon(s_1, s_2, u, v) = \frac{1}{(2\pi)^2}\frac{1}{s_1+\varepsilon}\cdot\frac{1}{s_2+\varepsilon-\frac{s_1^2}{s_1+\varepsilon}}\cdot$$

$$\cdot\exp\{-\frac{\|u\|^2}{2(s_1+\varepsilon)}\}\cdot\exp\{-\frac{\|v-\frac{s_1}{s_1+\varepsilon}u\|^2}{2((s_2+\varepsilon)-\frac{s_1^2}{s_1+\varepsilon})}\}.$$

For $\alpha > 0$, we put

$$\psi(\alpha, u, y) := \sum_{v:F(v)=y} e^{-\alpha\|v-u\|^2}\frac{1}{|\det F'(v)|}.$$

Following the definition of the integral over a manifold and changing the variable $u = z\sqrt{s_1+\varepsilon}$, we get

$$\int_M g_\varepsilon(s_1, s_2, u, v)\gamma(u, v)\sigma(du, dv) =$$

$$= \int_{\mathbb{R}^2}\frac{1}{(2\pi)^2}\cdot\frac{1}{s_2+\varepsilon-\frac{s_1^2}{s_1+\varepsilon}}\cdot e^{-\frac{\|z\|^2}{2}}.$$

$$\cdot\psi(\frac{1}{2(s_2+\varepsilon-\frac{s_1^2}{s_1+\varepsilon})}, z\frac{s_1}{\sqrt{s_1+\varepsilon}}, F(z\sqrt{s_1+\varepsilon}))dz.$$

To prove the theorem, we use the dominated convergence theorem. To apply it, let us check that

1) for $\alpha_0 > 0$:

$$\lim_{(\alpha,y,u)\to(\alpha_0,u_0,y_0)}\psi(\alpha, u, y) = \psi(\alpha_0, u_0, y_0).$$

2) there exists $c(s_1, s_2, z)$ such that:

$$\forall\varepsilon > 0 : |\frac{1}{s_2+\varepsilon-\frac{s_1^2}{s_1+\varepsilon}}\psi(\frac{1}{2(s_2+\varepsilon-\frac{s_1^2}{s_1+\varepsilon})}, z\frac{s_1}{\sqrt{s_1+\varepsilon}}, F(z\sqrt{s_1+\varepsilon}))| \leq$$

$$\leq c(s_1, s_2, z)$$

and

$$\int_t^1\int_0^t\int_{\mathbb{R}^2} c(s_1, s_2, z)dzds_1ds_2 < \infty.$$

Let us check the first condition of the dominated convergence theorem . We denote

$$\mathcal{K}_r(u_0) = \{v : r-1 \leq \|u_0 - v\| \leq r\},$$

$$\psi_{r_0}(\alpha, u, y) := \sum_{r=1}^{r_0}\sum_{v:F(v)=y,v\in K_r(u_0)} e^{-\alpha\|v-u\|^2}\frac{1}{|\det F'(v)|},$$

and

$$b_{r_0}(\alpha, u, y) := \sum_{r=r_0+1}^{\infty}\sum_{v:F(v)=y,v\in K_r(u_0)} e^{-\alpha\|v-u\|^2}\frac{1}{|\det F'(v)|}.$$

Then, using a new notation, $\psi(\alpha, u, y)$ can be rewritten as follows:

$$\psi(\alpha, u, y) = \psi_{r_0}(\alpha, u, y) + b_{r_0}(\alpha, u, y).$$

By the inverse function theorem [4], one can check that there exist some neighborhood of y_0 and the family of continuous functions $\{a_1(y), \cdots, a_n(y)\}$ which are the complete system of solutions of the equation $F(v) = y$ for y in this neighborhood of y_0, and $v \in \cup_{r=1}^{r_0} K_r(u_0)$.

This gives us a possibility to write that, in some neighborhood of (α_0, u_0, y_0),

$$\psi_{r_0}(\alpha, u, y) = \sum_{i=1}^{n} e^{-\alpha\|a_i(y)-u\|^2} \frac{1}{|\det F'(a_i(y))|}. \tag{8}$$

It follows from (8) that $\psi_{r_0}(\alpha, u, y)$ is a continuous function in some neighborhood of (α_0, u_0, y_0). Let us estimate $b_{r_0}(\alpha, u, y)$. We denote

$$N_r(u_0, y) = \#\{v : v \in \mathcal{K}_r(u_0), F(v) = y\}.$$

One can see that, for $\|u - u_0\| < r_0$,

$$b_{r_0}(\alpha, u, y) \le c_1 \sum_{r=r_0+1}^{\infty} N_r(u_0, y) e^{-\alpha(r-1-\|u-u_0\|)^2},$$

where c_1 is some positive constant. Let us check that the inequality

$$N_r(u_0, y) \le c_2(1 + (\|u_0\| + r)^4) \tag{9}$$

holds with some positive constant c_6. Put $q(x) = \frac{m \circ F^{-1}}{dx}(x)$, where

$$m(dx) = \frac{c_3}{1 + \|x\|^4} dx, \ c_3 > 0,$$

$$m \circ F^{-1}(A) = m\{x \in \mathbb{R}^2 : F(x) \in A\}$$

for some $A \subset \mathbb{R}^2$. Then, for some positive constants c_4 and c_5, we have

$$c_4 \ge q(x) = \sum_{y:F(y)=x} \frac{c_3}{1 + \|y\|^4} \frac{1}{|\det F'(y)|} \ge$$

$$\ge c_5 \sum_{y \in \mathcal{K}_r(u_0), F(y)=x} \frac{1}{1 + \|y\|^4} \ge$$

$$\ge N_r(u_0, x) \frac{1}{1 + (r + \|u_0\|)^4},$$

since for $y \in K_r(u_0)$

$$\|y\|^4 \le (\|y - u_0\| + \|u_0\|)^4 \le (r + \|u_0\|)^4.$$

Therefore, estimate (9) is true. It follows from (9) that, for $\|u - u_0\| < r_0$,

$$b_{r_0}(\alpha, u, y) \le c_1 c_2 \sum_{r=r_0}^{\infty} (1 + (\|u_0\| + r))^4) e^{-\alpha(r-1-\|u-u_0\|)^2}. \tag{10}$$

Estimate (10) implies that there exists the neighborhood V of (α_0, u_0, y_0) such that

$$\sup_V b_{r_0}(\alpha, u, y) \to 0, r_0 \to \infty. \tag{11}$$

Continuity of $\psi_{r_0}(\alpha, u, y)$ and (11) imply that, for $\alpha_0 > 0$,

$$\lim_{(\alpha,y,u)\to(\alpha_0,u_0,y_0)} \psi(\alpha, u, y) = \psi(\alpha_0, u_0, y_0). \tag{12}$$

It follows from (12) that

$$\begin{aligned} &\frac{1}{s_2 + \varepsilon - \frac{s_1^2}{s_1+\varepsilon}} \sum_{v:F(v)=F(z\sqrt{s_1+\varepsilon})} exp\{-\frac{\|v - \frac{s_1}{\sqrt{s_1+\varepsilon}} z\|^2}{2((s_2+\varepsilon) - \frac{s_1^2}{s_1+\varepsilon})}\} \longrightarrow \\ &\longrightarrow \frac{1}{s_2 - s_1} \sum_{v:F(v)=F(z\sqrt{s_1})} e^{-\frac{\|v-\sqrt{s_1}z\|^2}{2(s_2-s_1)}}, \varepsilon \to 0+. \end{aligned} \tag{13}$$

To check the second condition of the dominated convergence theorem, we note that, for $\varepsilon < 1$, the following estimate holds:

$$|\frac{1}{s_2+\varepsilon-\frac{s_1^2}{s_1+\varepsilon}}\exp\{-\frac{\|v-\frac{s_1}{\sqrt{s_1+\varepsilon}}z\|^2}{2((s_2+\varepsilon)-\frac{s_1^2}{s_1+\varepsilon})}\}|$$
$$\leq \frac{1}{(s_2-s_1)}\exp\{\frac{-\|v-\frac{s_1}{\sqrt{s_1+\varepsilon}}z\|^2}{4}\},$$

since

$$\frac{1}{s_2+\varepsilon-\frac{s_1^2}{s_1+\varepsilon}} \in (\frac{1}{2}, \frac{1}{s_2-s_1}).$$

This estimate implies that

$$\frac{1}{s_2+\varepsilon-\frac{s_1^2}{s_1+\varepsilon}}\cdot\psi(\frac{1}{2(s_2+\varepsilon-\frac{s_1^2}{s_1+\varepsilon})}, z\frac{s_1}{\sqrt{s_1+\varepsilon}}, F(z\sqrt{s_1+\varepsilon})) \leq$$
$$\leq \frac{1}{s_2-s_1}\cdot\psi(\frac{1}{4}, z\frac{s_1}{\sqrt{s_1+\varepsilon}}, F(z\sqrt{s_1+\varepsilon})).$$

To end the proof of the theorem, we note in view of (9) that, for $\kappa \in [0,1]$, the following inequality holds:

$$|\psi(\frac{1}{4}, \kappa z, y)| \leq c_6\sum_{r=1}^{\infty}(1+(\kappa\|z\|+r)^4)e^{-\frac{(r-1)^2}{4}} \leq c_7(1+\|z\|^4).$$

Since

$$\int_t^1\int_0^t\int_{\mathbb{R}^2} e^{-\frac{\|z\|^2}{2}}\frac{1}{s_2-s_1}(1+\|z\|^4)dzds_1ds_2 < \infty,$$

we have the statement of the theorem. The theorem is proved. □

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