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WEAK CONVERGENCE OF FIRST-RARE-EVENT TIMES FOR SEMI-MARKOV PROCESSES II

Necessary and sufficient conditions for the weak convergence of flows of rare events controlled by semi-Markov processes with a finite set of states in schemes of series are given. Applications of the obtained results to geometric sums, risk processes, and queueing systems are presented.

1. Introduction

Limit theorems for random functionals of similar first-rare-event times and flows of rare events known under such names as first hitting times, first passage times, first record times, etc., were studied by many authors. A survey of the literature related to the subject can be found in Drozdenko (2007, 2009), Silvestrov (2004), and papers by Silvestrov and Drozdenko (2005, 2006a, 2006b).

In addition to surveys of the literature presented in the mentioned papers, we would like to emphasize that the conditions of convergence of normalized first-rare-event times defined on a regenerative process to an exponentially distributed random variable were first obtained in the paper by Solov'yev (1971); however, the model considered by Solov'yev is slightly different from our settings. In particular, we define our first-rare-event time on a semi-Markov process. This means that we consider a multistage process with different distributions of staying in the taken positions for an embedded ergodic Markov chain, whereas Solov'yev defined his first-rare-event time as a sum of random number of i.i.d. random variables. There are other differences, for example, he normalized his first-rare-event by its own expectation, whereas we use more general normalization functions; there are also the differences in the definition of vanishing probabilities of stopping during one transition; it deserves to mention that we propose conditions of convergence to a much larger limit class which contains exponential distribution as an element.

First-rare-event times defined on a semi-Markov process become to be identical to geometric random sums in the case of a degenerated embedded Markov chain, i.e. in the case where the phase space of the Markov chain consists of only one element. Due to this fact, the very first paper directly relevant to our results was, probably, the paper by Rényi (1956) who first formulated the conditions of weak convergence of distributions of normalized geometric sums to the exponential law.

The main features for the most previous results are that they give sufficient conditions of convergence for first-rare-event times and flows of rare events. As a rule, those conditions involve assumptions, which imply the convergence of distributions for sums of i.i.d. random variables distributed as sojourn times for the semi-Markov process (for every state) to some infinitely divisible laws plus some ergodicity condition for the embedded Markov chain plus the condition of vanishing of probabilities of occurring a rare event during one transition step for the semi-Markov process.

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Our results are related to the model of semi-Markov processes with a finite set of states. In papers by Silvestrov and Drozdenko (2005, 2006a, 2006b), the necessary and sufficient conditions of first-rare-event times and flows of rare events for semi-Markov processes were obtained for the non-triangular-array case of stable type asymptotics for sojourn times distributions. In the present paper, we generalize results of those papers to a general triangular array model.

Instead of using the traditional approach based on conditions for "individual" distributions of sojourn times, we use more general and weaker conditions imposed on the distributions of sojourn times averaged by the stationary distribution of the limit embedded Markov chain. Moreover, we show that these conditions are not only sufficient, but also necessary for the weak convergence for first-rare-event times and flows of rare events, and describe the class of all possible limit laws non-concentrated at zero. The results presented in the paper give some kind of a "final solution" for the limit theorems for first-rare-event times for a semi-Markov process with a finite set of states in the triangular array mode.

The previously published part I, Drozdenko (2007), was organized in the following way. In Section 2, we formulated and proved our main Theorem 1, which describes the class of all possible limit distributions for first-rare-event times for semi-Markov processes and gave the necessary and sufficient conditions of weak convergence for distributions from this class. Several lemmas describing asymptotic solidarity cyclic properties for sum-processes defined on Markov chains were used in the proof of Theorem 1. These lemmas and their proofs were collected in Section 3.

The present paper, part II, is organized in the following way. To make this part self-readable, Section 2 gives a brief survey of the main results given in Part I. In Section 3, we give the necessary and sufficient conditions of weak convergence of counting processes generated by the flows of rare events. In Section 4, we give a comparative analysis of our results and the previously known results, by emphasizing the averaging principles. In Section 5, we give the necessary and sufficient conditions of weak convergence of geometric random sums and apply our results to the asymptotic analysis of non-ruin probabilities in the classical risk model. In Section 6, we give applications of Theorem 1 to the M/G type queueing system with quick service and give the necessary and sufficient conditions of weak convergence of failure times to the exponential law.

2. Survey of main results of Part I

To make the present paper self-readable, we give a short survey of the results given in Part I.

Let $\left(\eta_n^{(\varepsilon)},\ \varkappa_n^{(\varepsilon)},\ \zeta_n^{(\varepsilon)}\right)$, $n = 0, 1, \cdots$ be, for every $\varepsilon > 0$, a Markov renewal process, i.e. a homogeneous Markov chain with the phase space $Z = X \times [0, +\infty) \times Y$ (here $X = \{1, 2, \cdots, m\}$, and let Y be some measurable space with the σ–algebra of measurable sets B_Y) and transition probabilities,

$$
\begin{aligned}
&\mathsf{P}\left\{\eta_{n+1}^{(\varepsilon)} = j,\ \varkappa_{n+1}^{(\varepsilon)} \leq t,\ \zeta_{n+1}^{(\varepsilon)} \in A \Big/ \eta_n^{(\varepsilon)} = i,\ \varkappa_n^{(\varepsilon)} = s,\ \zeta_n^{(\varepsilon)} = y\right\} \\
(1) \qquad &= \mathsf{P}\left\{\eta_{n+1}^{(\varepsilon)} = j,\ \varkappa_{n+1}^{(\varepsilon)} \leq t,\ \zeta_{n+1}^{(\varepsilon)} \in A \Big/ \eta_n^{(\varepsilon)} = i\right\} \\
&= Q_{ij}^{(\varepsilon)}(t, A), \quad i, j \in X, \quad s, t \geq 0, \quad y \in Y, \quad A \in B_Y.
\end{aligned}
$$

The characterization property, which specifies Markov renewal processes in the class of general multivariate Markov chains $\left(\eta_n^{(\varepsilon)},\ \varkappa_n^{(\varepsilon)},\ \zeta_n^{(\varepsilon)}\right)$, is, as was shown in (1), that the transition probabilities do depend only on the current position of the first component $\eta_n^{(\varepsilon)}$.

As is known, the first component $\eta_n^{(\varepsilon)}$ of a Markov renewal process is also a homogeneous Markov chain with the phase space X and the transition probabilities $p_{ij}^{(\varepsilon)} = Q_{ij}^{(\varepsilon)}(+\infty, Y)$, $i, j \in X$.

Moreover, the first two components of the Markov renewal process (namely $\eta_n^{(\varepsilon)}$ and $\varkappa_n^{(\varepsilon)}$) can be associated with a semi-Markov process $\eta^{(\varepsilon)}(t)$, $t \geq 0$ defined as

$$\eta^{(\varepsilon)}(t) = \eta_n^{(\varepsilon)} \quad \text{for} \quad \tau_n^{(\varepsilon)} \leq t < \tau_{n+1}^{(\varepsilon)}, \quad n = 0, 1, \cdots,$$

where $\tau_0^{(\varepsilon)} = 0$ and $\tau_n^{(\varepsilon)} = \varkappa_1^{(\varepsilon)} + \ldots + \varkappa_n^{(\varepsilon)}$, $n \geq 1$.

The random variables $\varkappa_n^{(\varepsilon)}$ represent inter–jump times for the process $\eta^{(\varepsilon)}(t)$. As far as the random variables $\zeta_n^{(\varepsilon)}$ are concerned, they are the so-called "flag variables" and are used to record "rare" events.

Let D_ε, $\varepsilon > 0$ be a family of measurable subsets of Y which are "small" in some sense. Then the events $\left\{\zeta_n^{(\varepsilon)} \in D_\varepsilon\right\}$ can be considered as "rare".

Let us introduce the random variables

$$\nu_\varepsilon = \min\left(n \geq 1 : \ \zeta_n^{(\varepsilon)} \in D_\varepsilon\right)$$

and

$$\xi_\varepsilon = \sum_{n=1}^{\nu_\varepsilon} \varkappa_n^{(\varepsilon)}.$$

The random variable ν_ε counts the number of transitions of an embedded Markov chain $\eta_n^{(\varepsilon)}$ up to the first appearance of the "rare" event, while the random variable ξ_ε can be interpreted as the first-rare-event time for the semi-Markov process $\eta^{(\varepsilon)}(t)$.

Let us consider the distribution function of first-rare-event times ξ_ε, under a fixed initial state of the embedded Markov chain $\eta_n^{(\varepsilon)}$,

$$F_i^{(\varepsilon)}(u) = \mathsf{P}_i\{\xi_\varepsilon \leq u\}, \quad u \geq 0.$$

Here and henceforth, P_i and E_i denote, respectively, the conditional probability and the expectation calculated under the condition $\eta_0^{(\varepsilon)} = i$.

We give the necessary and sufficient conditions for the weak convergence of distribution functions $F_i^{(\varepsilon)}(uu_\varepsilon)$, where $u_\varepsilon > 0$, $u_\varepsilon \to \infty$ as $\varepsilon \to 0$ is a non-random normalizing function, and describe a class of possible limit distributions.

The problem is solved under four general model assumptions.

The first assumption **A** guarantees that the last summand in the random sum ξ_ε is negligible under any normalization u_ε, i.e. $\varkappa_{\nu_\varepsilon}^{(\varepsilon)} \Big/ u_\varepsilon \xrightarrow{\mathsf{P}} 0$ as $\varepsilon \to 0$:

A: $\lim\limits_{t\to\infty} \overline{\lim\limits_{\varepsilon\to 0}}\, \mathsf{P}_i \left\{ \varkappa_1^{(\varepsilon)} > t \Big/ \zeta_1^{(\varepsilon)} \in D_\varepsilon \right\} = 0, \quad i \in X.$

Let us introduce the probabilities of occurrence of a rare event during one transition step of the semi-Markov process $\eta^{(\varepsilon)}(t)$,

$$p_{i\varepsilon} = \mathsf{P}_i\left\{\zeta_1^{(\varepsilon)} \in D_\varepsilon\right\}, \quad i \in X.$$

The second assumption **B** imposed on probabilities $p_{i\varepsilon}$ specifies the interpretation of the event $\left\{\zeta_n^{(\varepsilon)} \in D_\varepsilon\right\}$ as "rare" and guarantees the possibility for such an event to occur:

B: $0 < \max_{1\leq i\leq m} p_{i\varepsilon} \to 0 \quad \text{as} \quad \varepsilon \to 0.$

The third assumption **C** is a condition of convergence of the transition matrix of an embedded perturbed Markov chain $\eta_n^{(\varepsilon)}$ to the transition matrix of an embedded limit Markov chain $\eta_n^{(0)}$:

C: $p_{ij}^{(\varepsilon)} \to p_{ij}^{(0)} \quad \text{as} \quad \varepsilon \to 0, \quad \text{for} \quad i, j \in X.$

The fourth assumption **D** is the standard ergodicity condition for a limit embedded Markov chain $\eta_n^{(0)}$:

D: Markov chain $\eta_n^{(0)}$ with the matrix of transition probabilities $\left\| p_{ij}^{(0)} \right\|$ is ergodic with the stationary distribution $\pi_i^{(0)}$, $i \in X$.

Let us define a probability which is a result of the averaging of probabilities of occurrence of a rare event in one transition step by the stationary distribution of a limit embedded Markov chain $\eta_n^{(0)}$,

$$p_\varepsilon = \sum_{i=1}^{m} \pi_i^{(0)} p_{i\varepsilon}.$$

Let us also introduce the distribution functions of sojourn times $\varkappa_1^{(\varepsilon)}$ for the semi-Markov processes $\eta^{(\varepsilon)}(t)$,

$$G_i^{(\varepsilon)}(t) = \mathsf{P}_i \left\{ \varkappa_1^{(\varepsilon)} \le t \right\}, \quad \text{for} \quad i \in X,$$

and the distribution function which is a result of the averaging of distribution functions of sojourn times by the stationary distribution of a limit embedded Markov chain $\eta_n^{(0)}$,

$$G^{(\varepsilon)}(t) = \sum_{i=1}^{m} \pi_i^{(0)} G_i^{(\varepsilon)}(t).$$

Now we are in position to formulate the necessary and sufficient conditions for the weak convergence of distribution functions of first-rare-event times ξ_ε. Mentioned conditions have the following form:

E: $p_\varepsilon^{-1} \left(1 - G^{(\varepsilon)}(u u_\varepsilon)\right) \to h(u)$ as $\varepsilon \to 0$ for all $u > 0$, which are points of continuity of the limit function $h(u)$.

F: $p_\varepsilon^{-1} \int_0^{u u_\varepsilon} s\, G^{(\varepsilon)}(ds) \to f(u)$ as $\varepsilon \to 0$ for some $u > 0$ which is a point of continuity of $h(u)$.

The limits here satisfy a number of conditions:

($\mathbf{a}_1$) $h(u)$ is a non-negative, non-increasing, and right-continuous function for $u > 0$ and $h(\infty) = 0$;

($\mathbf{a}_2$) a measure $\mathrm{H}(A)$ on the σ-algebra $\mathcal{H}^+$, i.e. the Borel σ-algebra of subsets of $(0, \infty)$, defined by the relation $\mathrm{H}((u_1, u_2]) = h(u_1) - h(u_2)$, $0 < u_1 \le u_2 < \infty$, satisfies the condition $\int_0^\infty \frac{s}{1+s} \mathrm{H}(ds) < \infty$;

($\mathbf{a}_3$) under condition **E**, condition **F** can only hold simultaneously for all continuity points of $h(u)$, and $f(u_1) = f(u_2) - \int_{u_1}^{u_2} s\mathrm{H}(ds)$ for any such points $0 < u_1 < u_2 < \infty$;

($\mathbf{a}_4$) $f(u)$ is a non-negative function.

We use symbol $\Longrightarrow$ to denote the weak convergence of random variables (pointwise convergence of corresponding distribution functions for all points of continuity of the limit distribution function).

Conditions **E** and **F** are the necessary and sufficient conditions for the following relation of weak convergence to hold:

$$\vartheta^{(\varepsilon)}(t) = \sum_{k=1}^{[t p_\varepsilon^{-1}]} \frac{\vartheta_k^{(\varepsilon)}}{u_\varepsilon},\ t \ge 0, \quad \Longrightarrow \quad \vartheta(t),\ t \ge 0, \quad \text{as} \quad \varepsilon \to 0, \tag{2}$$

where $\vartheta_k^{(\varepsilon)}$ are i.i.d. random variables with the joint distribution $G^{(\varepsilon)}(t)$; $\vartheta(t)$, $t \ge 0$ is a Lévy process with cumulant $a(s)$, i.e. $Ee^{-s\vartheta(t)} = e^{-a(s)t}$ which has the form

$$a(s) = as - \int_0^\infty (e^{-sx} - 1)\mathrm{H}(dx).$$

Here, the constant

$$a = f(u) - \int_0^u s\mathrm{H}(ds)$$

does not depend on the choice of a point u in condition **F**.

The main result obtained in Part I is the following theorem.

Theorem 2.1. *Let conditions* **A**, **B**, **C**, *and* **D** *hold. Then*:

(i): *the class of all possible, non-concentrated at zero, limit distribution functions (in the sense of weak convergence) for distribution functions of first-rare-event times $F_i^{(\varepsilon)}(uu_\varepsilon)$ coincides with the class of distribution functions $F(u)$ with the Laplace transform $\phi(s) = \frac{1}{1+a(s)}$.*

(ii): *conditions* **E** *and* **F** *are the necessary and sufficient for the following relation of weak convergence to hold (for some or every $i \in X$, respectively, in the statements of necessity and sufficiency):*

$$F_i^{(\varepsilon)}(uu_\varepsilon) \Longrightarrow F(u) \quad as \quad \varepsilon \to 0, \tag{3}$$

where $F(u)$ is a distribution function with the Laplace transform $\frac{1}{1+a(s)}$.

3. Flows of rare events

In this section, we study the conditions of convergence for flows of rare events within the model considered above.

Let us define recurrently the random variables

$$\nu_\varepsilon(k) = \min\Big(n \geq \nu_\varepsilon(k-1) : \ \zeta_n^{(\varepsilon)} \in D_\varepsilon\Big), \quad k = 1, 2, \cdots,$$

where $\nu_\varepsilon(0) = 0$. The random variable $\nu_\varepsilon(k)$ counts the number of transitions of an embedded Markov chain $\eta_n^{(\varepsilon)}$ up to the k-th appearance of the "rare" event $\Big\{\zeta_n^{(\varepsilon)} \in D_\varepsilon\Big\}$. Obviously,

$$\nu_\varepsilon(1) = \nu_\varepsilon = \min\Big\{n : \ \zeta_n^{(\varepsilon)} \in D_\varepsilon\Big\}.$$

Let us also define the inter-rare-event times,

$$\varkappa_\varepsilon(k) = \sum_{n=\nu_\varepsilon(k-1)+1}^{\nu_\varepsilon(k)} \varkappa_n^{(\varepsilon)}, \quad k = 1, 2, \cdots.$$

Let us also introduce the random variables showing positions of the embedded Markov chain $\eta_n^{(\varepsilon)}$ at moments $\nu_\varepsilon(k)$,

$$\eta_\varepsilon(k) = \eta_{\nu_\varepsilon(k)}^{(\varepsilon)}, \quad k = 0, 1, \cdots.$$

Obviously, $(\eta_\varepsilon(k), \ \varkappa_\varepsilon(k)), \ k = 0, 1, \cdots$ (here $\varkappa_\varepsilon(0) = 0$) is a Markov renewal process, i.e. a homogeneous Markov chain with the phase space $X \times [0, \infty)$ and the transition probabilities

$$\begin{aligned}
&\mathsf{P}\left\{\eta_\varepsilon(k+1) = j, \ \varkappa_\varepsilon(k+1) \leq t \ / \ \eta_\varepsilon(k) = i, \ \varkappa_\varepsilon(k) = s\right\} \\
&\quad = \mathsf{P}\{\eta_\varepsilon(k+1) = j, \ \varkappa_\varepsilon(k+1) \leq t \ / \ \eta_\varepsilon(k) = i\} \\
&\quad = \mathsf{P}_i\Big\{\eta_{\nu_\varepsilon}^{(\varepsilon)} = j, \ \xi_\varepsilon \leq t\Big\}, \quad i, j \in X, \quad s, t \geq 0.
\end{aligned}$$

Let us now define the random variables

$$\xi_\varepsilon(k) = \sum_{n=1}^{\nu_\varepsilon(k)} \varkappa_n^{(\varepsilon)} = \sum_{n=1}^{k} \varkappa_\varepsilon(n), \quad k = 0, 1, \cdots.$$

The random variable $\xi_\varepsilon(k)$ can be interpreted as the time of the k-th appearance of a rare-event for the semi-Markov process $\eta^{(\varepsilon)}(t)$. Obviously, $\xi_\varepsilon(0) = 0$ and

$$\xi_\varepsilon(1) = \xi_\varepsilon = \sum_{n=1}^{\nu_\varepsilon} \varkappa_n^{(\varepsilon)}.$$

We now define a counting stochastic process that describes the flow of rare events,

$$N_\varepsilon(t) = \max(k \geq 0 : \ \xi_\varepsilon(k) \leq t u_\varepsilon), \quad t \geq 0.$$

Note that the time scale for this counting process is stretched with the use of the scale parameter u_ε according to the asymptotic results given in Theorem 2.1.

Let us also define the corresponding limit counting process. Let $\varkappa(k)$, $k = 1, 2, \cdots$ be a sequence of positive i.i.d. random variables with distribution $F(u)$ (limit distribution from Theorem 2.1), and

$$\xi(k) = \sum_{n=1}^{k} \varkappa(n), \quad k = 0, 1, \cdots.$$

We denote, by V, the set of all discontinuity points of the distribution $F(u)$; by V_r, the set of discontinuity points of $F^{*r}(u)$, and $V_\infty := \bigcup_{r=1}^{\infty} V_r$. Note that V_∞ is an at most countable set; thus, $T := \overline{V_\infty}$ is the interval $[0, \infty)$ except for an at most countable set of points.

Let us also define the standard renewal counting process with i.i.d. inter-renewal times $\varkappa(k), k = 1, 2, \cdots$,

$$N(t) = \max\left(k \geq 0 : \ \xi(k) \leq t\right), \quad t \geq 0.$$

Theorem 3.1. *Let conditions* **A**, **B**, **C**, *and* **D** *hold. Then*:

- **(i):** *the class of all possible non-zero limit counting processes* (*in the sense of weak convergence of finite-dimensional distributions*) *for counting processes* $N_\varepsilon(t)$, $t \in T$ *coincides with the class of standard renewal counting processes* $N(t)$, $t \in T$ *with the distribution function of inter-renewal times* $F(u)$.
- **(ii):** *conditions* **E** *and* **F** *are necessary and sufficient for such a convergence in the case where the corresponding limit counting process has the distribution function of inter-renewal times* $F(u)$.

Proof. Obviously,

$$F_i^{(\varepsilon)}(u) = \mathsf{P}_i\{\xi_\varepsilon \leq u\} = \sum_{j \in X} Q_{ij}^{(\varepsilon)}(t), \quad u \geq 0.$$

Using the Markov property of the Markov renewal process $(\eta_\varepsilon(k), \ \varkappa_\varepsilon(k))$, we get the following formula for joint distributions of properly normalized inter-renewal times for the counting process $N_\varepsilon(t)$,

$$\begin{aligned} &\mathsf{P}_i\{\, \varkappa_\varepsilon(k)/\, u_\varepsilon \ \leq \ t_k, \quad k = 1, \cdots, n\} \\ &\quad = \sum_{j \in X} \mathsf{P}_i\{\varkappa_\varepsilon(k)/u_\varepsilon \leq t_k, \quad k = 1, \cdots, n-1, \quad \eta_\varepsilon(n-1) = j\} \times F_j^{(\varepsilon)}(t_n u_\varepsilon), \\ &\quad \text{for} \quad i \in X, \quad t_1, \cdots, t_n \in T, \quad n = 1, 2, \cdots. \end{aligned} \tag{4}$$

According to Theorem 2.1, under **A**, **B**, **C**, and **D** conditions, **E** and **F** imply that (**a**) $F_j^{(\varepsilon)}(t u_\varepsilon) \to F(t)$ as $\varepsilon \to 0$, for $t \in T$, $j \in X$.

Using (**a**) and relation (4) we get that, under **A**, **B**, **C**, and **D** conditions, **E** and **F** imply that, for every $i \in X$, $n = 1, 2, \cdots,\ t_1, \cdots, t_n \in T$,

$$\begin{aligned} \mathsf{P}_i\{\varkappa_\varepsilon(k)/u_\varepsilon \ \le\ t_k, \quad k = 1, \cdots, n\} \\ \to \prod_{k=1}^{n} F(t_k) \quad \text{as} \quad \varepsilon \to 0. \end{aligned} \tag{5}$$

Relation (5) means that the inter-renewal times $\varkappa_\varepsilon(k)/u_\varepsilon$, $k = 1, 2, \cdots$ are asymptotically independent. Note that the multivariate distribution function on the right-hand side of (5) is continuous. Due to this fact, relation (5) implies in an obvious way that, for every $i \in X$, and for a collection of nonnegative real numbers $t_1, \cdots, t_n \in T$, and nonnegative integers $r_1, \cdots, r_n$, $n = 1, 2, \cdots$,

$$\begin{aligned} &\mathsf{P}_i\{N_\varepsilon(t_k) \ \ge\ r_k, \quad k = 1, \cdots, n\} \\ &\quad = \mathsf{P}_i\{\xi_\varepsilon(r_k)/u_\varepsilon \ \le\ t_k, \quad k = 1, \cdots, n\} \\ &\quad\quad \to \mathsf{P}_i\{\xi(r_k) \ \le\ t_k, \quad k = 1, \cdots, n\} \\ &\quad\quad\quad = \mathsf{P}_i\{N(t_k) \ \ge\ r_k, \quad k = 1, \cdots, n\} \quad \text{as} \quad \varepsilon \to 0. \end{aligned}$$

The statement of necessity is trivial and follows from the formula $\mathsf{P}_i\{N_\varepsilon(t) \ge 1\} = \mathsf{P}_i\{\xi_\varepsilon/u_\varepsilon \le t\}$, $t \ge 0$. The proof is complete. □

It is interesting that, under **A**, **B**, **C**, and **D** conditions, **E** and **F** are not sufficient for the weak convergence of transition probabilities of the Markov renewal process $(\eta_\varepsilon(k), \varkappa_\varepsilon(k))$ that forms a counting process $N_\varepsilon(t)$.

This follows from the following lemma which describes the asymptotic behavior of the so-called absorption probabilities,

$$Q_{ij}^{(\varepsilon)}(\infty) = \mathsf{P}_i\left\{\eta_{\nu_\varepsilon}^{(\varepsilon)} = j\right\}, \quad i, j \in X.$$

Let us denote

$$p_{i\varepsilon}(r) = \mathsf{P}_i\left\{\zeta_1^{(\varepsilon)} \in D_\varepsilon, \quad \eta_1^{(\varepsilon)} = r\right\}, \quad i, r \in X,$$

and

$$p_\varepsilon(r) = \sum_{i=1}^{m} \pi_i^{(0)} p_{i\varepsilon}(r), \quad j \in X.$$

By definition,

$$p_\varepsilon = \sum_{i \in X} \pi_i^{(0)} \mathsf{P}_i\left\{\zeta_1^{(\varepsilon)} \in D_\varepsilon\right\} = \sum_{r \in X} p_\varepsilon(r). \tag{6}$$

Lemma 3.1. *Let conditions* **B**, **C**, *and* **D** *hold. Then, for every* $i \in X$,

$$Q_{ir}^{(\varepsilon)}(\infty) - \frac{p_\varepsilon(r)}{p_\varepsilon} \to 0 \quad \textit{as} \quad \varepsilon \to 0, \quad r \in X. \tag{7}$$

Proof. Let us define the probability that the first-rare-event will occur when the state of the embedded Markov chain will be r and before the first hitting of the embedded Markov chain of the state i, under condition that the initial state of this Markov chain is $\eta_0^{(\varepsilon)} = j$,

$$q_{ji\varepsilon}(r) = \mathsf{P}_j\left\{\nu_\varepsilon \le \tau_i^{(\varepsilon)}, \quad \eta_{\nu_\varepsilon}^{(\varepsilon)} = r\right\}, \quad i, j, r \in X.$$

Taking into account that the Markov renewal process $\left(\eta_n^{(\varepsilon)},\ \varkappa_n^{(\varepsilon)},\ \zeta_n^{(\varepsilon)}\right)$ regenerates at moments of return to every state i and that ν_ε is a Markov moment of this process, we can get the following cyclic representation for absorption probabilities $Q^{(\varepsilon)}_{ir}(\infty)$,

$$
\begin{aligned}
Q^{(\varepsilon)}_{ir}(\infty) &= \sum_{n=0}^{\infty} \mathsf{P}_i \left\{ \tau^{(\varepsilon)}_i(n) < \nu_\varepsilon \leq \tau^{(\varepsilon)}_i(n+1), \quad \eta^{(\varepsilon)}_{\nu_\varepsilon} = j \right\} \\
&= \sum_{n=0}^{\infty} (1 - q_{i\varepsilon})^n q_{ii\varepsilon}(r) = \frac{q_{ii\varepsilon}(r)}{q_{i\varepsilon}}, \quad i, r \in X.
\end{aligned} \tag{8}
$$

The probabilities $q_{ji\varepsilon}(r), j \in X$ satisfy, for every $i, r \in X$, the following system of linear equations similar to system (34) in Part I:

$$
\begin{cases}
q_{ji\varepsilon}(r) = p_{j\varepsilon}(r) + \sum_{k \neq i} p^{(\varepsilon)}_{jk} q_{ki\varepsilon}(r) \\
j \in X
\end{cases}
$$

This system has the matrix of coefficients ${}_i\mathbf{P}^{(\varepsilon)}$ (defined in the proof of Lemma 1, Part I) as the system of linear equations (34, Part I) and differs from this system only by the free terms. Thus, by repeating the reasoning given in the proof of Lemma 1, Part I, we can get the following formula similar to formula (42, Part I):

$$
q_{ii\varepsilon}(r) = \sum_{k=1}^{m} \mathsf{E}_i \delta_{ik\varepsilon}\, p_{k\varepsilon}(r).
$$

We recall that it was shown in the proof of Lemma 1, Part I, that (**b**) $\mathsf{E}_i \delta_{ik\varepsilon} \to \pi^{(0)}_k / \pi^{(0)}_i$ as $\varepsilon \to 0$, for $i, k \in X$. Using (**b**), relation (**c**) $\pi^{(0)}_i q_{i\varepsilon}/p_\varepsilon \to 1$ given in Lemma 1, Part I, and inequality (**d**) $p_\varepsilon(r) \leq p_\varepsilon$ following from formula (6), we get, for every $i, r \in X$,

$$
\begin{aligned}
&\frac{\left| q_{ii\varepsilon}(r) - p_\varepsilon(r) / \pi^{(0)}_i \right|}{q_{i\varepsilon}} \\
&\leq \sum_{k=1}^{m} \left| \mathsf{E}_i \delta_{ik\varepsilon} - \frac{\pi^{(0)}_k}{\pi^{(0)}_i} \right| \cdot \frac{\pi^{(0)}_i p_{k\varepsilon}(r)}{\sum_{j=1}^{m} \pi^{(0)}_j p_{j\varepsilon}} \cdot \frac{p_\varepsilon}{\pi^{(0)}_i q_{i\varepsilon}} \\
&\leq \sum_{k=1}^{m} \left| \mathsf{E}_i \delta_{ik\varepsilon} - \frac{\pi^{(0)}_k}{\pi^{(0)}_i} \right| \cdot \frac{\pi^{(0)}_i}{\pi^{(0)}_k} \cdot \frac{p_\varepsilon}{\pi^{(0)}_i q_{i\varepsilon}} \ \to \ 0 \quad \text{as} \quad \varepsilon \to 0.
\end{aligned} \tag{9}
$$

Using (**c**) and (**d**) once more, we get

$$
\begin{aligned}
\left| \frac{p_\varepsilon(r)}{\pi^{(0)}_i q_{i\varepsilon}} - \frac{p_\varepsilon(r)}{p_\varepsilon} \right| &\leq \frac{p_\varepsilon(r)}{p_\varepsilon} \cdot \frac{\left| q_{i\varepsilon} - \frac{p_\varepsilon}{\pi^{(0)}_i} \right|}{\pi^{(0)}_i q_{i\varepsilon}} \\
&\leq \frac{\left| q_{i\varepsilon} - \frac{p_\varepsilon}{\pi^{(0)}_i} \right|}{p_\varepsilon} \cdot \frac{p_\varepsilon}{\pi^{(0)}_i q_{i\varepsilon}} \ \to \ 0 \quad \text{as} \quad \varepsilon \to 0.
\end{aligned} \tag{10}
$$

Formula (8) together with relations (9) and (10) imply relation (7). □

Let us introduce the balancing condition

L: $\frac{p_\varepsilon(j)}{p_\varepsilon} \to Q_j$ as $\varepsilon \to 0, \quad j \in X$.

The constants Q_j automatically satisfy the conditions ($\mathbf{e}_1$) $Q_j \geq 0$, $j \in X$, and ($\mathbf{e}_2$) $\sum_{j \in X} Q_j = 1$.

Lemma 3.1 implies the following statement.

Lemma 3.2. *Let conditions* **B**, **C**, *and* **D** *hold. Then, condition* **L** *is necessary and sufficient for the following relation to hold* (*for some or every $i \in X$, respectively, in the statements of necessity and sufficiency*)*:*

$$Q_{ir}^{(\varepsilon)}(\infty) \to Q_r \quad \text{as} \quad \varepsilon \to 0, \quad r \in X. \tag{11}$$

The following theorem shows that the first-rare-event times ξ_ε and the random functional $\eta_{\nu_\varepsilon}^{(\varepsilon)}$ are asymptotically independent and completes the description of the asymptotic behavior of the transition probabilities $Q_{ij}^{(\varepsilon)}(t)$ for the Markov renewal process $(\eta_\varepsilon(k),\ \varkappa_\varepsilon(k))$.

Theorem 3.2. *Let conditions* **A**, **B**, **C**, *and* **D** *hold. Then, conditions* **E**, **F**, *and* **L** *are necessary and sufficient for the asymptotic relations* (3) *given in Theorem 2.1 and* (11) *given in Lemma 3.2 to hold. In this case, for every $u \in [0, \infty], i, r \in X$,*

$$Q_{ir}^{(\varepsilon)}(uu_\varepsilon) \to F(u)Q_r \quad \text{as} \quad \varepsilon \to 0. \tag{12}$$

Proof. The first statement of the theorem follows from Theorem 2.1 and Lemma 3.2. Let us prove that conditions **E**, **F**, and **L** imply the asymptotic relation (12).

Let us introduce, for $i, r \in X$, the Laplace transforms

$$\Phi_{ir\varepsilon}(s) = \mathsf{E}_i \exp\{-s\xi_\varepsilon\}\chi\left(\eta_{\nu_\varepsilon}^{(\varepsilon)} = r\right), \quad s \geq 0,$$

and

$$\widetilde{\psi}_{ir\varepsilon}(s) = \mathsf{E}_i\left\{\exp\left\{-s\tilde{\beta}_{i\varepsilon}\right\}\chi\left(\eta_{\nu_\varepsilon}^{(\varepsilon)} = r\right)\Big/\ \nu_\varepsilon \leq \tau_i^{(\varepsilon)}\right\}, \quad s \geq 0.$$

Analogously to formula (6, Part I) the following representation can be written down for the Laplace transforms $\Phi_{ir\varepsilon}(s)$:

$$\begin{aligned} \Phi_{ir\varepsilon}(s) &= \sum_{n=0}^{\infty}(1-q_{i\varepsilon})^n q_{i\varepsilon}\psi_{i\varepsilon}(s)^n\widetilde{\psi}_{ir\varepsilon}(s) \\ &= \frac{q_{i\varepsilon}\widetilde{\psi}_{ir\varepsilon}(s)}{1-(1-q_{i\varepsilon})\psi_{i\varepsilon}(s)} \\ &= \frac{1}{1+(1-q_{i\varepsilon})\frac{(1-\psi_{i\varepsilon}(s))}{q_{i\varepsilon}}}\cdot\widetilde{\psi}_{ir\varepsilon}(s), \quad s \geq 0. \end{aligned} \tag{13}$$

Let us now show that, under conditions **A**, **B**, and **C**, for every $s \geq 0$ and $i, r \in X$,

$$\widetilde{\psi}_{ir\varepsilon}(s/u_\varepsilon) - q_{ii\varepsilon}(r)/q_{i\varepsilon} \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{14}$$

Indeed, using Lemma 4, Part I, we get, for any $\delta > 0$,

$$\begin{aligned} &\frac{q_{ii\varepsilon}(r)}{q_{i\varepsilon}} - \widetilde{\psi}_{ir\varepsilon}(\frac{s}{u_\varepsilon}) \\ &\quad = \mathsf{E}_i\left\{(1-\exp\{-s\frac{\tilde{\beta}_{i\varepsilon}}{u_\varepsilon}\})\chi(\eta_{\nu_\varepsilon}^{(\varepsilon)} = r)\Big/\ \nu_\varepsilon \leq \tau_i^{(\varepsilon)}\right\} \\ &\quad \leq \left(1-e^{s\delta}\right) + \mathsf{P}_i\left\{\frac{\tilde{\beta}_{i\varepsilon}}{u_\varepsilon} \geq \delta\Big/\ \nu_\varepsilon \leq \tau_i^{(\varepsilon)}\right\} \to \left(1-e^{s\delta}\right) \quad \text{as} \quad \varepsilon \to 0. \end{aligned} \tag{15}$$

Relation (15) yields, due to the arbitrary choice of $\delta > 0$, relation (14).

Using formula (13) and relations (46, Part I) given in Lemma 2, Part I, and (14), Theorem 2.1 and Lemma 3.1, we get, for every $s \geq 0$ and $i, r \in X$,

$$
\begin{aligned}
\lim_{\varepsilon\to 0} \Phi_{ir\varepsilon}(s/u_\varepsilon) &= \lim_{\varepsilon\to 0} \frac{1}{1 + (1 - q_{i\varepsilon})\frac{(1-\psi_{i\varepsilon}(s))}{q_{i\varepsilon}}} \cdot \widetilde{\psi}_{ir\varepsilon}(s/u_\varepsilon) \\
&= \lim_{\varepsilon\to 0} \Phi_{i\varepsilon}(s/u_\varepsilon) \cdot Q^{(\varepsilon)}_{ir}(\infty) \\
&= \frac{1}{1+a(s)} \cdot Q_r.
\end{aligned} \tag{16}
$$

Relation (16) is equivalent to relation (12). □

4. Analysis of conditions

The non-triangular-array model of Markov renewal process

$$
\left(\eta^{(\varepsilon)}_n,\ \varkappa^{(\varepsilon)}_n,\ \zeta^{(\varepsilon)}_n\right) = (\eta_n,\ \varkappa_n,\ \zeta_n), \quad n = 0, 1, 2, \cdots, \tag{17}
$$

which does not depend on the perturbation parameter, was considered in papers by Silvestrov and Drozdenko (2005, 2006a, 2006b).

As was shown, conditions **E** and **F** are equivalent in this case to the following conditions:

$\mathbf{K}_\gamma$**:** $1 - G(t) \sim \frac{t^{-\gamma}L(t)}{\Gamma(1-\gamma)}$ as $t \to \infty$,
where $0 < \gamma \leq 1$, and $L(\cdot)$ is a slowly varying function;

$\mathbf{H}_{a,\gamma}$**:** $\frac{L(u_\varepsilon)}{p_\varepsilon u_\varepsilon^\gamma} \to$ as $\varepsilon \to 0$, where $a = \text{constant} > 0$,

and the limit cumulant $a(s)$ should be of the form $a(s) = as^\gamma$. Here,

$$
G(t) = \sum_{i=1}^m \pi^{(0)}_i G_i(t), \qquad G_i(t) = \mathsf{P}\{\varkappa_i | \eta_0 = i\}, \qquad p_\varepsilon = \sum_{i=1}^m \pi^{(0)}_i p_{i\varepsilon}.
$$

The previous results related to the stable type asymptotics of first-rare-event times known in the literature (see, for example, Silvestrov (1974)), were the sufficient (but not necessary) conditions similar to $\mathbf{K}_\gamma$ and $\mathbf{H}_{a,\gamma}$ but involving "individual" distributions of sojourn times and absorption probabilities. Those conditions have the form

$\mathbf{B}''$**:** $p_{i\varepsilon} \sim c_i b(\varepsilon) + o_i(b(\varepsilon))$ as $\varepsilon \to 0$, $i = \overline{1,m}$,
where: ($\mathbf{b}_1$) $0 < b(\varepsilon) \to 0$ as $\varepsilon \to 0$; ($\mathbf{b}_2$) $c_i \geq 0$, $i = \overline{1,m}$;
($\mathbf{b}_3$) $\sum_{i=1}^m c_i > 0$; ($\mathbf{b}_4$) $o_i(b(\varepsilon))/b(\varepsilon) \to 0$ as $\varepsilon \to 0$, $i = \overline{1,m}$,

and

$\mathbf{K}''_\gamma$**:** $1 - G_i(t) \sim \frac{g_i t^{-\gamma}\widetilde{L}(t)}{\Gamma(1-\gamma)}$ as $t \to \infty$, $i = \overline{1,m}$, $0 < \gamma \leq 1$,
where: ($\mathbf{k}_1$) $g_i \geq 0$, $i = \overline{1,m}$; ($\mathbf{k}_2$) $\sum_{i=1}^m g_i > 0$;
($\mathbf{k}_3$) $\widetilde{L}(\cdot)$ is a slowly varying function.

$\mathbf{H}''_{a,\gamma}$**:** $\frac{\widetilde{L}(u_\varepsilon)}{b(\varepsilon)u_\varepsilon^\gamma} \to \tilde{a}$ as $\varepsilon \to 0$, where $\tilde{a}$ is a positive constant.

Our conditions based on the behavior of averaged probabilities p_ε and on the behavior of averaged distribution functions $G(t)$ are weaker and thus improve the previously known results.

We recall that condition **B** describing the limit behavior of the probability of appearance of a rare event for the non-perturbed Markov renewal process (17) studied in papers by Silvestrov and Drozdenko (2005, 2006a, 2006b) is identical to condition **B** studied in the present paper, i.e.,

B: $0 < \max_{1\leq i\leq m} p_{i\varepsilon} \to 0$ as $\varepsilon \to 0$, this was the case where the renewal process itself was unperturbed, but the probability of dropping into a stopping domain D_ε was changed.

In the rest of the section, we give the proofs for the following implications: $\mathbf{B}''$ implies $\mathbf{B}$; $\mathbf{K}''_\gamma$ implies $\mathbf{K}_\gamma$; $\mathbf{K}''_\gamma$ together with $\mathbf{H}''_{a,\gamma}$ imply $\mathbf{H}_{a,\gamma}$. We also give examples which show that the inverse implications do not always hold.

Let us show that condition $\mathbf{B}''$ implies condition $\mathbf{B}$. Indeed,

$$\max_{1\le i\le m} p_{i\varepsilon} \ \le\ b(\varepsilon)\max_{1\le i\le m} c_i + \max_{1\le i\le m} o_i(b(\varepsilon)) \ \to\ 0 \quad \text{as} \quad \varepsilon\to 0,$$

and, moreover, for all ε small enough,

$$\max_{1\le i\le m} p_{i\varepsilon} \ \ge\ b(\varepsilon)\max_{1\le i\le m} c_i - \max_{1\le i\le m} o_i(b(\varepsilon)) > 0.$$

Hence, the implication holds.

Condition $\mathbf{K}''_\gamma$ implies condition $\mathbf{K}_\gamma$. Indeed,

$$1-G(t)=\sum_{i=1}^m \pi_i^{(0)}[1-G_i(t)] \ \sim\ \sum_{i=1}^m \pi_i^{(0)}\frac{g_i t^{-\gamma}\widetilde{L}(t)}{\Gamma(1-\gamma)} = \frac{t^{-\gamma}L(t)}{\Gamma(1-\gamma)} \quad \text{as} \quad t\to+\infty,$$

where

$$L(t) := \widetilde{L}(t)\sum_{i=1}^m \pi_i^{(0)} g_i.$$

Let us finally show that, under conditions $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$, conditions $\mathbf{B}''$ and $\mathbf{H}''_{a,\gamma}$ imply condition $\mathbf{H}_{a,\gamma}$. Indeed,

$$\frac{p_{i\varepsilon}}{p_\varepsilon} = \frac{c_i b(\varepsilon)+o_i(b(\varepsilon))}{\sum_{i=1}^m \pi_i^{(\varepsilon)}[c_i b(\varepsilon)+o_i(b(\varepsilon))]} \ \to\ \frac{c_i}{\sum_{i=1}^m c_i\pi_i^{(0)}} \quad \text{as} \quad \varepsilon\to 0. \tag{18}$$

Condition $\mathbf{B}''$ and relation (18) imply that

$$p_\varepsilon \sim b(\varepsilon)\sum_{i=1}^m c_i\pi_i^{(0)} + o(b(\varepsilon)) \quad \text{as} \ \varepsilon\to 0.$$

It follows from conditions $\mathbf{B}''$ and $\mathbf{H}''_{a,\gamma}$ that

$$\frac{L(u_\varepsilon)}{p_\varepsilon u_\varepsilon^\gamma} \ \sim\ \frac{\sum_{i=1}^m g_i\pi_i^{(0)}\widetilde{L}(u_\varepsilon)}{b(\varepsilon)\sum_{i=1}^m c_i\pi_i^{(0)}u_\varepsilon^\gamma} \ \to\ \frac{\tilde{a}\sum_{i=1}^m g_i\pi_i^{(0)}}{\sum_{i=1}^m c_i\pi_i^{(0)}} \quad \text{as} \quad \varepsilon\to 0.$$

Thus, conditions $\mathbf{B}''$ and $\mathbf{H}''_{a,\gamma}$ imply condition $\mathbf{H}_{a,\gamma}$ with

$$a := \frac{\tilde{a}\sum_{i=1}^m g_i\pi_i^{(0)}}{\sum_{i=1}^m c_i\pi_i^{(0)}}.$$

We will now consider several examples which show that inverse implications do not always hold. We start from the example, in which condition $\mathbf{B}$ holds, but condition $\mathbf{B}''$ does not.

Example 4.1. Let us consider the case where $m=2$ and

$$p_{1\varepsilon} = b(\varepsilon)\left[c_1 + c\sin(1/\varepsilon)\right],$$
$$p_{2\varepsilon} = b(\varepsilon)\left[c_2 - c\sin(1/\varepsilon)\right],$$

where the parameters are chosen so that

$$c_1 > c > 0; \quad c_2 > c > 0; \quad \text{and} \quad 0 < b(\varepsilon)\to 0 \quad \text{as} \quad \varepsilon\to 0,$$

and the stationary distribution of the embedded Markov chain is

$$\pi_1^{(0)} = \pi_2^{(0)} = \frac{1}{2}.$$

Then

$$\frac{p_{i\varepsilon}}{b(\varepsilon)} = c_i + (-1)^{i+1} c \sin(1/\varepsilon) \not\to \text{ constant} \quad \text{as} \quad \varepsilon \to 0, \quad i = \overline{1,2},$$

which means that condition $\mathbf{B}''$ is not satisfied, but

$$0 \; < \; p_\varepsilon \; = \; \pi_1^{(0)} p_{\varepsilon 1} + \pi_2^{(0)} p_{\varepsilon 2} \; = \; b(\varepsilon)(a_1 \pi_1^{(0)} + a_2 \pi_2^{(0)}) \to 0 \quad \text{as} \quad \varepsilon \to 0,$$

and, thus, condition $\mathbf{B}$ holds.

The next example shows the case where condition $\mathbf{K}_\gamma$ holds, but condition $\mathbf{K}''_\gamma$ does not.

Example 4.2. Let us consider the survival functions $R_i(t)$, $i = \overline{1,3}$, such that

$$1 - R_i(t) = \min\{c_i/t^\gamma, 1\}, \quad t > 0, \quad \gamma > 0, \quad i = \overline{1,3},$$

where

$$c_2 := \frac{c_1 + c_3}{2} \quad \text{and} \quad c_3 > c_1 > 0.$$

To build the example, we take $m = 2$ and assume that the stationary probabilities of the embedded Markov chain

$$\pi_1^{(0)} = \pi_2^{(0)} = 1/2.$$

We construct the distribution functions $G_1(t)$ and $G_2(t)$ with the following tail behavior:

$$1 - G_1(t) = \begin{cases} \min\{c_2/t^\gamma, 1\} & \text{for} \quad t \le t_0 \\ y_{2k} & \text{for} \quad t_{2k} < t \le t_{2k+1}, \quad k = 0,1,2,\cdots, \\ (c_1 + c_3)/t^\gamma - y_{2k+1} & \text{for} \quad t_{2k+1} < t \le t_{2k+2}, \quad k = 0,1,2,\cdots, \end{cases}$$

and

$$1 - G_2(t) = \begin{cases} \min\{c_2/t^\gamma, 1\} & \text{for} \quad t \le t_0 \\ (c_1 + c_3)/t^\gamma - y_{2k} & \text{for} \quad t_{2k} < t \le t_{2k+1}, \quad k = 0,1,2,\cdots, \\ y_{2k+1} & \text{for} \quad t_{2k+1} < t \le t_{2k+2}, \quad k = 0,1,2,\cdots, \end{cases}$$

where the constants y_k and t_k are defined recursively as

$$\begin{array}{lll} t_0 = c_3^{1/\gamma}; & y_0 = c_2/t_0^\gamma; & \\ t_{2k+1} = (c_3/y_{2k})^{1/\gamma}, & y_{2k+1} = c_1/t_{2k+1}^\gamma, & k = 0,1,2,\cdots; \\ t_{2k+2} = (c_2/y_{2k+1})^{1/\gamma}, & y_{2k+2} = y_{2k+1}, & k = 0,1,2,\cdots. \end{array}$$

Then

$$\begin{aligned} 1 - G(t) &= \frac{1}{2}\left[(1 - G_1(t)) + (1 - G_2(t))\right] \\ &= R_2(t) = c_2 t^{-\gamma} \quad \text{for} \quad t \ge t_0, \end{aligned}$$

and, thus, condition $\mathbf{K}_\gamma$ holds.

Note that the function $G_1(t)$ is constructed so that it coincides with the function $R_2(t)$ at the points t_{2k}, $k = 0,1,2,\cdots$, and coincides with the function $R_3(t)$ at the points t_{2k+1}, $k = 0,1,2,\cdots$. In a similar way, the function $G_2(t)$ coincides with the function $R_2(t)$ at the points t_{2k}, $k = 0,1,2,\cdots$, and coincides with the function $R_1(t)$ at the points t_{2k+1}, $k = 0,1,2,\cdots$. Due to this,

$$\frac{1 - G_i(t_{2k})}{t_{2k}^{-\gamma}} \; = \; c_2, \quad i = \overline{1,2}, \quad k = 0,1,2,\cdots, \tag{19}$$

and

$$\frac{1 - G_i(t_{2k+1})}{t_{2k+1}^{-\gamma}} \; = \; c_{-2i+5}, \quad i = \overline{1,2}, \quad k = 0,1,2,\cdots. \tag{20}$$

We are going to show that the functions $G_1(t)$ and $G_2(t)$ do not satisfy conditions $\mathbf{K}''_\gamma$. Namely, we are going to show that, for any slowly varying function $\widetilde{L}(\cdot)$,

$$\frac{1 - G_i(t)}{t^{-\gamma}\widetilde{L}(t)} \nsim 1 \quad \text{as} \quad t \to \infty, \quad i \in \overline{1,2}.$$

Indeed, it follows from (19) and (20) that

$$\frac{1 - G_i(t_{2k})}{t_{2k}^{-\gamma}\widetilde{L}(t_{2k})} \Bigg/ \frac{1 - G_i(t_{2k+1})}{t_{2k+1}^{-\gamma}\widetilde{L}(t_{2k+1})} \to \frac{c_2 \cdot c_2}{c_3 \cdot c_{-2i+5}} \neq 1 \quad \text{as} \quad k \to \infty, \quad i \in \overline{1,2},$$

since

$$\frac{t_{2k+1}}{t_{2k}} = \frac{(c_3/y_{2k})^{1/\gamma}}{t_{2(k-1)+2}} = \frac{(c_3/y_{2(k-1)+2})^{1/\gamma}}{(c_2/y_{2(k-1)+1})^{1/\gamma}} = (c_3/c_2)^{1/\gamma} \quad \text{for} \quad k \geq 1$$

and

$$\frac{\widetilde{L}(t_{2k+1})}{\widetilde{L}(t_{2k})} = \frac{\widetilde{L}((c_3/c_2)^{1/\gamma} t_{2k})}{\widetilde{L}(t_{2k})} \to 1 \quad \text{as} \quad k \to \infty.$$

Since the required slowly varying function $\widetilde{L}(\cdot)$, does not exist, condition $\mathbf{K}''_\gamma$ does not hold.

5. Geometric sums and asymptotics for non-ruin probabilities

In this section, we apply our results to the so-called geometric random sums. This is a reduction of our model to the case where the embedded Markov chain $\eta_n^{(\varepsilon)}$ has a degenerate set of states, namely $X = \{1\}$.

In this case, the first-rare-event time $\xi_\varepsilon = \sum_{n=1}^{\nu_\varepsilon} \varkappa_n^{(\varepsilon)}$ is a geometric sum.

Indeed, $\left(\varkappa_n^{(\varepsilon)},\ \zeta_n^{(\varepsilon)}\right)$, $n = 1, 2, \cdots$ is a sequence of i.i.d. random vectors. Therefore, the random variable $\nu_\varepsilon = \min\left(n \geq 1 :\ \zeta_n^{(\varepsilon)} \in D_\varepsilon\right)$ has a geometric distribution with success probability $p_\varepsilon = \mathsf{P}\left\{\zeta_n^{(\varepsilon)} \in D_\varepsilon\right\}$.

However, the geometric random index ν_ε and the random summands $\varkappa_n^{(\varepsilon)}$, $n = 1, 2, \cdots$ are, in this case, dependent random variables. They depend via the indicators of rare events $\chi_{n\varepsilon} = \chi\left(\zeta_n^{(\varepsilon)} \in D_\varepsilon\right)$, $n = 1, 2, \cdots$. More precisely, $\left(\varkappa_n^{(\varepsilon)},\ \chi_{n\varepsilon}\right)$, $n = 1, 2, \cdots$ is a sequence of i.i.d. random vectors.

Conditions **A** and **B** take, in this case, the following form:

A′: $\lim\limits_{t\to\infty} \overline{\lim\limits_{\varepsilon\to 0}}\, \mathsf{P}\left\{\varkappa_1^{(\varepsilon)} > t \Big/ \zeta_1^{(\varepsilon)} \in D_\varepsilon\right\} = 0;$

and

B′: $0 < p_\varepsilon = \mathsf{P}\left\{\zeta_1^{(\varepsilon)} \in D_\varepsilon\right\} \to 0 \quad \text{as} \quad \varepsilon \to 0.$

Conditions **C** and **D** hold automatically.

Conditions **E** and **F** remain and should be imposed on the distribution function $G^{(\varepsilon)}(t) = \mathsf{P}\left\{\varkappa_1^{(\varepsilon)} \leq t\right\}$ (no averaging is involved).

A standard geometric sum is a particular case of the model described above, which corresponds to the case where two sequences of random variables $\varkappa_n^{(\varepsilon)}$, $n = 1, 2, \cdots$ and $\zeta_n^{(\varepsilon)}$, $n = 1, 2, \cdots$ are independent. In this case, the random index ν_ε and the summands $\varkappa_n^{(\varepsilon)}$, $n = 1, 2, \cdots$ are also independent.

In this case, condition **A′** has the following form:

A″: $\lim\limits_{t\to\infty} \overline{\lim\limits_{\varepsilon\to 0}} [1 - G^{(\varepsilon)}(t)] = 0.$

Note that a standard geometric sum with any distribution of summands $G^{(\varepsilon)}(t)$ and the geometric random index $p_\varepsilon \in (0,1]$ can be modeled in the way described above. Indeed, it is enough to consider the geometric sum $\xi_\varepsilon = \sum_{n=1}^{\nu_\varepsilon} \varkappa_n^{(\varepsilon)}$ defined above, where ($\mathbf{a}_1$) $\varkappa_n^{(\varepsilon)}$, $n = 1, 2, \cdots$ is a sequence of i.i.d. random variables with the joint distribution function $G^{(\varepsilon)}(t)$; ($\mathbf{a}_2$) $\nu_\varepsilon = \max\left(n \geq 1 : \zeta_n^{(\varepsilon)} \in D_\varepsilon\right)$, where $\zeta_n^{(\varepsilon)}$, $n = 1, 2, \cdots$ is a sequence of i.i.d. random variables uniformly distributed on the interval $[0,1]$ and domains $D_\varepsilon = [0, p_\varepsilon)$; ($\mathbf{a}_3$) two sequences of random variables $\varkappa_n^{(\varepsilon)}$, $n = 1, 2, \cdots$ and $\zeta_n^{(\varepsilon)}$, $n = 1, 2, \cdots$ are independent.

Theorem 2.1 reduces in this case to the result equivalent to those obtained by Kruglov and Korolev (1990).

Let us illustrate applications of Theorem 2.1 by giving the necessary and sufficient conditions for the weak convergence of non-ruin distribution functions. Let us consider a process used in the classical risk theory to model the activity of an insurance company,

$$X_\varepsilon(t) = c_\varepsilon t - \sum_{n=1}^{N_{\lambda_\varepsilon}(t)} Z_n^{(\varepsilon)}, \ t \geq 0.$$

Here, a positive constant c_ε (depending on the parameter $\varepsilon > 0$) is the gross premium rate; $N_{\lambda_\varepsilon}(t)$, $t \geq 0$ is a Poisson process with parameter λ_ε counting the number of claims on the time-interval $[0,t]$; and $Z_n^{(\varepsilon)}$, $n = 1, 2, \cdots$ is a sequence of nonnegative i.i.d. random variables which are independent of the process $N_{\lambda_\varepsilon}(t)$, $t \geq 0$. The random variable $Z_k^{(\varepsilon)}$ is the amount of the k^{th} claim.

In this model, an important object for studies is the non-ruin probabilities on the infinite time interval for a company with an initial capital $u \geq 0$,

$$F_\varepsilon(u) = \mathsf{P}\left\{u + \inf_{t\geq 0} X_\varepsilon(t) \geq 0\right\}, \quad u \geq 0.$$

Let $H^{(\varepsilon)}(x) = \mathsf{P}\left\{Z_1^{(\varepsilon)} \leq x\right\}$ be the claim distribution function. We assume the standard condition:

M: $\mu_\varepsilon = \int_0^\infty s H^{(\varepsilon)}(ds) < \infty$.

The crucial role here is played by the so-called safety loading coefficient $\alpha_\varepsilon = \lambda_\varepsilon \mu_\varepsilon / c_\varepsilon$. If $\alpha_\varepsilon \geq 1$ then $F_\varepsilon(u) = 0$, $u \geq 0$. The only non-trivial case is given by $\alpha_\varepsilon < 1$. We assume the following condition:

N: $0 < \alpha_\varepsilon < 1 \quad \text{for} \quad \varepsilon > 0; \quad \text{and} \quad \alpha_\varepsilon \to 1 \quad \text{as} \quad \varepsilon \to 0.$

According to the Pollaczek–Khinchine formula (see, for example, Asmussen (2000)), the non-ruin distribution function $F_\varepsilon(u)$ coincides with the distribution function of a geometric random sum which is slightly different from the standard geometric sums considered above, namely,

$$F_\varepsilon(u) = \mathsf{P}\left\{\xi'_\varepsilon = \sum_{n=1}^{\nu_\varepsilon - 1} \varkappa_n^{(\varepsilon)} \ \leq \ u\right\}, \quad u \geq 0,$$

where ($\mathbf{a}$) $\varkappa_n^{(\varepsilon)}$, $n = 1, 2, \cdots$ is a sequence of non-negative i.i.d. random variables with the distribution function $G^{(\varepsilon)}(u) = \frac{1}{\mu_\varepsilon} \int_0^u \left(1 - H^{(\varepsilon)}(s)\right) ds$, for $u \geq 0$, (the so-called steady claim distribution); ($\mathbf{b}$) $\nu_\varepsilon = \min_{n\geq 1} \{n : \chi_{n\varepsilon} = 1\}$; ($\mathbf{c}$) $\chi_{n\varepsilon}$, $n = 1, 2, \cdots$ is a sequence of i.i.d. random variables taking values 1 and 0 with probabilities $p_\varepsilon = 1 - \alpha_\varepsilon$ and $1 - p_\varepsilon$; ($\mathbf{d}$) random sequences $\varkappa_n^{(\varepsilon)}$, $n = 1, 2, \cdots$, and $\chi_{n\varepsilon}$, $n = 1, 2, \cdots$, are independent.

In this case, condition **A** takes the form

A‴: $\lim_{u\to\infty} \overline{\lim_{\varepsilon\to 0}} \frac{1}{\mu_\varepsilon} \int_u^\infty [1 - H^{(\varepsilon)}(s)] ds = 0.$

Theorem 2.1, with specification to the geometric sums described above, can be applied to the geometric random sums ξ'_ε.

Conditions **C** and **D** can be omitted. Condition **B** is equivalent to condition **N**.

Conditions **E** and **F** take, in this case, the form

E′: $\frac{\int_{uu_\varepsilon}^{+\infty}[1-H^{(\varepsilon)}(s)]ds}{\mu_\varepsilon(1-\alpha_\varepsilon)} \to h(u) \quad \text{as} \quad \varepsilon \to 0,$
for all positive points of continuity of the limit function $h(u)$.

F′: $\frac{\int_0^{uu_\varepsilon} s(1-H^{(\varepsilon)}(s))ds}{\mu_\varepsilon(1-\alpha_\varepsilon)} \to f(u) \quad \text{as} \quad \varepsilon \to 0,$
for some positive point of continuity of $h(u)$.

Limit functions $h(u)$ and $f(u)$ satisfy conditions ($\mathbf{a}_1$)-($\mathbf{a}_4$):

($\mathbf{a}_1$) $h(u)$ is a non-negative, non-increasing, and right-continuous function of $u > 0$, such that $h(\infty) = 0$;

($\mathbf{a}_2$) The measure $\mathrm{H}(A)$ on the σ-algebra $\mathcal{H}^+$, the Borel σ-algebra of subsets of $(0, \infty)$, defined by the relation $\mathrm{H}((u_1, u_2]) = h(u_1) - h(u_2)$, $0 < u_1 \le u_2 < \infty$;

($\mathbf{a}_3$) Under $\mathbf{E}'_\gamma$, condition $\mathbf{F}'_{a\gamma}$ can only hold simultaneously for all continuity points of $h(u)$ and $f(u_1) = f(u_2) - \int_{u_1}^{u_2} s\mathrm{H}(ds)$ for any points $0 < u_1 < u_2 < \infty$;

($\mathbf{a}_4$) $f(u)$ is a non-negative function.

Conditions **E′** and **F′** are the necessary and sufficient conditions for the weak convergence,

$$\vartheta^{(\varepsilon)}(t) = \sum_{k=1}^{[tp_\varepsilon^{-1}]} \frac{\vartheta_k^{(\varepsilon)}}{u_\varepsilon}, \ t \ge 0 \quad \Longrightarrow \quad \vartheta(t), \ t \ge 0 \quad \text{as} \quad \varepsilon \to 0,$$

where $\vartheta_k^{(\varepsilon)}$ are i.i.d. random variables with the joint distribution $G^{(\varepsilon)}(t)$; $\vartheta(t)$, $t \ge 0$ is a Lévy process with cumulant $a(s)$, i.e. $Ee^{-s\vartheta(t)} = e^{-a(s)t}$, which has the form

$$a(s) = as - \int_0^\infty (e^{-sx} - 1)\mathrm{H}(dx),$$

where the constant

$$a = f(u) - \int_0^u s\mathrm{H}(ds)$$

does not depend on the choice of a point u in condition **F′**.

Let us summarize the discussion above in the form of the following theorem which gives the necessary and sufficient conditions for the approximation of non-ruin probabilities.

Theorem 5.1. *Let conditions* **M**, **N** *and* **A‴** *hold. Then the class of all possible, non-concentrated at zero, limit distribution functions* $F(u)$ (*in the sense of weak convergence*), *such that the non-ruin distribution functions* $F_\varepsilon(uu_\varepsilon) \Longrightarrow F(u)$ *as* $\varepsilon \to 0$, *coincides with the class of distributions* $F(u)$ *with Laplace transforms* $\frac{1}{1+a(s)}$. *Conditions* **E′** *and* **F′** *are necessary and sufficient for the weak convergence* $F_\varepsilon(uu_\varepsilon) \Longrightarrow F(u)$ *as* $\varepsilon \to 0$.

6. Application to queueing systems

Let us consider a controlled M/G queueing system with m different types of customers in the input flow. When a customer of type i is coming to the system, its service time has distribution $H_i^{(\varepsilon)}(\cdot)$, and the interarrival time (difference in arrival times for the next and the current customers) is exponentially distributed with parameter $\lambda_{i\varepsilon}$. The appearance of the customer of a certain type is modeled by the ergodic Markov chain $\eta_n^{(\varepsilon)}$, with the phase space $X = \{1, 2, \cdots, m\}$, and the transition probabilities $\left\| p_{ij}^{(\varepsilon)} \right\|$, $i, j \in X$. Thus, the input flow of customers in the system is modeled by the semi-Markov process with transition probabilities

$$Q_{ij}^{(\varepsilon)}(t) = p_{ij}^{(\varepsilon)}\left(1 - e^{-\lambda_{i\varepsilon} t}\right), \quad t \geq 0, \quad i, j \in X.$$

The failure in the system occurs if the next customer arrives before the service for the previous customer is finished. It is the so-called system with no buffer.

Let $\varkappa'_{n\varepsilon}$ and $\varkappa''_{n\varepsilon}$ be random variables representing, respectively, the interarrival time and the service time for the n-th customer. Our service system can be described by a Markov renewal process $(\eta_n^{(\varepsilon)},\ \varkappa'_{n\varepsilon},\ \varkappa''_{n\varepsilon})$, with the phase space $\{1, 2, \ldots, m\} \times [0, \infty) \times [0, \infty)$ and the transition probabilities

$$\begin{aligned} &\mathsf{P}\left\{ \eta_{n+1}^{(\varepsilon)} = j,\ \varkappa'_{n+1\varepsilon} \leq u,\ \varkappa''_{n+1\varepsilon} \leq v \Big/ \eta_n^{(\varepsilon)} = i,\ \varkappa'_{n\varepsilon} = k,\ \varkappa''_{n\varepsilon} = l \right\} \\ &\quad = \mathsf{P}\left\{ \eta_{n+1}^{(\varepsilon)} = j,\ \varkappa'_{n+1\varepsilon} \leq u,\ \varkappa''_{n+1\varepsilon} \leq v \Big/ \eta_n^{(\varepsilon)} = i \right\} \\ &\quad = p_{ij}^{(\varepsilon)}\left(1 - e^{-\lambda_{i\varepsilon} u}\right) H_i^{(\varepsilon)}(v). \end{aligned}$$

We consider the model with quick service and impose the following “quick-service” condition:

O: $(\mathbf{o}_1)$ $\min_{i \in X} H_i^{(\varepsilon)}(0) < 1$ for $\varepsilon > 0$;

$(\mathbf{o}_2)$ $\overline{\lim_{\varepsilon \to 0}} \left[1 - H_i^{(\varepsilon)}(t)\right] = 0, \quad i \in X, \quad t > 0.$

The second condition **P** is a condition of convergence of the transition matrix of the embedded perturbed Markov chain $\eta_n^{(\varepsilon)}$ to the transition matrix of the embedded limit Markov chain $\eta_n^{(0)}$:

P: $p_{ij}^{(\varepsilon)} \to p_{ij}^{(0)}$ as $\varepsilon \to 0, \quad i, j \in X.$

The third condition **Q** is a standard ergodicity condition for the embedded limit Markov chain $\eta_n^{(0)}$:

Q: Markov chain $\eta_n^{(0)}$ with the matrix of transition probabilities $\|p_{ij}^{(0)}\|$ is ergodic with the stationary distribution $\pi_i^{(0)}$, $i \in X$.

Condition **R** is a regularity condition for the intensities of interarrival times:

R: $0 < \underline{\lim}_{\varepsilon \to 0} \lambda_{i\varepsilon} \leq \overline{\lim}_{\varepsilon \to 0} \lambda_{i\varepsilon} < \infty, \quad i \in X.$

The total service time, starting from the moment when the first customer comes to the system, can be defined as

$$\xi_\varepsilon = \sum_{k=1}^{\nu_\varepsilon} \varkappa'_{k\varepsilon},$$

where the random variable

$$\nu_\varepsilon = \min\{n :\ \varkappa'_{n\varepsilon} < \varkappa''_{n\varepsilon}\}$$

represents the number of service cycles before the failure in the system occurs.

The probability that a customer of type i will not have enough time to finish his/her service is

$$p_{i\varepsilon} = \int_0^\infty \left[1 - H_i^{(\varepsilon)}(s)\right] \lambda_{i\varepsilon} e^{-\lambda_{i\varepsilon} s} ds, \quad i \in X, \tag{21}$$

and we define the corresponding averaged probability in our usual way as

$$p_\varepsilon = \sum_{i=1}^m \pi_i^{(0)} p_{i\varepsilon}.$$

Since the interarrival distributions are exponential, the corresponding averaged interarrival distribution function will have the form

$$G^{(\varepsilon)}(t) = 1 - \sum_{i=1}^{m} \pi_i^{(0)} e^{-\lambda_{i\varepsilon} t}, \quad t > 0. \tag{22}$$

The following condition is necessary and sufficient for the weak convergence of the distribution functions of normalized first-rare-event times $p_\varepsilon \xi_\varepsilon$ to the exponential distribution:

S: $\lambda_\varepsilon = \sum_{i=1}^{m} \pi_i^{(0)} \frac{1}{\lambda_{i\varepsilon}} \to a \quad \text{as} \quad \varepsilon \to 0, \quad \text{where} \quad a > 0.$

Theorem 6.1. *Let conditions* **O**, **P**, **Q**, *and* **R** *hold. Then condition* **S** *is necessary and sufficient for the convergence relation* $p_\varepsilon \xi_\varepsilon \Longrightarrow \xi$ *as* $\varepsilon \to 0$ *to hold, where* ξ *has the exponential distribution with parameter* a.

Proof. To reduce the model to our general settings, we introduce the Markov renewal process $\left(\eta_n^{(\varepsilon)},\ \varkappa_n^{(\varepsilon)},\ \zeta_n^{(\varepsilon)}\right)$, $n = 0, 1, \cdots$, with the phase space $X \times [0, +\infty) \times \{0, 1\}$, where

$$\begin{cases} \eta_n^{(\varepsilon)} & := & \eta_n^{(\varepsilon)} \\ \varkappa_n^{(\varepsilon)} & := & \varkappa'_{n\varepsilon} \\ \zeta_n^{(\varepsilon)} & := & \chi\left(\varkappa'_{n\varepsilon} < \varkappa''_{n\varepsilon}\right). \end{cases}$$

In this case, the stopping domain is $D_\varepsilon = \{1\}$, and the transition probabilities for the process have the form

$$\begin{aligned} &\mathsf{P}\left\{ \eta_{n+1}^{(\varepsilon)} = j,\ \varkappa_{n+1}^{(\varepsilon)} \le u,\ \zeta_{n+1}^{(\varepsilon)} = \frac{1 \pm 1}{2} \Big/ \eta_n^{(\varepsilon)} = i,\ \varkappa_n^{(\varepsilon)} = s,\ \zeta_n^{(\varepsilon)} = y \right\} \\ &\quad = \mathsf{P}\left\{ \eta_{n+1}^{(\varepsilon)} = j,\ \varkappa_{n+1}^{(\varepsilon)} \le u,\ \zeta_{n+1}^{(\varepsilon)} = \frac{1 \pm 1}{2} \Big/ \eta_n^{(\varepsilon)} = i \right\} \\ &\quad = \begin{cases} p_{ij}^{(\varepsilon)} \int_0^u \left[1 - H_i^{(\varepsilon)}(s)\right] \lambda_{i\varepsilon} e^{-\lambda_{i\varepsilon} s} ds, & \text{for } \zeta_{n+1}^{(\varepsilon)} = 1; \\ p_{ij}^{(\varepsilon)} \int_0^u H_i^{(\varepsilon)}(s) \lambda_{i\varepsilon} e^{-\lambda_{i\varepsilon} s} ds, & \text{for } \zeta_{n+1}^{(\varepsilon)} = 0, \end{cases} \end{aligned}$$

where $i, j \in X$, and $s, t \ge 0$.

The number of served customers including the customer, whose service was started but not completed, can be redefined as

$$\nu_\varepsilon = \min\left\{ n :\ \zeta_n^{(\varepsilon)} = 1 \right\},$$

and the total service time of the system, starting from the moment of arrival of the first customer, can be redefined as

$$\xi_\varepsilon = \sum_{k=1}^{\nu_\varepsilon} \varkappa_k^{(\varepsilon)}.$$

To prove the current theorem, we are going to show that the statement of the theorem follows from Theorem 2.1. Let us check that condition **A** holds. Indeed, we have, for every $i \in X$,

$$\begin{aligned}
\mathsf{P}_i\left\{\varkappa_1^{(\varepsilon)} > t \Big/ \zeta_1^{(\varepsilon)} \in D_\varepsilon\right\} &= \mathsf{P}_i\{\varkappa'_{1\varepsilon} > t \,/\, \varkappa'_{1\varepsilon} < \varkappa''_{1\varepsilon}\} \\
&= \frac{\mathsf{P}_i\{t < \varkappa'_{1\varepsilon} < \varkappa''_{1\varepsilon}\}}{\mathsf{P}_i\{\varkappa'_{1\varepsilon} < \varkappa''_{1\varepsilon}\}} = \frac{\int_t^{+\infty}[1 - H_i^{(\varepsilon)}(s)]\lambda_{i\varepsilon}e^{-\lambda_{i\varepsilon}s}ds}{\int_0^{+\infty}[1 - H_i^{(\varepsilon)}(s)]\lambda_{i\varepsilon}e^{-\lambda_{i\varepsilon}s}ds} \\
&= \frac{1}{\frac{\int_0^t[1-H_i^{(\varepsilon)}(s)]\lambda_{i\varepsilon}e^{-\lambda_{i\varepsilon}s}ds}{\int_t^{+\infty}[1-H_i^{(\varepsilon)}(s)]\lambda_{i\varepsilon}e^{-\lambda_{i\varepsilon}s}ds} + 1} \leq \frac{1}{\frac{[1-H_i^{(\varepsilon)}(t)]\int_0^t\lambda_{i\varepsilon}e^{-\lambda_{i\varepsilon}s}ds}{[1-H_i^{(\varepsilon)}(t)]\int_t^{+\infty}\lambda_{i\varepsilon}e^{-\lambda_{i\varepsilon}s}ds} + 1} \qquad (23) \\
&= \frac{\int_t^{+\infty}\lambda_{i\varepsilon}e^{-\lambda_{i\varepsilon}s}ds}{\int_0^{+\infty}\lambda_{i\varepsilon}e^{-\lambda_{i\varepsilon}s}ds} = \int_t^{+\infty}\lambda_{i\varepsilon}e^{-\lambda_{i\varepsilon}s}ds = e^{-\lambda_{i\varepsilon}t},
\end{aligned}$$

and, using condition **R**,

$$\lim_{t\to\infty}\overline{\lim_{\varepsilon\to 0}}\, e^{-\lambda_{i\varepsilon}t} = \lim_{t\to\infty} e^{-\underline{\lim}_{\varepsilon\to 0}\lambda_{i\varepsilon}t} = 0. \qquad (24)$$

Relations (23) and (24) imply that condition **A** of Theorem 2.1 holds.

Let us check condition **B** of Theorem 2.1. It follows from condition $\mathbf{o}_1$ that there exists $j \in X$ such that $H_j^{(\varepsilon)}(0) < 1$ for $\varepsilon > 0$, thus

$$\begin{aligned}
\max_{i\in X} p_{i\varepsilon} &= \max_{i\in X}\int_0^\infty\left[1 - H_j^{(\varepsilon)}(s)\right]\lambda_{j\varepsilon}e^{-\lambda_{j\varepsilon}s}ds \\
&\geq \max_{i\in X}\left[1 - H_i^{(\varepsilon)}(0)\right]\lambda_{j\varepsilon} > 0.
\end{aligned}$$

Moreover, due to conditions **R** and $\mathbf{o}_2$, for every $i \in X$ and every $\delta > 0$,

$$\overline{\lim_{\varepsilon\to 0}}\int_0^\delta[1 - H_i^{(\varepsilon)}(s)]\lambda_{i\varepsilon}e^{-\lambda_{i\varepsilon}s}ds \leq \overline{\lim_{\varepsilon\to 0}}\,\lambda_{i\varepsilon}\delta \to 0 \quad \text{as} \quad \delta \to 0. \qquad (25)$$

Due to the same conditions, for every $i \in X$ and every $\delta > 0$,

$$\begin{aligned}
&\overline{\lim_{\varepsilon\to 0}}\int_\delta^{+\infty}[1 - H_i^{(\varepsilon)}(s)]\lambda_{i\varepsilon}e^{-\lambda_{i\varepsilon}s}ds \\
&\quad\leq \overline{\lim_{\varepsilon\to 0}}[1 - H_i^{(\varepsilon)}(\delta)]\int_\delta^{+\infty}\lambda_{i\varepsilon}e^{-\lambda_{i\varepsilon}s}ds \qquad (26) \\
&\quad\leq \overline{\lim_{\varepsilon\to 0}}[1 - H_i^{(\varepsilon)}(\delta)] = 0.
\end{aligned}$$

Taking relations (21), (25), and (26) into account, we get

$$p_{i\varepsilon} \to 0 \quad \text{as} \quad \varepsilon \to 0, \quad i \in X.$$

This means that condition **B** holds.

Conditions **P** and **Q** are identical to conditions **C** and **D**.

Due to relation (2), we have to search for the necessary and sufficient conditions of convergence

$$\vartheta^{(\varepsilon)}(t) = p_\varepsilon\sum_{k=1}^{[tp_\varepsilon^{-1}]}\vartheta_k^{(\varepsilon)},\ t \geq 0, \quad \Longrightarrow \quad \vartheta(t),\ t \geq 0, \quad \text{as} \quad \varepsilon \to 0, \qquad (27)$$

where $\vartheta_k^{(\varepsilon)}$ are i.i.d. random variables with the joint distribution

$$G^{(\varepsilon)}(t) = 1 - \sum_{i=1}^m \pi_i^{(0)}e^{-\lambda_{i\varepsilon}t},$$

and $\vartheta(t)$, $t \geq 0$ is some Lévy process.

We are going to prove that, under condition **R**, the only possible variant of the limit process is $\vartheta(t) = at$, $t \geq 0$, and that condition **S** is the necessary and sufficient condition for weak convergence in (27) to such a process.

Using the Chebyshev inequality, condition **R**, relation $p_\varepsilon \to 0$ as $\varepsilon \to 0$, and formula (22), we have, for every $t \geq 0$ and $\delta > 0$,

$$\begin{aligned} \mathsf{P}\{|p_\varepsilon \sum_{k=1}^{[tp_\varepsilon^{-1}]} \vartheta_k^{(\varepsilon)} - p_\varepsilon[tp_\varepsilon^{-1}]\mathsf{E}\vartheta_1^{(\varepsilon)}| > \delta\} &\leq \delta^{-2}p_\varepsilon^2[tp_\varepsilon^{-1}]\mathsf{Var}\vartheta_1^{(\varepsilon)}, \\ &\leq \delta^{-2}p_\varepsilon^2[tp_\varepsilon^{-1}]\sum_{i=1}^{m}\frac{\pi_i^{(0)}}{\lambda_{i\varepsilon}^2} \to 0 \quad \text{as} \quad \varepsilon \to 0. \end{aligned} \tag{28}$$

It is easy to see that

$$|p_\varepsilon[tp_\varepsilon^{-1}]\mathsf{E}\vartheta_1^{(\varepsilon)} - at| \to 0 \quad \text{as} \quad \varepsilon \to 0, \tag{29}$$

if and only if there exists a constant $a > 0$ such that $E\vartheta_1^{(\varepsilon)} \to a$ as $\varepsilon \to 0$.

It follows from (28) and (29) that relation (27) can hold if and only if there exist a constant $a > 0$ such that

$$\lim_{\varepsilon\to 0}\mathsf{E}\vartheta_1^{(\varepsilon)} = a.$$

In this case, the limit process $\vartheta(t) = at$, $t \geq 0$. But

$$\mathsf{E}\vartheta_1^{(\varepsilon)} = \sum_{i=1}^{m}\pi_i^{(0)}\frac{1}{\lambda_{i\varepsilon}}.$$

Thus, condition **S** is, indeed, the necessary and sufficient condition for relation (27) to hold, and the limit process $\vartheta(t)$ should be of the form $\vartheta(t) = at$, $t \geq 0$. □

Note that conditions of Theorem 6.1 do not require the convergence of the coefficients $\lambda_{i\varepsilon}$ as $\varepsilon \to 0$, $i \in X$.

Example 6.1. Let us consider the case where $m = 2$ and

$$\lambda_{i\varepsilon} = \frac{1}{a(\varepsilon)[c_i + (-1)^{i+1}c\sin(1/\varepsilon)]}, \quad i \in X.$$

The parameters are chosen so that

$$c_i > c > 0, \quad i \in X,$$

and

$$0 < a(\varepsilon) \to \frac{2a}{c_1 + c_2} \quad \text{as} \quad \varepsilon \to 0, \quad \text{where} \quad 0 < a < \infty.$$

In this case, for all $i \in X$,

$$0 < \underline{\lim}_{\varepsilon\to 0}\lambda_{i\varepsilon} = \frac{c_1 + c_2}{2a[c_i + c]} < \frac{c_1 + c_2}{2a[c_i - c]} = \overline{\lim}_{\varepsilon\to 0}\lambda_{i\varepsilon} < \infty.$$

Thus, condition **R** holds, but $\underline{\lim}_{\varepsilon\to 0}\lambda_{i\varepsilon} \neq \overline{\lim}_{\varepsilon\to 0}\lambda_{i\varepsilon}$.

If the stationary distribution of the embedded Markov chain is

$$\pi_1^{(0)} = \pi_2^{(0)} = \frac{1}{2},$$

then

$$\begin{aligned} \mathsf{E}\vartheta_1^{(\varepsilon)} &= \sum_{i=1}^{2}\frac{a(\varepsilon)[c_i + c(-1)^{i+1}\sin(1/\varepsilon)]}{2} \\ &= a(\varepsilon)\frac{c_1 + c_2}{2} \to a \quad \text{as} \quad \varepsilon \to 0. \end{aligned}$$

and condition **S** also holds.

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