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ON ASYMPTOTIC BEHAVIOR OF CROSS-CORRELOGRAM ESTIMATORS OF RESPONSE FUNCTIONS IN LINEAR VOLTERRA SYSTEMS

The problem of estimation of an unknown response function of a linear system with inner noises is considered. We suppose that the response function of the system belongs to $L_2(\mathbf{R})$. Integral-type sample input-output cross-correlograms are taken as estimators of the response function. The inputs are supposed to be zero-mean stationary Gaussian processes close, in some sense, to a white noise. Both the asymptotic normality of finite-dimensional distributions of the centered estimators and their asymptotic normality in the space of continuous functions are studied.

1. Introduction

In this paper, we consider a time-invariant casual continuous linear Volterra system with inner noises and a response function $H(\tau), \tau \in \mathbf{R}$. It means that the real-valued function H satisfies the condition $H(\tau) = 0,\ \tau < 0$, and the response of the system to an input process $X(t),\ t \in \mathbf{R}$, has the form

$$U(t) = \int_0^\infty H(\tau)X(t-\tau)\,d\tau + Z(t), \tag{1}$$

where the process $Z(t),\ t \in \mathbf{R}$, describes inner noises of the system.

One of the problems arising in the theory of such systems is to estimate or identify the function H by observations of responses of the system to certain input signals. While solving this problem, different statistical approaches along with various deterministic methods are used. These statistical approaches are based on a perturbation of the system by stationary stochastic processes and the further analysis of some characteristics of both input and output processes [2, 4, 5, 14]. The output process of the stable system ($H \in L_1(\mathbf{R})$) has a spectral density, and a method of periodograms may be used for the estimation [1, 3]. For an unstable system, it is reasonable to use other methods, in particular, a method of correlograms. This method is based on constructing a sample cross-correlogram between the input stochastic process similar to the white noise and the response of the system ([6], [10]). Such an approach is not applicable in practice because the simulation of the white noise is impossible. In fact, we always deal with a sequence of stationary Gaussian processes that disturb the system and depend on a certain parameter $\Delta \in (0, \infty)$ and such that their spectral densities converge to a constant as $\Delta \to \infty$. Sample correlograms between input and output processes are taken as estimators for H ([9], [11]-[13]).

Here, we use the method of correlograms for the estimation of the response function $H \in L_2(\mathbf{R})$. Such an assumption makes it possible to consider unstable systems with resonant singularities. This paper continues the research of ([7], [9]) and focuses on the asymptotic normality of integral-type cross-correlogram estimators in the space of continuous functions.

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2. Preliminaries

Assume that $(X_\Delta(t),\ t \in \mathbf{R})$, $\Delta > 0$, is a family of measurable real-valued stationary zero-mean Gaussian processes that disturb the system (1). Let $(f_\Delta(\lambda),\ \lambda \in \mathbf{R})$, $\Delta > 0$, be a family of spectral densities of the processes X_Δ. We suppose that these functions are nonnegative, continuous, and satisfy the conditions

(2a) $$f_\Delta(\lambda) = f_\Delta(-\lambda),\ \lambda \in \mathbf{R};$$

(2b) $$\sup_{\Delta>0} \|f_\Delta\|_\infty < \infty;$$

(2c) $$f_\Delta \in L_1(\mathbf{R});$$

(2d) $$\exists c \in (0,\infty)\ \ \forall a \in (0,\infty):\ \lim_{\Delta\to\infty} \sup_{-a\le\lambda\le a} \left| f_\Delta(\lambda) - \frac{c}{2\pi} \right| = 0;$$

(2e) $$K_{X_\Delta} \in L_1(\mathbf{R}),$$

where $K_{X_\Delta}(t) = \int\limits_{-\infty}^{\infty} e^{i\lambda t} f_\Delta(t)\, dt,\ t \in \mathbf{R}$, is the correlation function of X_Δ.

These conditions are satisfied, for example, by the function

$$f_\Delta(\lambda) = \frac{c}{2\pi} \exp\left(-\frac{\lambda^2}{\Delta}\right),\ \lambda \in \mathbf{R}.$$

By (1), the reaction of the system on an input signal X_Δ is represented by

(3) $$U_\Delta(t) = \int_0^\infty H(\tau) X_\Delta(t-\tau)\, d\tau + Z(t),\ t \in \mathbf{R}.$$

We assume that the inner noise $(Z(t),\ t \in \mathbf{R})$ is a separable real-valued stationary zero-mean Gaussian process which is orthogonal to X_Δ; that is, $\mathbf{E} X_\Delta(s) Z(t) = 0,\ s,t \in \mathbf{R}$. Let $(g(\lambda),\ \lambda \in \mathbf{R})$ be the spectral density of the process Z. It is a nonnegative measurable function which satisfies the conditions

(4a) $$g(\lambda) = g(-\lambda);$$

(4b) $$g \in L_1(\mathbf{R}).$$

The so-called *cross-correlogram* (or *sample cross-correlation function*)

(5) $$\widehat{H}_{T,\Delta}(\tau) = \frac{1}{cT} \int_0^T U_\Delta(t+\tau) X_\Delta(t)\, dt,\ \tau \ge 0,$$

will be used as an estimate for H. Here, c is the constant from (2d), and T is the length of the averaging interval. The integrals in (3) and (5) are interpreted as mean square Riemann integrals.

Denote, by

$$H^*(\lambda) = \int\limits_{-\infty}^{\infty} e^{i\lambda t} H(t)\, dt,\ \lambda \in \mathbf{R},$$

the *Fourier–Plancherel transform* of $H \in L_2(\mathbf{R})$ (see [17]).

Along with the process $(\widehat{H}_{T,\Delta}(\tau),\ \tau \ge 0)$, we consider the process

(6) $$A_{T,\Delta}(\tau) = \sqrt{T}[\widehat{H}_{T,\Delta}(\tau) - \mathbf{E}\widehat{H}_{T,\Delta}(\tau)],\ \tau \ge 0,$$

where $\mathbf{E}\widehat{H}_{T,\Delta}(\tau) = \frac{1}{c}\int_0^T K_{X_\Delta}(\tau - s) H(s)\, ds$.

In the statement below, we obtain a form of the correlation function of $(A_{T,\Delta}(\tau),\ \tau \ge 0)$.

Lemma 2.1. *Assume that $H \in L_2(\mathbf{R})$ and $g \in L_1(\mathbf{R})$. Then the equality*

(7) $$\boldsymbol{E} A_{T,\Delta}(\tau_1) A_{T,\Delta}(\tau_2) = \frac{2\pi}{c^2} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \Big[e^{i(\tau_1-\tau_2)\lambda_2} \left(|H^*(\lambda_2)|^2 f_\Delta(\lambda_2) + g(\lambda_2) \right) +$$

$$+e^{i(\tau_1\lambda_1+\tau_2\lambda_2)}H^*(\lambda_1)H^*(\lambda_2)f_\Delta(\lambda_2)\Big]\Phi_T(\lambda_2-\lambda_1)f_\Delta(\lambda_1)\,d\lambda_1 d\lambda_2$$

holds for all $\tau_1, \tau_2 \geq 0$. Here, Φ_T is the Fejer kernel; that is,

$$\Phi_T(\lambda) = \frac{1}{2\pi T}\left(\frac{\sin(T\lambda/2)}{\lambda/2}\right)^2, \ \lambda \in \mathbf{R}.$$

Proof. The proof of Lemma 2.1 is standard (see [12]). □

3. Asymptotic behavior of the correlation function of $A_{T,\Delta}$

In this section, we consider the asymptotic behavior of the correlation function of $A_{T,\Delta}$ as T and Δ tend to infinity. In what follows, we write $(T,\Delta) \to \infty$ if both $T \to \infty$ and $\Delta \to \infty$.

For all τ_1, $\tau_2 \geq 0$, set

(8)
$$C_\infty(\tau_1,\tau_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[e^{i(\tau_1-\tau_2)\lambda}\left(|H^*(\lambda)|^2 + \frac{2\pi}{c}g(\lambda)\right) + e^{i(\tau_1+\tau_2)\lambda}(H^*(\lambda))^2\right]d\lambda.$$

Note that the function C_∞ is well-defined and continuous, since $H \in L_2(\mathbf{R})$ and $g \in L_1(\mathbf{R})$.

Theorem 3.1. *Assume that $H \in L_2(\mathbf{R})$ and $g \in L_1(\mathbf{R})$. Then the equality*

$$\lim_{(T,\Delta)\to\infty} \boldsymbol{E}A_{T,\Delta}(\tau_1)A_{T,\Delta}(\tau_2) = C_\infty(\tau_1,\tau_2)$$

holds for all $\tau_1, \tau_2 \geq 0$.

Corollary 3.1. *Assume that $H \in L_2(\mathbf{R})$ and $g \in L_1(\mathbf{R})$. Then the equality*

$$\lim_{(T,\Delta)\to\infty} T\boldsymbol{E}|\widehat{H}_{T,\Delta}(\tau) - \boldsymbol{E}\widehat{H}_{T,\Delta}(\tau)|^2 = \frac{1}{c}\|g\|_1 + \|H\|_2^2 + \int_0^{2\tau} H(t)H(2\tau-t)\,dt$$

holds for all $\tau \geq 0$.

Remark 3.1. Theorem 3.1 weakens conditions of Lemma 4 in [9].

Proof. To prove Theorem 3.1, we use some general results stated in [11, 12] and break up the proof into three steps.

At Step 1, we prove the equality

(9)
$$\lim_{(T,\Delta)\to\infty} \widehat{C}^{(1)}_{T,\Delta}(\tau_1,\tau_2) = C^{(1)}_\infty(\tau_1,\tau_2),$$

where

$$\widehat{C}^{(1)}_{T,\Delta}(\tau_1,\tau_2) = \frac{2\pi}{c^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\tau_1-\tau_2)\lambda_2}|H^*(\lambda_2)|^2 f_\Delta(\lambda_1)f_\Delta(\lambda_2)\Phi_T(\lambda_2-\lambda_1)\,d\lambda_1 d\lambda_2$$

and

$$C^{(1)}_\infty(\tau_1,\tau_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\tau_1-\tau_2)\lambda}|H^*(\lambda)|^2\,d\lambda, \quad \tau_1,\tau_2 \geq 0.$$

At Step 2, we prove the equality

(10)
$$\lim_{(T,\Delta)\to\infty} \widehat{C}^{(2)}_{T,\Delta}(\tau_1,\tau_2) = C^{(2)}_\infty(\tau_1,\tau_2),$$

where

$$\widehat{C}^{(2)}_{T,\Delta}(\tau_1,\tau_2) = \frac{2\pi}{c^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\tau_1\lambda_1+\tau_2\lambda_2)}H^*(\lambda_1)H^*(\lambda_2)f_\Delta(\lambda_1)f_\Delta(\lambda_2)\Phi_T(\lambda_2-\lambda_1)\,d\lambda_1 d\lambda_2$$

and

$$C_\infty^{(2)}(\tau_1,\tau_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\tau_1+\tau_2)\lambda}(H^*(\lambda))^2\,d\lambda, \quad \tau_1,\tau_2\ge 0.$$

At Step 3, we prove the equality

$$\lim_{(T,\Delta)\to\infty} \widehat{C}_{T,\Delta}^{(3)}(\tau_1,\tau_2) = C_\infty^{(3)}(\tau_1,\tau_2), \tag{11}$$

where

$$\widehat{C}_{T,\Delta}^{(3)}(\tau_1,\tau_2) = \frac{2\pi}{c^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\tau_1-\tau_2)\lambda_2} f_\Delta(\lambda_1)g(\lambda_2)\Phi_T(\lambda_2-\lambda_1)\,d\lambda_1 d\lambda_2$$

and

$$C_\infty^{(3)}(\tau_1,\tau_2) = \frac{1}{c}\int_{-\infty}^{\infty} e^{i(\tau_1-\tau_2)\lambda} g(\lambda)\,d\lambda, \quad \tau_1,\tau_2\ge 0.$$

Step 1. Observe that, for all $\tau_1,\ \tau_2\ge 0,\ T>0,\ \Delta>0$ and each $b>0$, the equality

$$C_\infty^{(1)}(\tau_1,\tau_2) - \widehat{C}_{T,\Delta}^{(1)}(\tau_1,\tau_2) = \frac{1}{2\pi}[d_1(b)+d_2(b,T,\Delta)+d_3(b,T,\Delta)]$$

holds, where

$$d_1(b) = \int_{-\infty}^{\infty} e^{i(\tau_1-\tau_2)\lambda}|H^*(\lambda)|^2[1-I_{[-b/2,b/2]}(\lambda)]\,d\lambda,$$

$$d_2(b,T,\Delta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\tau_1-\tau_2)\lambda_2}|H^*(\lambda_2)|^2 I_{[-b/2,b/2]}(\lambda_2)\Big[1- \\ -\left(\frac{2\pi}{c}\right)^2 f_\Delta(\lambda_1)f_\Delta(\lambda_2)\Big]\Phi_T(\lambda_2-\lambda_1)\,d\lambda_1 d\lambda_2,$$

$$d_3(b,T,\Delta) = \left(\frac{2\pi}{c}\right)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\tau_1-\tau_2)\lambda_2}|H^*(\lambda_2)|^2[I_{[-b/2,b/2]}(\lambda_2)- \\ -1]f_\Delta(\lambda_1)f_\Delta(\lambda_2)\Phi_T(\lambda_2-\lambda_1)\,d\lambda_1 d\lambda_2$$

and $I_{[-b/2,b/2]}$ denotes the indicator of $[-b/2,b/2]$.

By the inequality $|d_1(b)|\le \int_{|\lambda|>b/2}|H^*(\lambda)|^2\,d\lambda$, one has

$$d_1(b)\to 0 \quad as \quad b\to\infty, \tag{12}$$

since $|H^*|^2\in L_1(\mathbf{R})$.

Since $\|\Phi_T\|_1 = 1$, the following inequality holds for any $b>0,\ T>0$ and $\Delta>0$:

$$|d_2(b,T,\Delta)|\le B_1(T,\Delta)+B_2(T,\Delta).$$

Here,

$$B_1(T,\Delta) = \Big|\int\int_{D_b} e^{i(\tau_1-\tau_2)\lambda_2}|H^*(\lambda_2)|^2 I_{[-b/2,b/2]}(\lambda_2)\Big[1- \\ -\left(\frac{2\pi}{c}\right)^2 f_\Delta(\lambda_1)f_\Delta(\lambda_2)\Big]\Phi_T(\lambda_2-\lambda_1)\,d\lambda_1 d\lambda_2\Big|,$$

$$B_2(T,\Delta) = \Big|\int\int_{\mathbf{R}^2\setminus D_b} e^{i(\tau_1-\tau_2)\lambda_2}|H^*(\lambda_2)|^2 I_{[-b/2,b/2]}(\lambda_2)\Big[1- \\ -\left(\frac{2\pi}{c}\right)^2 f_\Delta(\lambda_1)f_\Delta(\lambda_2)\Big]\Phi_T(\lambda_2-\lambda_1)\,d\lambda_1 d\lambda_2\Big|$$

and $D_b = [-b,b]\times[-b,b]$.

Since

$$B_1(T,\Delta) \le \sup_{(\lambda_1,\lambda_2)\in D_b} \left| 1 - \left(\frac{2\pi}{c}\right)^2 f_\Delta(\lambda_1) f_\Delta(\lambda_2) \right| \|H^*\|_2^2$$

and

$$\sup_{(\lambda_1,\lambda_2)\in D_b} \left| 1 - \left(\frac{2\pi}{c}\right)^2 f_\Delta(\lambda_1) f_\Delta(\lambda_2) \right| \le \tag{13}$$

$$\le \left[1 + \frac{2\pi}{c} \sup_{\Delta>0} |f_\Delta\|_\infty \right] \sup_{-b\le\lambda\le b} \left| 1 - \frac{2\pi}{c} f_\Delta(\lambda) \right|,$$

one has, by (2d), that $B_1(T,\Delta) \to 0$ as $(T,\Delta) \to \infty$.

For fixed $b > 0$, put $\Pi(b/2) = \{(\lambda_1,\lambda_2) \in \mathbf{R}^2 : |\lambda_2 - \lambda_1| \le b/2\}$. Since

$$B_2(T,\Delta) \le \int\int_{\mathbf{R}^2\setminus\Pi(b/2)} |H^*(\lambda_2)|^2 I_{[-b/2,b/2]}(\lambda_2) \Big| 1 -$$

$$- \left(\frac{2\pi}{c}\right)^2 f_\Delta(\lambda_1) f_\Delta(\lambda_2) \Big| \Phi_T(\lambda_2 - \lambda_1)\, d\lambda_1 d\lambda_2 \le$$

$$\le \left[1 + \left(\frac{2\pi}{c} \sup_{\Delta>0} |f_\Delta\|_\infty \right)^2 \right] \|H^*\|_2^2 \int\limits_{|\lambda|>b/2} \Phi_T(\lambda)\, d\lambda,$$

and, for any $b > 0$,

$$\lim_{T\to\infty} \int_{|\lambda|>b/2} \Phi_T(\lambda)\, d\lambda = 0,$$

we have $B_2(T,\Delta) \to 0$ as $(T,\Delta) \to \infty$. Hence,

$$\lim_{(T,\Delta)\to\infty} d_2(T,\Delta) = 0 \tag{14}$$

since $|d_2| \le B_1(T,\Delta) + B_2(T,\Delta)$.

For any $b > 0,\ T > 0$, and $\Delta > 0$, one can obtain that

$$d_3(b,T,\Delta) \le \left(\frac{2\pi}{c} \sup_{\Delta>0} |f_\Delta\|_\infty \right)^2 \int_{|\lambda|>b/2} |H^*(\lambda)|^2\, d\lambda.$$

Since the inequality above is uniform in $T > 0$, $\Delta > 0$, and $|H^*|^2 \in L_1(\mathbf{R})$, we have

$$\sup_{T,\Delta>0} |d_3(b,T,\Delta)| \to 0 \quad as \quad b \to \infty. \tag{15}$$

By formulas (12), (14), and (15),

$$\limsup_{(T,\Delta)\to\infty} |C_\infty^{(1)}(\tau_1,\tau_2) - \widehat{C}_{T,\Delta}^{(1)}(\tau_1,\tau_2)| \le$$

$$\le \frac{1}{2\pi} \Big[\limsup_{b\to\infty} |d_1(b)| + \limsup_{b\to\infty} \left(\limsup_{(T,\Delta)\to\infty} |d_2(b,T,\Delta)| \right) +$$

$$+ \limsup_{b\to\infty} \left(\limsup_{(T,\Delta)\to\infty} |d_3(b,T,\Delta)| \right) \Big] = 0.$$

Thus, formula (9) holds.

Step 2. Consider the space $C_0(\mathbf{R})$ of all complex-valued continuous functions with compact support defined on $\mathbf{R}$. This means that if $h \in C_0(\mathbf{R})$, then h is continuous on $\mathbf{R}$, and there exists a positive number $a_0(h)$ such that $h(\lambda) = 0$ for $|\lambda| > a_0(h)$. Note that any $h \in C_0(\mathbf{R})$ is a uniformly continuous function.

In the integral representation of the correlation function of $A_{T,\Delta}$ (see (7)), the functions $\big(H^*(\lambda)e^{i\tau_k\lambda},\ \lambda \in \mathbf{R}\big) \in L_2(\mathbf{R})$, $k = 1,2$, appear. Since $C_0(\mathbf{R})$ is dense in $L_2(\mathbf{R})$, for any $\varepsilon > 0$ and all $\lambda \in \mathbf{R}$, we can choose functions $h_k^\varepsilon \in C_0(\mathbf{R})$, $k = 1, 2$, such that $\|H^*(\lambda)e^{i\tau_k\lambda} - h_k^\varepsilon(\lambda)\|_2 < \varepsilon$.

For all $\tau_1,\ \tau_2 \geq 0,\ T > 0,\ \Delta > 0$ and any $\varepsilon > 0$, the equality

$$C_\infty^{(2)}(\tau_1,\tau_2) - C_{T,\Delta}^{(2)}(\tau_1,\tau_2) = \frac{1}{2\pi}[d_1(\varepsilon) + d_2(\varepsilon,T,\Delta) + d_3(\varepsilon,T,\Delta)]$$

holds, where

$$d_1(\varepsilon) = \int\limits_{-\infty}^{\infty} [e^{i(\tau_1+\tau_2)\lambda}(H^*(\lambda))^2 - h_1^\varepsilon(\lambda)h_2^\varepsilon(\lambda)]\, d\lambda,$$

$$d_2(\varepsilon,T,\Delta) =$$

$$= \int\limits_{-\infty}^{\infty} h_1^\varepsilon(\lambda)h_2^\varepsilon(\lambda)\, d\lambda - \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} h_1^\varepsilon(\lambda_1)h_2^\varepsilon(\lambda_2)\left[\left(\frac{2\pi}{c}\right)^2 f_\Delta(\lambda_1)f_\Delta(\lambda_2)\right]\Phi_T(\lambda_2-\lambda_1)\, d\lambda_1 d\lambda_2,$$

$$d_3(\varepsilon,T,\Delta) =$$

$$= \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} [h_1^\varepsilon(\lambda_1)h_2^\varepsilon(\lambda_2) - e^{i(\tau_1\lambda_1+\tau_2\lambda_2)}H^*(\lambda_1)H^*(\lambda_2)]\times$$

$$\times\left[\left(\frac{2\pi}{c}\right)^2 f_\Delta(\lambda_1)f_\Delta(\lambda_2)\right]\Phi_T(\lambda_2-\lambda_1)\, d\lambda_1 d\lambda_2.$$

The Cauchy–Schwarz inequality implies that $|d_1(\varepsilon)| < \varepsilon\,[\,2\|H^*\|_2 + \varepsilon\,]$, hence

(16) $$d_1(\varepsilon) \to 0 \quad as \quad \varepsilon \to \infty.$$

By the Young inequality for convolution [16] applied to d_3, we have

$$d_3(\varepsilon,T,\Delta) \leq \left(\frac{2\pi}{c}\right)^2 \|f_\Delta\|_\infty^2\, \varepsilon\,[\,2\|H^*\|_2 + \varepsilon\,]\,.$$

Since, for any $\varepsilon > 0$, the inequality above is uniform in $T > 0$, $\Delta > 0$, one has

(17) $$\sup_{T,\Delta>0} |d_3(\varepsilon,T,\Delta)| \to 0 \quad as \quad \varepsilon \to \infty.$$

Now, we consider the value of d_2 and note that, for any $\varepsilon > 0$, $T > 0$, and $\Delta > 0$, the inequality

$$|d_2(\varepsilon,T,\Delta)| \leq E_1(\varepsilon,T) + E_2(\varepsilon,T,\Delta)$$

holds, where

$$E_1(\varepsilon,T) = \left|\int\limits_{-\infty}^{\infty} h_1^\varepsilon(\lambda)h_2^\varepsilon(\lambda)\, d\lambda - \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} h_1^\varepsilon(\lambda_1)h_2^\varepsilon(\lambda_2)\Phi_T(\lambda_2-\lambda_1)\, d\lambda_1 d\lambda_2\right|$$

and

$$E_2(\varepsilon,T,\Delta) = \left|\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} h_1^\varepsilon(\lambda_1)h_2^\varepsilon(\lambda_2)\left[1 - \left(\frac{2\pi}{c}\right)^2 f_\Delta(\lambda_1)f_\Delta(\lambda_2)\right]\Phi_T(\lambda_2-\lambda_1)\, d\lambda_1 d\lambda_2\right|.$$

For fixed $\varepsilon > 0$, define the set $\Pi(\varepsilon) = \{(\lambda_1,\lambda_2) \in \mathbf{R}^2 : |\lambda_2 - \lambda_1| \leq \varepsilon\}$. From the inequality

$$E_1(\varepsilon,T) \leq \int\int\limits_{\Pi(\varepsilon)} |h_1^\varepsilon(\lambda_1)||h_2^\varepsilon(\lambda_1) - h_2^\varepsilon(\lambda_2)|\Phi_T(\lambda_2-\lambda_1)\, d\lambda_1 d\lambda_2 +$$

$$+ \int\int\limits_{\mathbf{R}^2\setminus\Pi(\varepsilon)} |h_1^\varepsilon(\lambda_1)||h_2^\varepsilon(\lambda_1) - h_2^\varepsilon(\lambda_2)|\Phi_T(\lambda_2-\lambda_1)\, d\lambda_1 d\lambda_2 \leq$$

$$\le \|h_1^\varepsilon\|_1 \left[\max_{(\lambda_1,\lambda_1)\in\Pi(\varepsilon)} |h_2^\varepsilon(\lambda_1) - h_2^\varepsilon(\lambda_2)| + 2\|h_2^\varepsilon\|_\infty \int\limits_{|\lambda|>\varepsilon} \Phi_T(\lambda)\,d\lambda \right],$$

it follows that

$$\limsup_{(T,\Delta)\to\infty} E_1(\varepsilon,T) = \limsup_{\varepsilon\to\infty} \left(\limsup_{(T,\Delta)\to\infty} E_1(\varepsilon,T) \right) = 0,$$

since h_2^ε is uniformly continuous, and, for any $\varepsilon > 0$, $\lim\limits_{T\to\infty} \int\limits_{|\lambda|>\varepsilon} \Phi_T(\lambda)\,d\lambda = 0 holds.$

Put $b = \max\{a_0(h_1^\varepsilon), a_0(h_2^\varepsilon)\}$. Since $\{h_1^\varepsilon,\ h_2^\varepsilon\} \subset C_0(\mathbf{R})$ and $\|\Phi_T\|_1 = 1$, the following inequality holds for all $\varepsilon > 0,\ T > 0,$ and $\Delta > 0:$

$$E_2(\varepsilon,T,\Delta) \le \int\int\limits_{D_b} |h_1^\varepsilon(\lambda_1)||h_2^\varepsilon(\lambda_2)| \left| 1 - \left(\frac{2\pi}{c}\right)^2 f_\Delta(\lambda_1) f_\Delta(\lambda_2) \right| \Phi_T(\lambda_2-\lambda_1)\,d\lambda_1 d\lambda_2 +$$

$$+ \int\int\limits_{\mathbf{R}^2\setminus\Pi(b/2)} |h_1^\varepsilon(\lambda_1)||h_2^\varepsilon(\lambda_2)| \left| 1 - \left(\frac{2\pi}{c}\right)^2 f_\Delta(\lambda_1) f_\Delta(\lambda_2) \right| \Phi_T(\lambda_2-\lambda_1)\,d\lambda_1 d\lambda_2 \le$$

$$\le \sup_{(\lambda_1,\lambda_2)\in D_b} \left| 1 - \left(\frac{2\pi}{c}\right)^2 f_\Delta(\lambda_1) f_\Delta(\lambda_2) \right| \|h_1^\varepsilon\|_2 \|h_2^\varepsilon\|_2 +$$

$$+ \left[1 + \left(\frac{2\pi}{c} \sup_{\Delta>0} \|f_\Delta\|_\infty \right)^2 \right] \|h_2^\varepsilon\|_\infty \|h_1^\varepsilon\|_1 \int\limits_{|\lambda|>b/2} \Phi_T(\lambda)\,d\lambda.$$

By (13), the equality

$$\limsup_{(T,\Delta)\to\infty} E_2(\varepsilon,T,\Delta) = 0$$

holds for any $\varepsilon > 0$, since, for any $b > 0$,

$$\lim_{T\to\infty} \int\limits_{|\lambda|>b/2} \Phi_T(\lambda)\,d\lambda = 0.$$

Thus, the inequality $|d_2| \le E_1(\varepsilon,T) + E_2(\varepsilon,T,\Delta)$ yields

$$\limsup_{(T,\Delta)\to\infty} d_2(\varepsilon,T,\Delta) = 0. \tag{18}$$

By formulas (16)-(18), we have, for $\tau_1,\tau_2 \ge 0$,

$$\limsup_{(T,\Delta)\to\infty} |C_\infty^{(2)}(\tau_1,\tau_2) - \widehat{C}_{T,\Delta}^{(2)}(\tau_1,\tau_2)| \le$$

$$\le \frac{1}{2\pi} \Big[\limsup_{\varepsilon\to 0} |d_1(\varepsilon)| + \limsup_{\varepsilon\to 0} \left(\limsup_{(T,\Delta)\to\infty} |d_2(\varepsilon,T,\Delta)| \right) +$$

$$+ \limsup_{\varepsilon\to 0} \left(\limsup_{(T,\Delta)\to\infty} |d_3(\varepsilon,T,\Delta)| \right) \Big] = 0.$$

Thus, the formula (10) holds true.

Step 3. Observe that, for all $\tau_1,\tau_2 \ge 0$, $T > 0$, and $\Delta > 0$, the equality

$$C_\infty^{(3)}(\tau_1,\tau_2) - \widehat{C}_{T,\Delta}^{(3)}(\tau_1,\tau_2) =$$

$$= \frac{2\pi}{c} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} e^{i(\tau_1-\tau_2)\lambda_2} \left[\frac{c}{2\pi} - f_\Delta(\lambda_1) \right] g(\lambda_2) \Phi_T(\lambda_2-\lambda_1)\,d\lambda_1 d\lambda_2$$

holds, since Φ_T is an even function, and $\|\Phi_T\|_1 = 1$.

For any $b>0$, we define the set $P(b,b)=\{(\lambda_1,\lambda_2)\in\mathbf{R}^2:\,|\lambda_2-\lambda_1|\le b\}$. Since

$$J_1(b,T,\Delta)=\left|\int\int\limits_{P(b,b)} e^{i(\tau_1-\tau_2)\lambda_2}\left[\frac{c}{2\pi}-f_\Delta(\lambda_1)\right]g(\lambda_2)\Phi_T(\lambda_2-\lambda_1)\,d\lambda_1 d\lambda_2\right|\le$$

$$\le\sup_{-b\le\lambda\le b}\left|\frac{c}{2\pi}-f_\Delta(\lambda)\right|\|g\|_1,$$

we have, by (2d) and (4b):

$$\limsup_{(T,\Delta)\to\infty} J_1(b,T,\Delta)=0. \tag{19}$$

For all $\tau_1,\tau_2\ge 0$, $T>0$, $\Delta>0$, and given $b>0$, consider the relation

$$J_2(b,T,\Delta)=\left|\int\int\limits_{\mathbf{R}^2\setminus P(b,b)} e^{i(\tau_1-\tau_2)\lambda_2}\left[\frac{c}{2\pi}-f_\Delta(\lambda_1)\right]g(\lambda_2)\Phi_T(\lambda_2-\lambda_1)\,d\lambda_1 d\lambda_2\right|\le$$

$$\le\left[\frac{c}{2\pi}+\sup_{\Delta>0}\|f_\Delta\|_\infty\right]\|g\|_1\int\limits_{|\lambda|>b}\Phi_T(\lambda)\,d\lambda.$$

Since, for any $b>0$,

$$\lim_{T\to\infty}\int\limits_{|\lambda|>b}\Phi_T(\lambda)\,d\lambda=0,$$

the inequality above yields, by (2b) and (4b),

$$\limsup_{(T,\Delta)\to\infty} J_2(b,T,\Delta)=\limsup_{b\to\infty}\left(\limsup_{(T,\Delta)\to\infty} J_2(b,T,\Delta)\right)=0. \tag{20}$$

From formulas (19) and (20), one can obtain for all $\tau_1,\tau_2\ge 0$ that

$$\limsup_{(T,\Delta)\to\infty}|C^{(3)}_\infty(\tau_1,\tau_2)-\widehat{C}^{(3)}_{T,\Delta}(\tau_1,\tau_2)|\le$$

$$\le\frac{2\pi}{c^2}\left[\limsup_{b\to\infty}\left(\limsup_{(T,\Delta)\to\infty} J_1(b,T,\Delta)\right)+\limsup_{b\to\infty}\left(\limsup_{(T,\Delta)\to\infty} J_1(b,T,\Delta)\right)\right]=0.$$

Thus, formula (11) holds true.

Summarizing, we have, for all $\tau_1,\tau_2\ge 0$:

$$\lim_{(T,\Delta)\to\infty}\widehat{C}_{T,\Delta}(\tau_1,\tau_2)=\sum_{j=1}^{3}\lim_{(T,\Delta)\to\infty}\widehat{C}^{(j)}_{T,\Delta}(\tau_1,\tau_2)=\sum_{j=1}^{3}C^{(j)}_\infty(\tau_1,\tau_2)=C_\infty(\tau_1,\tau_2).$$

Theorem 3.1 is proved. □

4. Asymptotic normality of finite-dimensional distributions of $A_{T,\Delta}$

Theorem 3.1 demonstrates that the function C_∞ defined in (8) is positive semidefinite on $[0,\infty)\times[0,\infty)$. So, there exists a zero-mean real-valued Gaussian process $(A(\tau),\tau\ge 0)$ with a correlation function C_∞; that is,

$$\mathbf{E}A(\tau_1)A(\tau_2)=C_\infty(\tau_1,\tau_2).$$

Without loss of generality, we assume that the process A is defined on the same probability space as the processes $A_{T,\Delta}$.

Theorem 4.1. *Assume that $H\in L_2(\mathbf{R})$ and $g\in L_1(\mathbf{R})$. Then the equality*

$$\lim_{(T,\Delta)\to\infty}\boldsymbol{E}\left[\prod_{j=1}^{m}A_{T,\Delta}(\tau_j)\right]=\boldsymbol{E}\left[\prod_{j=1}^{m}A(\tau_j)\right] \tag{21}$$

holds for any $m\in\mathbb{N}$ and any τ_1, ..., $\tau_m\ge 0$.

In particular, all finite-dimensional distributions of the process $(A_{T,\Delta}(\tau), \tau \geq 0)$ converge weakly to the corresponding finite-dimensional distributions of the Gaussian process $(A(\tau), \tau \geq 0)$.

Remark 4.1. Theorem 4.1 refines results of [9] (see Theorems 1 and 2).

Proof. To prove Theorem 4.1, we apply the Brillinger's method of cumulants [5] reinforced by the results concerning the integrals involving the cyclic products of kernels [13].

Let $m \in \{1, 2\}$. The statement of Theorem 4.1 is true, since the processes $A_{T,\Delta}$ and A are centered, and the conditions of Theorem 3.1 are satisfied.

Let $m \in \mathbb{N} \setminus \{1, 2\}$; $\tau_j \geq 0$, $j = 1, ..., m$, and let

$$\mathrm{cum}_{A_{T,\Delta}}(\tau_1, ..., \tau_m) = \mathrm{cum}(A_{T,\Delta}(\tau_1), ..., A_{T,\Delta}(\tau_m))$$

be a *joint simple cumulant* of the family of random variables $A_{T,\Delta}(\tau_1), ..., A_{T,\Delta}(\tau_m)$. Since the moments of a random vector are uniquely determined by its cumulants [5], and because the Gaussian distribution is uniquely determined by its mean and correlation matrix, we need to show that

$$\lim_{(T,\Delta)\to\infty} \mathrm{cum}_{A_{T,\Delta}}(\tau_1, ..., \tau_m) = 0. \tag{22}$$

General properties of cumulants and definitions of the process $A_{T,\Delta}$ and the estimator $\widehat{H}_{T,\Delta}$ (see (6) and (5), respectively) imply that

$$\mathrm{cum}_{A_{T,\Delta}}(\tau_1, ..., \tau_m) = \tag{23}$$

$$= \left(\frac{1}{c^2 T}\right)^{\frac{m}{2}} \int_0^T ... \int_0^T \mathrm{cum}(U_\Delta(t_j + \tau_j) X_\Delta(t_j),\ j = 1, ..., m)\, dt_1 ... dt_m.$$

Let us apply Theorem 2.3.2 in [5] to the integrand in (23). Because X_Δ and U_Δ are zero-mean jointly Gaussian processes, we obtain

$$\mathrm{cum}(U_\Delta(t_j + \tau_j) X_\Delta(t_j),\ j = 1, ..., m) = \sum \prod_{p=1}^{m} \mathrm{cum}(D_p^{(2)}), \tag{24}$$

where the summation is extended over all unordered indecomposable partitions of the table

$$\mathrm{D}_{m\times 2} = \begin{bmatrix} U_\Delta(t_1 + \tau_1) & X_\Delta(t_1) \\ U_\Delta(t_2 + \tau_2) & X_\Delta(t_2) \\ \vdots & \vdots \\ U_\Delta(t_m + \tau_m) & X_\Delta(t_m) \end{bmatrix}$$

into the pairs $\{D_1^{(2)}, ..., D_m^{(2)}\}$.

Since the order of elements in the partition $\{D_1^{(2)}, ..., D_m^{(2)}\}$ is of no importance, we can always assume that this partition satisfies the following conditions (see [13]): (1) $D_p^{(2)} \bigcap D_q^{(2)} = \varnothing$ for $p \neq q$; (2) $\mathrm{D}_{m\times 2} = D_1^{(2)} \bigcup ... \bigcup D_m^{(2)}$; (3) if $m \geq 3$, then, for any p, the set $D_p^{(2)}$ does not coincide with any of the rows $\mathrm{R}_1, ..., \mathrm{R}_m$ of the table $\mathrm{D}_{m\times 2}$; if, moreover, $1 \leq \nu < m$, then the union of any set of elements in the partition $\{D_1^{(2)}, ..., D_m^{(2)}\}$ should not coincide with the union of any ν rows of the table $\mathrm{D}_{m\times 2}$; (4) for any $p = 1, ..., m-1$, there exists exactly one row $\widetilde{\mathrm{R}}_p$ of the table $\mathrm{D}_{m\times 2}$ such that $\widetilde{\mathrm{R}}_p \subset D_p^{(2)} \bigcup D_{p+1}^{(2)}$; (5) $\mathrm{R}_1 \subset D_1^{(2)} \bigcup D_m^{(2)}$. This means that the elements of the unordered indecomposable partition $\{D_1^{(2)}, ..., D_m^{(2)}\}$ hook each other sequentially and exhaust all the table $\mathrm{D}_{m\times 2}$, and $D_m^{(2)}$ hooks back to $D_1^{(2)}$.

In what follows, for a given $D_p^{(2)}$, we write $D_p^{(2)} = D_{j,\widetilde{j}}^{(2)}$, where j and $\widetilde{j}$ are row numbers of those two rows of $\mathrm{D}_{m\times 2}$, whose elements form the set $D_p^{(2)}$. Thus, any unordered indecomposable partition $\{D_1^{(2)}, ..., D_m^{(2)}\}$ can be written as

$$\overrightarrow{D}^{(2)} = \{D^{(2)}_{j_1, j_2},\, D^{(2)}_{j_2,\, j_3}, ..., D^{(2)}_{j_{m-1},\, j_m},\, D^{(2)}_{j_m,\, j_{m+1}}\},$$

where $(j_1, ..., j_m)$ is such that $j_1 = 1$, $(j_2, j_3, ..., j_m)$ is a perturbation of $\{2, 3, ..., m\}$, and $j_{m+1} = j_1 = 1$.

The further analysis of the structure of $\overrightarrow{D}^{(2)}$ shows that we can distinguish three groups of its elements. The first group is formed by unordered sets having the form $D^{(2)}_{j,\widetilde{j}} = \{X_\Delta(t_j),\, X_\Delta(t_{\widetilde{j}})\}$, $j \neq \widetilde{j}$. The second group contains unordered sets which can be represented as follows: $D^{(2)}_{j,\widetilde{j}} = \{U_\Delta(t_j + \tau_j),\, U_\Delta(t_{\widetilde{j}} + \tau_{\widetilde{j}})\},\ j \neq \widetilde{j}$. Finally, the third group is formed by unordered sets of the form $D^{(2)}_{j,\widetilde{j}} = \{U_\Delta(t_j + \tau_j),\, X_\Delta(t_{\widetilde{j}})\},\ j \neq \widetilde{j}$. Denote these three groups by $G_1(\overrightarrow{D}^{(2)})$, $G_2(\overrightarrow{D}^{(2)})$, , and $G_3(\overrightarrow{D}^{(2)})$, respectively, and let $m_\nu(\overrightarrow{D}^{(2)})$ be the cardinality of $G_\nu(\overrightarrow{D}^{(2)})$, $\nu = 1, 2, 3$. It is clear that, for any $\overrightarrow{D}^{(2)}$,

$$m_1(\overrightarrow{D}^{(2)}) = m_2(\overrightarrow{D}^{(2)});\quad \sum_{\nu=1}^{3} m_\nu(\overrightarrow{D}^{(2)}) = m. \tag{25}$$

Further, if $D^{(2)}_{j,\widetilde{j}} \in G_1(\overrightarrow{D}^{(2)})$, then

$$\mathrm{cum}(D^{(2)}_{j,\widetilde{j}}) = \mathbf{E} X_\Delta(t_j) X_\Delta(t_{\widetilde{j}}) = K_{X_\Delta}(t_j - t_{\widetilde{j}}) = \int_{-\infty}^{\infty} e^{i(t_j - t_{\widetilde{j}})\lambda_j} f_\Delta(\lambda_j)\, d\lambda_j;$$

if $D^{(2)}_{j,\widetilde{j}} \in G_2(\overrightarrow{D}^{(2)})$, then

$$\mathrm{cum}(D^{(2)}_{j,\widetilde{j}}) = \mathbf{E} U_\Delta(t_j + \tau_j) U_\Delta(t_{\widetilde{j}} + \tau_{\widetilde{j}}) =$$

$$= \int_{-\infty}^{\infty} e^{i(t_j - t_{\widetilde{j}})\lambda_j} \cdot e^{i(\tau_j - \tau_{\widetilde{j}})\lambda_j} \left(|H^*(\lambda_j)|^2 f_\Delta(\lambda_j) + g(\lambda_j)\right)\, d\lambda_j;$$

if $D^{(2)}_{j,\widetilde{j}} \in G_3(\overrightarrow{D}^{(2)})$, then

$$\mathrm{cum}(D^{(2)}_{j,\widetilde{j}}) = \mathbf{E} U_\Delta(t_j + \tau_j) X_\Delta(t_{\widetilde{j}}) = \int_{-\infty}^{\infty} e^{i(t_j - t_{\widetilde{j}})\lambda_j} \cdot e^{i\tau_j \lambda_j} H^*(\lambda_j) f_\Delta(\lambda_j)\, d\lambda_j.$$

These formulas imply that

$$\prod_{p=1}^{m} \mathrm{cum}(D_p^{(2)}) = \tag{26}$$

$$= \int \cdots \int_{\mathbf{R^m}} \left[\prod_{k=1}^{m} e^{i(t_{j_k} - t_{j_{k+1}})\lambda_{j_k}}\right] \varphi_0(\overrightarrow{\lambda}, \overrightarrow{\tau}, \overrightarrow{D}^{(2)}) \left(\prod_{k=1}^{m} \varphi_{j_k}(\lambda_{j_{k+1}}, \Delta, \overrightarrow{D}^{(2)})\right) d\lambda_{j_1} ... d\lambda_{j_m}.$$

Here, $\overrightarrow{\lambda} = (\lambda_1, ..., \lambda_m)$; $\overrightarrow{\tau} = (\tau_1, ..., \tau_m)$; $(j_1, ..., j_m)$ is such that $j_1 = 1$, $(j_2, ..., j_m)$ is a permutation of $\{2, ..., m\}$, and $j_{m+1} = j_1 = 1$. For every partition $\overrightarrow{D}^{(2)}$, the function $\varphi_0(\cdot, \overrightarrow{D}^{(2)})$ is a product of some of the functions $e^{i(\tau_{j_k} - \tau_{j_{k+1}})\lambda_{j_k}}$, $e^{i\tau_{j_k}\lambda_{j_k}}$ and the indicator functions $I_{\mathbf{R}}(\lambda_{j_k})$. Therefore,

$$\sup_{\overrightarrow{D}^{(2)}} \sup_{\overrightarrow{\lambda}, \overrightarrow{\tau}} |\varphi_0(\overrightarrow{\lambda}, \overrightarrow{\tau}, \overrightarrow{D}^{(2)})| = 1; \tag{27}$$

the function $\varphi_{j_k}(\cdot, \Delta, \overrightarrow{D}^{(2)})$ is one of the following functions: f_Δ, $|H^*| f_\Delta + g_\Delta$, or $H^* f_\Delta$.

Note that, for any $\Delta > 0$ and a partition $\overrightarrow{D}^{(2)}$, the set of functions $F(\Delta, \overrightarrow{D}^{(2)}) = \{\varphi_1(\cdot, \Delta, \overrightarrow{D}^{(2)}), ..., \varphi_m(\cdot, \Delta, \overrightarrow{D}^{(2)})\}$ is divided into three classes

$$M_\infty(\overrightarrow{D}^{(2)}) = \{\varphi \in F(\overrightarrow{D}^{(2)}) :\ \varphi = f_\Delta\},$$

$$M_1(\overrightarrow{D}^{(2)}) = \{\varphi \in F(\overrightarrow{D}^{(2)}) : \ \varphi = |H^*| f_\Delta + g\},$$

$$M_2(\overrightarrow{D}^{(2)}) = \{\varphi \in F(\overrightarrow{D}^{(2)}) : \ \varphi = |H^*| f_\Delta\},$$

which satisfy the relations (see (25))

$$\text{card}[M_1(\overrightarrow{D}^{(2)})] = \text{card}[M_\infty(\overrightarrow{D}^{(2)})], \tag{28}$$

$$\text{card}[M_1(\overrightarrow{D}^{(2)})] + \text{card}[M_2(\overrightarrow{D}^{(2)})] + \text{card}[M_\infty(\overrightarrow{D}^{(2)})] = m.$$

Conditions $H \in L_2(\mathbf{R})$, (2b), and (4b) imply that

$$M_\infty(\overrightarrow{D}^{(2)}) \subset L_\infty(\mathbf{R}), \quad M_1(\overrightarrow{D}^{(2)}) \subset L_1(\mathbf{R}), \quad M_2(\overrightarrow{D}^{(2)}) \subset L_2(\mathbf{R}). \tag{29}$$

Moreover, these embeddings are uniform in the parameter $\Delta > 0$.

By some algebra, from (23), (24), and (26), we obtain

$$\text{cum}_{A_{T,\Delta}}(\tau_1, ..., \tau_m) = \tag{30}$$

$$= \left(\frac{2\pi}{c^2}\right)^{\frac{m}{2}} \sum_{\overrightarrow{D}^{(2)}} \int \cdots \int_{\mathbf{R^m}} \left[\prod_{k=1}^{m} \widehat{\Phi}^{(T)}(\lambda_{k+1} - \lambda_k)\right] \varphi_0(\overrightarrow{\lambda}, \overrightarrow{\tau}, \overrightarrow{D}^{(2)}) \times$$

$$\times \left(\prod_{k=1}^{m} \varphi_k(\lambda_k, \Delta, \overrightarrow{D}^{(2)})\right) d\lambda_1 ... d\lambda_m,$$

where $\lambda_{m+1} = \lambda_1$, and $\widehat{\Phi}^{(T)}(\lambda) = \left(\frac{1}{2\pi T}\right)^{\frac{1}{2}} \frac{e^{iT\lambda} - 1}{i\lambda}, \ \lambda \in \mathbf{R}$.

Formula (30) shows that $\text{cum}_{A_{T,\Delta}}(\tau_1, ..., \tau_m)$ can be represented as a finite sum of integrals involving the cyclic products of kernels (see definitions in [8] or [13]). All kernels are equal to $\widehat{\Phi}^{(T)}$. They depend on the parameter $T > 0$ and are independent of the parameter $\Delta > 0$.

By virtue of (27), for any $m \in \mathbb{N} \setminus \{1, 2\}$ and any numbers $\tau_j \geq 0$, $j = 1, ..., m$, from (30), we obtain the bound

$$|\text{cum}_{A_{T,\Delta}}(\tau_1, ..., \tau_m)| \leq \tag{31}$$

$$\leq \left(\frac{2\pi}{c^2}\right)^{\frac{m}{2}} \sum_{\overrightarrow{D}^{(2)}} \int \cdots \int_{\mathbf{R^m}} \left|\prod_{k=1}^{m} \widehat{\Phi}^{(T)}(\lambda_{k+1} - \lambda_k)\right| \sup_{\Delta > 0} \left|\prod_{k=1}^{m} \varphi_k(\lambda_k, \Delta, \overrightarrow{D}^{(2)})\right| d\lambda_1 ... d\lambda_m,$$

where $\lambda_{m+1} = \lambda_1$.

Fix $\overrightarrow{D}^{(2)}$. Define

$$\mathrm{I}^{(T)}(\overrightarrow{D}^{(2)}) =$$

$$= \int \cdots \int_{\mathbf{R^m}} \prod_{k=1}^{m} \left|\widehat{\Phi}^{(T)}(\lambda_{k+1} - \lambda_k)\right| \prod_{k=1}^{m} \left|\varphi_k(\lambda_k, \Delta, \overrightarrow{D}^{(2)})\right| d\lambda_1 ... d\lambda_m, \quad \lambda_{m+1} = \lambda_1.$$

The next step of the proof is based on the following *analog of the Young inequality for the integrals involving the cyclic products of kernels* (see, Theorem 5.2 [13]):

$$\mathrm{I}^{(T)}(\overrightarrow{D}^{(2)}) \leq \prod_{k=1}^{m} \left[\left\|\widehat{\Phi}^{(T)}\right\|_{p_k}^{n_k} \prod_{\varphi_j \in M_{q_k}} \|\varphi_j\|_{q_k}\right].$$

Here, $n \in \mathbb{N} \setminus \{1, 2\}$; for all n, the set of functions $M = \{\varphi_1, ..., \varphi_n\}$ becomes the union of some disjoint sets $M_{q_1}, ..., M_{q_m}$, where $M_{q_k} = \{\varphi \in M : \ \varphi \in L_{q_k}(\mathbf{R})\}$, $n_k = \text{card}(M_{q_k})$, $k = 1, ..., m$, and $n_1 \geq 2$; $1 \leq q_1 \leq q_2 \leq ... \leq q_m \leq \infty$, and $q_1 \leq 2$. We suppose also that $\widehat{\Phi}^{(T)} \in \bigcap_{k=1}^{m} L_{p_k}(\mathbf{R})$, where $p_1, ..., p_m$ are the conjugate numbers of $q_1, ..., q_m$, respectively.

To show the convergence of $\mathrm{I}^{(T)}(\overrightarrow{D}^{(2)})$ to zero, we use a special case of the above-stated inequality, namely Theorem 5.3, Part B [13]. Since $m \in \mathbb{N} \setminus \{1, 2\}$ and

(i) for any $p \in (1, \infty]$, all of kernels $\widehat{\Phi}^{(T)}$ satisfy the *majorant condition*

$$\|\widehat{\Phi}^{(T)}\|_p \le T^{\frac{1}{2}-\frac{1}{p}} C(p),$$

where $C(p) = \frac{1}{\sqrt{2\pi}} \left\| \frac{\sin(\lambda/2)}{\lambda/2} \right\|_p$ is a positive constant independent of $T > 0$;

(ii) conditions (28)-(29) hold true,

we have

$$\mathrm{I}^{(T)}(\overrightarrow{D}^{(2)}) \to 0 \quad as \quad T \to \infty. \tag{32}$$

Moreover, $\lim\limits_{T\to\infty} \left[\sup\limits_{\Delta>0} \mathrm{I}^{(T)}(\overrightarrow{D}^{(2)}) \right] = 0$. Because the sum in (31) contains a finite number of terms which satisfy (32) uniformly in the parameter $\Delta > 0$, then (22) holds true.

Thus, Theorem 4.1 is proved. □

5. Asymptotic normality of $A_{T,\Delta}$ in the space of continuous functions

In addition to Theorem 4.1, it is natural to study the asymptotic normality of our centered estimator (see (6)) in the space of continuous functions. Assume that $A_{T,\Delta}$, $T > 0$, $\Delta > 0$, and A are separable processes. We use the notation $C[0, a]$, $a > 0$, for the space of real-valued continuous functions defined on $[0, a]$ endowed with uniform norm. In what follows, we write $A_{T,\Delta} \overset{C[0,a]}{\Longrightarrow} A$ to denote the weak convergence of the process $A_{T,\Delta}$ to the process A in the space $C[0, a]$ as $(T, \Delta) \to \infty$.

Now we recall some tools related to Gaussian stochastic processes (see, for example, [10]). Let S be a parameter set. A function $\rho(t, s)$, $t, s \in S$, is called pseudometric on S, if it satisfies all axioms of a metric, with the exception for that the set $\{(t, s) \in S \times S : \rho(t, s) = 0\}$ may be wider than the diagonal $\{(t, s) \in S \times S : \ t = s\}$. We write $N_\rho(S, \varepsilon)$ for the minimal number of closed ρ-balls of radius $\varepsilon > 0$, whose centers lie in S and which cover S. If there is no finite covering of S, then $N_\rho(S, \varepsilon) = \infty$. Further, let, as usual, $\mathsf{H}_\rho(S, \varepsilon) = \log N_\rho(S, \varepsilon)$ be a metric entropy of the set S with respect to ρ. For any $\beta > 0$, the inequality $\int_{0+} \mathsf{H}^\beta_\rho(S, \varepsilon)\, d\varepsilon < \infty$ is always interpreted in the sense that, for some (and, hence, for all) $u > 0$, we have $\int\limits_0^u \mathsf{H}^\beta_\rho(S, \varepsilon)\, d\varepsilon < \infty$.

Consider the function

$$\sigma(\tau) = \left[\int\limits_{-\infty}^{\infty} \sin^2 \frac{\tau\lambda}{2} (|H^*(\lambda)|^2 + g(\lambda))\, d\lambda \right]^{\frac{1}{2}}, \ \tau \ge 0.$$

Since $H \in L_2(\mathbf{R})$ and $g \in L_1(\mathbf{R})$, this function is well-defined and generates the following two pseudometrics: $\sigma(\tau_1, \tau_2) = \sigma(|\tau_1 - \tau_2|)$ and $\sqrt{\sigma}(\tau_1, \tau_2) = \sqrt{\sigma(\tau_1, \tau_2)}$, $\tau_1, \tau_2 \ge 0$. Note that if $H^*(\lambda) \ne 0$ and $g(\lambda) \ne 0$ simultaneously on the set of positive Lebesgue measure, then σ and $\sqrt{\sigma}$ are metrics. For all $\varepsilon > 0$, put $\mathsf{H}_\sigma(\varepsilon) = \mathsf{H}_\sigma([0, 1], \varepsilon)$, $\mathsf{H}_{\sqrt{\sigma}}(\varepsilon) = \mathsf{H}_{\sqrt{\sigma}}([0, 1], \varepsilon)$. Since the pseudometrics σ and $\sqrt{\sigma}$ depend on $|\tau_1 - \tau_2|$ only, one has

$$\int_{0+} \mathsf{H}^\beta_\sigma(\varepsilon)\, d\varepsilon < \infty \Longleftrightarrow \int_{0+} \mathsf{H}^\beta_\sigma([0, a], \varepsilon)\, d\varepsilon < \infty;$$

$$\int_{0+} \mathsf{H}_{\sqrt{\sigma}}(\varepsilon)\, d\varepsilon < \infty \Longleftrightarrow \int_{0+} \mathsf{H}_{\sqrt{\sigma}}([0, a], \varepsilon)\, d\varepsilon < \infty,$$

for any $a > 0$ and $\beta > 0$.

In the theorem below, we state the sufficient conditions for the continuity almost surely of the processes $A_{T,\Delta}$ and A and for the weak convergence of $A_{T,\Delta}$ to A in $C[0, a]$ as $(T, \Delta) \to \infty$.

Theorem 5.1. *Assume that $H \in L_2(\mathbf{R})$, $g \in L_1(\mathbf{R})$, and*

$$\int_{0+} \mathsf{H}_{\sqrt{\sigma}}(\varepsilon)\, d\varepsilon < \infty. \tag{33}$$

Then, for any $a > 0$, the following statements hold true:

I) $A \in C[0,a]$ almost surely;

II) $A_{T,\Delta} \in C[0,a]$ almost surely;

III) $A_{T,\Delta} \overset{C[0,a]}{\Longrightarrow} A$ as $(T,\Delta) \to \infty$.

In particular, for all $x > 0$ and $a > 0$,

$$\lim_{(T,\Delta)\to\infty} P\left\{\sup_{\tau\in[0,a]} |A_{T,\Delta}(\tau)| > x\right\} = P\left\{\sup_{\tau\in[0,a]} \left|A_(\tau)\right| > x\right\}$$

Remark 5.1. Statement I) of Theorem 5.1 holds true under a weaker condition than (33), namely

$$\int_{0+} \mathsf{H}_{\sigma}^{\frac{1}{2}}(\varepsilon)\, d\varepsilon < \infty. \tag{34}$$

Note that (34) always holds if there exists $\beta > 0$ such that (see [15])

$$\int_0^\infty \left(|H^*(\lambda)|^2 + g(\lambda)\right) \log^{1+\beta}(1+\lambda)\, d\lambda < \infty.$$

Remark 5.2. Condition (33) holds if there exists $\beta > 0$ such that (see [13])

$$\int_0^\infty \left(|H^*(\lambda)|^2 + g(\lambda)\right) \log^{4+\beta}(1+\lambda)\, d\lambda < \infty.$$

To prove Theorem 5.1, we need some auxiliary statements. First of all, consider the following relation between pseudometrics σ and $\sqrt{\sigma}$. Since, for all $\tau_1, \tau_2 \geq 0$,

$$\sigma(\tau_1,\tau_2) \leq \left[\max_{\tau_1,\tau_2\leq 0} \sigma(\tau_1,\tau_2)\right]^{\frac{1}{2}} \sqrt{\sigma}(\tau_1,\tau_2) \leq \left[\|H^*\|_2^2 + \|g\|_1\right]^{\frac{1}{4}} \sqrt{\sigma}(\tau_1,\tau_2), \tag{35}$$

condition (33) yields $\int_{0+} \mathsf{H}_\sigma(\varepsilon)\, d\varepsilon < \infty$, which implies (34).

For all $T > 0$, $\Delta > 0$, we introduce a family of pseudometrics

$$\rho_{(T,\Delta)}(\tau_1,\tau_2) = \left(\mathbf{E}|A_{T,\Delta}(\tau_2) - A_{T,\Delta}(\tau_2)|^2\right)^{\frac{1}{2}}, \quad \tau_1,\tau_2 \geq 0.$$

Lemma 5.1. *Assume that $H \in L_2(\mathbf{R})$ and $g \in L_1(\mathbf{R})$. Then the inequality*

$$\rho_{(T,\Delta)}(\tau_1,\tau_2) \leq \tag{36}$$

$$\leq \frac{2\sqrt{2\pi M}}{c}\left(Q\left(\|H^*\|_2^2 + \|g\|_1\right)^{\frac{1}{2}} + M\|H^*\|_2\right)^{\frac{1}{2}} \sqrt{\sigma}(\tau_1,\tau_2), \quad \tau_1,\tau_2 \geq 0,$$

holds for all $T > 0$, $\Delta > 0$. Here, c is the constant from (2d), $M = \sup_{\Delta>0} \|f_\Delta\|_\infty$, and $Q = \max\{M,\ 1\}$. Moreover, the pseudometric $\rho_{(T,\Delta)}$ is continuous with respect to the pseudometic σ.

Proof. From (7), applying the Cauchy–Schwarz inequality, the Young inequality for convolution [16], and the fact that $\|\Phi_T\|_1 = 1$, we obtain

$$\rho^2_{(T,\Delta)}(\tau_1,\tau_2) = \mathbf{E}|A_{T,\Delta}(\tau_2) - A_{T,\Delta}(\tau_2)|^2 \leq$$

$$\leq \frac{8\pi}{c^2} MQ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|\sin\frac{(\tau_2-\tau_1)\lambda_2}{2}\right| \left(|H^*(\lambda_2)|^2 + g(\lambda_2)\right) \Phi_T(\lambda_1-\lambda_2)\, d\lambda_1 d\lambda_2 +$$

$$+ \frac{8\pi}{c^2} M^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|\sin\frac{(\tau_2-\tau_1)\lambda_2}{2}\right| |H^*(\lambda_1)||H^*(\lambda_2)| \Phi_T(\lambda_1-\lambda_2)\, d\lambda_1 d\lambda_2 \leq$$

$$\leq \frac{8\pi}{c^2} MQ \left(\|H^*\|_2^2 + \|g\|_1\right)^{\frac{1}{2}} \left[\int_{-\infty}^{\infty} \left| \sin \frac{(\tau_2 - \tau_1)\lambda}{2} \right|^2 \left(|H^*(\lambda)|^2 + g(\lambda)\right) d\lambda \right]^{\frac{1}{2}} +$$

$$+ \frac{8\pi}{c^2} M^2 \int_{-\infty}^{\infty} \left| \sin \frac{(\tau_2 - \tau_1)\lambda_2}{2} \right| |H^*(\lambda_2)| \, |H^* * \Phi_T| \, (\lambda_2) \, d\lambda_2.$$

Since g is a nonnegative function defined on $\mathbf{R}$, we have

$$\rho^2_{(T,\Delta)}(\tau_1, \tau_2) \leq \frac{8\pi}{c^2} MQ \left(\|H^*\|_2^2 + \|g\|_1\right)^{\frac{1}{2}} \left[\int_{-\infty}^{\infty} \sin^2 \frac{(\tau_2 - \tau_1)\lambda}{2} \left(|H^*(\lambda)|^2 + g(\lambda)\right) d\lambda \right]^{\frac{1}{2}} +$$

$$+ \frac{8\pi}{c^2} M^2 \|H^* * \Phi_T\|_2 \left[\int_{-\infty}^{\infty} \left| \sin \frac{(\tau_2 - \tau_1)\lambda}{2} \right|^2 |H^*(\lambda)|^2 \, d\lambda \right]^{\frac{1}{2}} \leq$$

$$\leq \frac{8\pi}{c^2} M \left(Q \left(\|H^*\|_2^2 + \|g\|_1\right)^{\frac{1}{2}} + M \|H^*\|_2 \right) \sigma(\tau_1, \tau_2).$$

The latter implies (36). Note that this inequality is uniform in $T > 0$, $\Delta > 0$. □

Proof of Theorem 5.1. By the Dudley's theorem on the continuity of Gaussian processes [15], statement I) of Theorem 5.1 holds if, for any $a > 0$,

$$\int_{0+} \mathsf{H}^{\frac{1}{2}}_{d_A}([0, a], \varepsilon) \, d\varepsilon < \infty, \tag{37}$$

where $d_A(\tau_1, \tau_2) = \left(\mathbf{E}|A(\tau_2) - A(\tau_1)|^2\right)^{\frac{1}{2}}$.

In calculations below, we use the notation $B = \max\{1, \ \frac{\pi}{c}\}$. Applying the Cauchy–Schwarz inequality to (8), we obtain, for all $\tau_1, \tau_2 \geq 0$, that

$$d_A^2(\tau_1, \tau_2) \leq \frac{4B}{\pi} \int_{-\infty}^{\infty} \left| \sin \frac{(\tau_2 - \tau_1)\lambda}{2} \right| \left(|H^*(\lambda)|^2 + g(\lambda)\right) d\lambda \leq$$

$$\leq \frac{4B}{\pi} \left(\|H^*\|_2^2 + \|g\|_1\right)^{\frac{1}{2}} \sigma(\tau_1, \tau_2).$$

Formula (37) holds true if $\int_{0+} \mathsf{H}^{\frac{1}{2}}_{\sigma}(\varepsilon) \, d\varepsilon < \infty$. The last condition follows from (33). Thus, we proved statement I); that is, the process A is continuous on $[0, a]$ almost surely.

Since the process $A_{T,\Delta}$ is quadratically Gaussian, we have, by Theorem 6.2.2 [10] for any $T > 0, \ \Delta > 0:$

$$\sup_{T,\Delta > 0} \sup_{\tau_1, \tau_2 \geq 0} \mathbf{E} \exp \left\{ \frac{|A_{T,\Delta}(\tau_2) - A_{T,\Delta}(\tau_1)|}{\sqrt{8} \, \rho_{(T,\Delta)}(\tau_1, \tau_2)} \right\} < \infty. \tag{38}$$

From the Lebesgue dominated convergence, it follows that the pseudometric $\sqrt{\sigma}$ is continuous with respect to the metric $d(\tau_1, \tau_2) = |\tau_1 - \tau_2|$. By Lemma 5.1, the pseudometric

$$\rho_\infty(\tau_1, \tau_2) = \sup_{T,\Delta > 0} \rho_{(T,\Delta)}(\tau_1, \tau_2), \quad \tau_1, \tau_2 \in [0, a], \tag{39}$$

is continuous with respect to the metric d.

By inequality (36), the condition (33) implies that, for any $a > 0$,

$$\lim_{u \downarrow 0} \sup_{T,\Delta > 0} \int_0^u \mathsf{H}^{\frac{1}{2}}_{\rho_{(T,\Delta)}}([0, a], \varepsilon) \, d\varepsilon = 0. \tag{40}$$

Relations (38)-(40) satisfy all the conditions of Lemma 4.2.1 [10]. Thus, statement II) of Theorem 5.1 holds true. Moreover, for any $\varepsilon > 0$ and any $a > 0$,

$$\lim_{h\downarrow 0} \sup_{T,\Delta>0} \mathsf{P}\left\{ \sup_{\substack{\tau_1,\tau_2\in[0,a]\\ |\tau_1-\tau_2|<h}} |A_{T,\Delta}(\tau_1) - A_{T,\Delta}(\tau_2)| > \varepsilon \right\} = 0. \tag{41}$$

Theorem 4.1 together with (41) satisfy all the conditions of the Prokhorov theorem on the weak convergence of stochastic processes in the space $C[0,a]$ (see [10]). Thus, statement III) of Theorem 5.1 holds true. □

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