Theory of Stochastic Processes
Vol. 15 (31), no. 2, 2009, pp. 54–61

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ON OPTIMAL STOPPING FOR TIME-DEPENDENT GAIN FUNCTION

General questions of optimal stopping for an inhomogeneous Markov process for a time-dependent gain function are investigated. The connection between the optimal stopping problems for an inhomogeneous standard Markov process and the corresponding homogeneous Markov process constructed in the extended state space is established. A detailed characterization of a value-function and the limit procedure for its construction in the problem of optimal stopping of an inhomogeneous Markov process is given. The form of ε-optimal (optimal) stopping times is also found.

1. Introduction

General questions of the theory of optimal stopping of a homogeneous Markov process are set forth in monograph [1]. An excessive characterization of the payoff, the methods of its construction, the form of ε-optimal and optimal stopping times are given in various restrictions on the gain function. Excessive and also supermartingale characterizations of costs present two fundamental directions in the theory of optimal stopping [3,5].

In the present work, the questions of the optimal stopping theory for an inhomogeneous (with infinite lifetime) Markov process for a time-dependent gain function and with the observation cost are studied. By extending the state space and the space of elementary events, the problems of optimal stopping for the inhomogeneous case can be reduced to the corresponding problems for homogeneous standard Markov processes, from which an excessive characterization of a value-function, the method of its construction, and the form of ε-optimal (optimal) stopping times for the initial problem are found.

It should be noted that the form of ε-optimal stopping times was established in the case of the optimal stopping of homogeneous Markov processes on a bounded time interval in papers [2,5,7] with the use of the method of state space extension.

Here, we consider an inhomogeneous (with infinite lifetime) Markov process

$$X = (\Omega, \mathcal{M}^s, \mathcal{M}_t^s, X_t, P_{s,x}), \ \ 0 \le s \le t < +\infty,$$

in the state space $(\mathsf{E}, \mathcal{B})$, i.e., it is assumed that (see [3,6])

1) E is a locally compact Hausdorff space with a countable base, $\mathcal{B}$ is the σ-algebra for Borel sets of the space;

2) for every $s \ge 0$, $x \in \mathsf{E}$, $P_{s,x}$ is a probability measure on the σ-algebra $\mathcal{M}^s$, $\mathcal{M}_t^s$, $t \ge s$, is the increasing family of sub-σ-algebras of the σ-algebra $\mathcal{M}^s$, where

$$\mathcal{M}^{s_1} \supseteq \mathcal{M}^{s_2}, \ \ \mathcal{M}_t^s \subseteq \mathcal{M}_v^u \ \text{for} \ s_1 \le s_2, \ \ u \le s \le t \le v,$$

it is assumed as well that

$$\overline{\mathcal{M}}^s = \mathcal{M}^s, \ \ \overline{\mathcal{M}}_t^s = \mathcal{M}_t^s = \mathcal{M}_{t+}^s, \ \ 0 \le s \le t < \infty,$$

2000 *Mathematics Subject Classification.* Primary 60J05, 91B28, 91B70.

Key words and phrases. Inhomogeneous Markov process, stopping time, payoff, excessive function, extension of state space, universal completion.

where $\overline{\mathcal{M}}^s$ is a completion of $\mathcal{M}^s$ with respect to the family of measures $\{P_{u,x}, u \le s, x \in \mathsf{E}\}$, $\overline{\mathcal{M}}_t^s$ is the completion of $\mathcal{M}_t^s$ in $\overline{\mathcal{M}}^s$ with respect to the same family of measures ([3], Ch. 1, Sect. 5);

3) the paths of the process $X = (X_t(\omega))$, $t \ge 0$, are right continuous on the time interval $[0, \infty)$;

4) for each $t \ge 0$, the random variables $X_t(\omega)$ (with values in $(\mathsf{E}, \mathcal{B})$) are $\mathcal{M}_t^s$-measurable, $t \ge s$, where it is supposed that

$$P_{s,x}(\omega:\ X_s(\omega) = x) = 1,$$

and the function $P_{s,x}(X_{s+h} \in B)$ is measurable in (s,x) for the fixed $h \ge 0$, $B \in \mathcal{B}$ (with respect to $\mathcal{B}[0,\infty) \otimes \mathcal{B}$);

5) the process X is strong Markov: for every $\mathcal{M}_t^s$, $t \ge s$-stopping time τ (i.e. $\tau(\omega)$, $\{\omega:\ \tau(\omega) \le t\} \in \mathcal{M}_t^s, t \ge s$), we should have

$$P_{s,x}(X_{\tau+h} \in B |\ \mathcal{M}_\tau^s) = P(\tau, X_\tau, \tau + h, B) \ \ (\{\tau < \infty\}, P_{s,x}\text{-a.s.}),$$

where

$$P(s, x, s+h, B) \equiv P_{s,x}(X_{s+h} \in B);$$

6) the process X is quasi-left-continuous: for every non-decreasing sequence of $(\mathcal{M}_t^s)$, $t \ge s$-stopping times $\tau_n \uparrow \tau$ should be

$$X_{\tau_n} \to X_\tau \ \ (\{\tau < \infty\}, P_{s,x}\text{-a.s.}).$$

Let the gain function $\varphi(t,x)$ and the observation cost $c(t,x) \ge 0$ be Borel measurable functions (i.e. measurable with respect to the product σ-algebra $\mathcal{B}' = \mathcal{B}[0,+\infty) \otimes \mathcal{B}$) which is defined on $\mathsf{E}' = [0,+\infty) \times \mathsf{E}$, and $\varphi(t,x)$ takes its values in $(-\infty, +\infty]$. It is assumed that, for the observation stopping time t, we obtain a gain

$$g(t,x) = \varphi(t,x) - \int_0^t c(s, X_s)\, ds.$$

It is further assumed that the following integrability condition of a random process $g(t, X_t(\omega))$, $t \ge 0$, is fulfilled:

$$M_{s,x} \sup_{t \ge s} g^-(t, X_t) < +\infty, \quad s \ge 0, \quad x \in \mathsf{E}. \tag{1}$$

The optimal stopping problem for the process X with gain $g(t,x)$ is stated as follows: the value-function (payoff) $v(s,x)$ is introduced in the form

$$v(s,x) = \sup_{\tau \in \mathfrak{M}_s} M_{s,x} g(\tau, X_\tau), \tag{2}$$

where $\mathfrak{M}_s$ is the class of all finite ($P_{s,x}$-a.s.) M_t^s, $t \ge s$-stopping times; it is required to find the stopping time τ_ε (for each $\varepsilon \ge 0$), for which

$$M_{s,x} g(\tau_\varepsilon, X_{\tau_\varepsilon}) \ge v(s,x) - \varepsilon$$

for any $x \in \mathsf{E}$.

Such a stopping time is called ε-optimal. In the case where $\varepsilon = 0$, it is called simply an optimal stopping time.

To construct ε-optimal (optimal) stopping times, it is necessary to characterize the value $v(s,x)$. For this purpose, the following notion of an excessive function turns out to be fundamental.

A function $f(t,x)$ which is given on E', takes its values in $(-\infty, +\infty]$, and is measurable with respect to the universal completion $\mathcal{B}'^*$ of the σ-algebra $\mathcal{B}'$ is called excessive (with respect to X) if

$$
\begin{aligned}
&1)\quad M_{s,x} f^{-}(t, X_t) < +\infty, \quad 0 \le s \le t < +\infty, \quad x \in \mathsf{E},\\
&2)\quad M_{s,x} f(t, X_t) \le f(s,x), \quad t \ge s, \quad x \in \mathsf{E},\\
&3)\quad M_{s,x} f(t, X_t) \to f(s,x), \quad \text{if } t \downarrow s, \quad x \in \mathsf{E}.
\end{aligned}
\tag{3}
$$

2. Construction of a Homogeneous Standard Markov Process in the Extended State Space

Let us introduce now a new space of elementary events $\Omega' = [0, +\infty) \times \Omega$ with elements $\omega' = (s, \omega)$, a new state space (extended state space) $\mathsf{E}' = [0, +\infty) \times \mathsf{E}$ with the σ-algebra $\mathcal{B}' = \mathcal{B}[0, +\infty) \otimes \mathcal{B}$, the new random process X' with values in $(\mathsf{E}', \mathcal{B}')$

$$X'_t(\omega') = X'_t(s, \omega) = (s + t, X_{s+t}(\omega)), \quad s \ge 0, \quad t \ge 0,$$

and the translation operators Θ'_t

$$\Theta'_t(s, \omega) = (s + t, \omega), \quad s \ge 0, \quad t \ge 0,$$

where it is obvious that

$$X'_u(\Theta'_t(\omega')) = X'_{u+t}(\omega'), u \ge 0, \quad t \ge 0.$$

In the space Ω', we introduce an σ-algebra

$$N^0 = \sigma(X'_u, u \ge 0), \quad N^0_t = \sigma(X'_u, 0 \le u \le t).$$

On the σ-algebra N^0, the probability measures

$$P'_{x'}(A) = P'_{(s,x)}(A) \equiv P_{s,x}(A_s),$$

where $A \in N^0$, and A_s is the section of A at the point s,

$$A_s = \{\omega : \ (s, x) \in A\},$$

where it is easy to see that $A_s \in \mathcal{F}^s \equiv \sigma(X_u, u = s)$, and if $a \in N^0_t$, then $A_s \in \mathcal{F}^s_{s+t} \equiv \sigma(X_u, s \le u \le s + t)$.

Consider the function

$$P'(h, x', B') \equiv P'_{x'}(X'_h \in B').$$

We have to verify that this function is measurable in x' for a fixed $h \ge 0$. For the rectangles $B' = \Gamma \times B$ which generate the σ-algebra $\mathcal{B}'$, we have

$$
\begin{aligned}
P'(h, x', B') &= P_{s,x}\big(\omega : \ (s + h, X_{s+h}(\omega)) \in \Gamma \times B\big) = \\
&= I_{(s+h \in \Gamma)} P_{s,x}(X_{s+h} \in B).
\end{aligned}
$$

The function $P'(h, x', B')$ is measurable in x', and, hence, we can introduce measures $P'_{\mu'}$ on $(\mathsf{E}', \mathcal{B}')$. Let us perform the completion of the σ-algebra N^0 with respect to the family of all measures $P'_{\mu'}$. We denote this completion by N' and then perform the completion of each σ-algebra N^0_t in N' with respect to the same family of measures, by denoting them by N'_t.

The following key result (in a somewhat different form) was proved in [2].

Theorem 1. *The random process*

$$X' = (\Omega', N', N'_t, X'_t, \Theta'_t, P'_{x'}), \quad t \ge 0,$$

is a homogeneous standard Markov process in the space $(\mathsf{E}', \mathcal{B}')$.

Proof. The main step in the proof is to verify that the process X'_t, $t \geq 0$ is strong Markov, i.e. we have to show that

$$M'_{x'}\big[f'(X'_{\tau'+h}) \cdot I_{(\tau'<\infty)}\big] = M'_{x'}\big[M'_{X'_{\tau'}} f'(X'_h) I_{(\tau'<\infty)}\big], \tag{4}$$

where $f'(x')$is an arbitrary bounded $\mathcal{B}'$-measurable function, and τ' is an arbitrary N^0_{t+}-stopping time. Using the monotone class theorem, it suffices to prove this relation for the indicator functions

$$f'(x') = I_{(s\in\Gamma)} \cdot I_{(x\in B)}.$$

Note that if $\tau'(\omega')$ is an N^0_{t+}-stopping time, then $\tau(\omega) = s + \tau'(s,\omega)$ is a $\mathcal{F}^0_{t+}$, $t \geq s$-stopping time, where $\mathcal{F}^s_t = \sigma(X_u, s \leq u \leq t)$, $t \geq s$.

We have

$$\big(\omega : \ \tau(\omega) < t\big) = \big(\omega : \ \tau'(s,\omega) < t - s\big) = \\ = \big(\omega' : \ \tau'(\omega') < t - s\big)_s,$$

but $\big(\omega' : \ \tau'(\omega') < t - s\big) \in N^0_{t-s}$. Therefore, the section $(\omega' : \ \tau'(\omega') < t - s)_s$ belongs to $\mathcal{F}^s_t$. Thus, $\tau(\omega)$ is a $\mathcal{F}^s_{t+}$, $t \geq s$-stopping time, and the variable $\tau(\omega) = s + \tau'(s,\omega)$ is a $\mathcal{M}^s_t$, $t \geq s$-stopping time.

We know from Proposition 7.3, Ch. I in [3] that the strong Markov property (4) of the process X' remains true for arbitrary N'_t, $t \geq 0$-stopping times τ'. From Proposition 8.12, Ch. I in [3], we get $N'_t = N'_{t+}$. The quasi-left-continuity of the process X' now easily follows from the same property of X. Theorem 1 is proved.

3. The Optimal Stopping Problem for Processes X and X' and the Connection Between Them

Let $f(x') = f(s,x)$ be an arbitrary Borel measurable function (i.e. $\mathcal{B}'$-measurable) which is given on E' and takes its values in $(-\infty, +\infty]$. Consider the sets

$$A = \Big\{\omega' : \ \lim_{t\downarrow 0} f(X'_t) = f(X'_0)\Big\},$$

$$B = \Big\{\omega' : \ \text{the path } f(X'_t(\omega')) \text{ is right continuous on } [0, +\infty)\Big\}.$$

Obviously, the sections A_s and B_s can be written in the form

$$A_s = \Big\{\omega : \ \lim_{t\downarrow s} f(t, X_t(\omega)) = f(s, X_s(\omega))\Big\},$$

$$B_s = \Big\{\omega : \ \text{the path } f(t, X_t(\omega)) \text{ is right continuous on } [s, +\infty)\Big\}.$$

Theorem 2. *The sets A and B belong to N^{0*} (N^{0*} is the universal completion of N^0), and the sections A_s and B_s belong to $\mathcal{F}^{s*}$ ($\mathcal{F}^{s*}$ is the universal completion of $\mathcal{F}^s = \sigma(X_u, u \geq s)$).*

Further, we have

$$P'_{s,x}(A) = P_{s,x}(A_s), \quad P'_{s,x}(B) = P_{s,x}(B_s). \tag{5}$$

Proof. The set A can be written as

$$A = \Big\{\omega' : \ \lim_{k\to\infty} \sup_{0<t<\frac{1}{k}} f(X'_t(\omega')) = \lim_{k\to\infty} \inf_{0<t<\frac{1}{k}} f(X'_t(\omega')) = f(X'_0(\omega'))\Big\}.$$

We get from Theorem 13, Ch. III in [4] that the latter sets are N^0-analytic, and, hence, they belong to the universal completion of N^0. Thus, the set A itself belongs to N^{0*}. As for the set B, we get from Theorem 34, Ch. IV in [4] that this set is a completion of the N^0-analytic set; hence, $B \in N^{0*}$. The same reasoning shows that A_s and B_s belong to the universal completion $\mathcal{F}^{s*}$ of the σ-algebra $\mathcal{F}^s$. For the measure $P'_{s,x}$ and for the sets A and B belonging to the universal completion of N^0, there obviously exist sets A^1, A^2, B^1, and B^2 belonging to N^0 such that

$$A^1 \subseteq A \subseteq A^2, \quad B^1 \subseteq B \subseteq B^2,$$
$$P'_{s,x}(A^1) = P'_{s,x}(A) = P'_{s,x}(A^2),$$
$$P'_{s,x}(B^1) = P'_{s,x}(B) = P'_{s,x}(B^2).$$

But, by the definition of the measure $P'_{s,x}$, we have

$$P'_{s,x}(A^1) = P'_{s,x}(A^1_s), \quad P'_{s,x}(A^2) = P_{s,x}(A^2_s),$$
$$P'_{s,x}(B^1) = P'_{s,x}(B^1_s), \quad P'_{s,x}(B^2) = P_{s,x}(B^2_s).$$

From these relations and the inclusions $A^1_s \subseteq A_s \subseteq A^2_s$, $B^1 \subseteq B \subseteq B^2$, it easily follows that

$$P'_{s,x}(A) = P'_{s,x}(A_s), \quad P'_{s,x}(B) = P_{s,x}(B_s). \tag{6}$$

Theorem 2 is proved.

Let us consider the optimal stopping problem for the process X' with the same gain $g(x') = g(s,x)$ $(x' = (s,x))$ satisfying the conditions

$$M'_{x'} \sup_{t\geq 0} g^-(X'_t) < \infty, \quad x' \in E', \tag{7}$$

$$P'_{x'}\big\{\omega' : \lim_{t\downarrow 0} g(X'_t) = g(x')\big\} = 1, \quad x' \in \mathsf{E}', \tag{8}$$

and with the value $v'(x')$ defined by

$$v'(x') = v'(s,x) = \sup_{\tau'\in\mathfrak{M}'} M'_{x'} g(X'_{\tau'}), \tag{9}$$

where $\mathfrak{M}'$ is the class of all finite ($P'_{x'}$-a.s.) N'_t, $t \geq 0$-stopping times.

Our next step consists in establishing the connection between the value-functions $v(s,x)$ and $v'(s,x)$.

Theorem 3. *The values of the initial optimal stopping problem* (9) *coincide*

$$v(s,x) = v'(s,x), \quad s \geq 0, \quad x \in \mathsf{E}. \tag{10}$$

Proof. First, consider the N'_t, $t \geq 0$-stopping time τ'. By Proposition 7.3, Ch. I in [3], for τ' and fixed $x' = (s,x)$, there exists an N^0_{t+}, $t \geq 0$-stopping time $\widetilde{\tau}'$ such that $P'_{x'}(\tau' = \widetilde{\tau}') = 1$. We have

$$M'_{x'} g(X'_{\widetilde{\tau}'}) = M_{s,x} g\big(s + \widetilde{\tau}'(s,\omega), X_{s+\widetilde{\tau}'(s,\omega)}\big) =$$
$$= M_{s,x} g\big(\tau(\omega), X_{\tau(\omega)}\big),$$

where $s + \widetilde{\tau}'(s,\omega) \equiv \tau(\omega)$ is an $\mathcal{M}^s_t$, $t \geq s$-stopping time. Hence, it is obvious that

$$v'(s,x) \leq v(s,x). \tag{11}$$

It remains to establish that the opposite inequality it true. Denote, by $\mathfrak{M}^n_s$, the class of all $\mathcal{M}^s_t$, $t \geq s$-stopping times taking their values from the finite set

$$s, s+2^{-n}, \dots, s+k\cdot 2^{-n}, \dots, s+n.$$

Obviously,

$$\mathfrak{M}^n_s \subseteq \mathfrak{M}^{n+1}_s, \quad n = 1,2,\dots .$$

For every $\tau \in \mathfrak{M}_s$, we define the sequence τ_n of stopping times

$$\tau_n = \begin{cases} s + k2^{-n}, & \text{if } s + (k-1)2^{-n} \leq \tau < s + k2^{-n}, \\ s + n & \text{if } \tau \geq s + n. \end{cases}$$

It is clear that $\tau_n \in \mathfrak{M}_s^n$, and the sequence $\tau_n(\omega)$ decreases to $\tau(\omega)$ starting from some $n(\omega)$. Using the right continuity of paths $g(t, X_t(\omega))$, $t \geq s$ ($P_{s,x}$-a.s.), we can write

$$g(\tau, X_\tau) = \lim_{n\to+\infty} g(\tau_n, X_{\tau_n}) \ \ (P_{s,x}\text{-a.s.}).$$

Hence, by Fatou's lemma, we get

$$M_{s,x} g(\tau, X_\tau) \leq \varliminf_n M_{s,x} g(\tau_n, X_{\tau_n}).$$

Consequently,

$$v(s,x) = \sup_{\tau \in \bigcup\limits_n \mathfrak{M}_s^n} M_{s,x} g(\tau, X_\tau) = \lim_{n\to+\infty} \sup_{\tau\in\mathfrak{M}_s^n} M_{s,x} g(\tau, X_\tau).$$

Consider the expression

$$\sup_{\tau\in\mathfrak{M}_s^n} M_{s,x} g(\tau, X_\tau),$$

for which the optimal stopping problem represents the value of the sequence

$$\big\{ g(s+k2^{-n}, X_{s+k2^{-n}}), \mathcal{M}_{s+k2^{-n}} \big\}, \ \ k = 0, 1, \dots, n2^{-n}.$$

It is well known that, for this problem, there always exists an optimal stopping time having form

$$\sigma_n = \min\big\{ s + k2^{-n} : \ \gamma_k^n = g(s+k2^{-n}, X_{s+k2^{-n}}) \big\},$$

where the sequence γ_k^n is constricted recursively:

$$\gamma_k^n = \max\Big\{ g(s+k2^{-n}, X_{s+k2^{-n}}), M_{s,x}(\gamma_{k+1}^n / \mathcal{M}_{s+k2^{-n}}) \Big\}.$$

It easily follows from these recursion relations that γ_k^n is a Borel function of $X_{s+k2^{-n}}$. Therefore, σ_n has the form

$$\sigma_n = \min\big\{ s+k2^{-n} : \ X_{s+k2^{-n}} \in B_k^n \big\},$$

where the sets B_k^n belong to the σ-algebra $\mathcal{B}$.

Thus, we get

$$v(s,x) = \lim_{n\to+\infty} \uparrow M_{s,x} g(\sigma_n, X_{\sigma_n}).$$

We now define the corresponding N_t^0, $t \geq 0$-stopping times

$$\sigma_n' = \min\big\{ k2^{-n} : \ X'_{k2^{-n}} \in [0,+\infty) \times B_k^n \big\}.$$

We have

$$M'_{s,x} g(X'_{\sigma_n'}) = M_{s,x} g\big(X'_{\sigma_n'(s,\omega)}(s,\omega)\big) =$$
$$= M_{s,x} g\big(s + \sigma_n'(s,\omega), X_{s+\sigma_n'(s,\omega)}(\omega)\big) = M_{s,x} g(\sigma_n, X_{\sigma_n})$$

as $s + \sigma_n'(s,\omega) = \sigma_n(\omega)$. Therefore,

$$M_{s,x} g(\sigma_n, X_{\sigma_n}) = M'_{s,x} g(X'_{\sigma_n'}) \leq v'(s,x).$$

Thus, $v(s,x) \leq v'(s,x)$ and, finally, $v(s,x) = v'(s,x)$. Theorem 3 is proved.

The next purpose is the excessive characterization of a payoff $v(s,x)$. Let us note (as can be easily seen) that our definition of an excessive function (with respect to X) coincides exactly with the usual definition of an excessive function (with respect to X'). Therefore, we can directly use Theorem 1, Ch. III in [1] and get the following result.

Theorem 4. *Suppose that condition* (1) *is satisfied. Then the value $v(s,x)$ is a minimal excessive majorant of the function $g(s,x)$. The value $v(s,x)$ is a Borel measurable function* (*i.e.* $\mathcal{B}'$*-measurable*) *which can be found by the limit procedure*

$$v(s,x)=\lim_{n\to+\infty}\lim_{N\to+\infty}Q_n^N g(s,x), \tag{12}$$

where

$$Q_n g(s,x)=\max\big\{g(s,x),M_{s,x}g(s+2^{-n},X_{s+2^{-n}})\big\},$$

and Q_n^N is the N-th power of the operator Q_n.

Proof. The assertion is a consequence of the coincidence of the values $v(s,x)$ and $v'(s,x)$ and of Lemma 3, Ch. III in [3] which states that

$$v'(x')=\lim_{n\to+\infty}\lim_{N\to+\infty}Q_n^N g(x'),$$

where

$$Q_n g(x')=\max\big\{g(x'),M'_{x'}g(X'_{2^{-n}})\big\}.$$

Note also that $M'_{x'}g(X'_{2^{-n}})$ is $\mathcal{B}'$-measurable in x'. Hence, the functions $Q_n g(x')$, $Q_n^N g(x')$ and the function $v'(x')$, being the limit of these functions, are also $\mathcal{B}'$-measurable.

Thus, the value $v'(x')$ is a Borel measurable excessive function (with respect to X') which obviously satisfies the condition

$$M'_{x'}\sup_{t\ge0}v'(X'_t)<+\infty,\ \ x'\in\mathsf{E}'.$$

Then, as is well known (Theorem 2.12, Ch. II in [3]), the paths $v(X'_t(\omega'))$ are right continuous with the left-hand limits on $[0,+\infty)$ ($P'_{x'}$-a.s.). Using Theorem 2, we obtain

$$P_{s,x}\Big\{\omega:\ \text{the path } v(t,X_t(\omega)) \text{ is right continuous on } [s,\infty)\Big\}=1,$$
$$s\ge0,\ \ x\in\mathsf{E}.$$

To prove the main result of the present work, we can now apply Theorem 3, Ch. III in [1]. Theorem 4 is proved.

Theorem 5. *Let the gain $g(t,x)$ satisfy* (*with respect to X*) *the conditions*

1) $M_{s,x}\sup\limits_{t\ge s}|g(t,X_t)|<+\infty$, $s\ge0$, $x\in E$;

2) $P_{s,x}\{\omega:\lim\limits_{t\downarrow s}g(t,X_t(\omega))=g(s,x)\}=1$, $s\ge0$, $x\in E$.

Then

i) *for every $\varepsilon>0$, the stopping times*

$$\tau_\varepsilon=\inf\big\{t\ge s:\ v(t,X_t)\le g(t,X_t)+\varepsilon\big\} \tag{13}$$

are ε-optimal;

ii) *if the function $g(t,x)$ is upper semicontinuous, i.e.*

$$g(s,x)\ge\varlimsup_{\substack{t\to s\\ y\to x}}g(t,y),$$

and the stopping time

$$\tau_0(\omega)=\inf\big\{t\ge s:\ v(t,X_t)=g(t,X_t)\big\} \tag{14}$$

is finite ($P_{s,x}$*-a.s.*), *then $\tau_0(\omega)$ is an optimal stopping time.*

Proof. From Theorem 3, Ch.III in [1], we know that, for every $\varepsilon>0$, the stopping time

$$\tau'_\varepsilon=\inf\big\{t:\ v(X'_t)\le g(X'_t)+\varepsilon\big\}$$

is ε-optimal:

$$M'_{x'}g(X'_{\tau'_\varepsilon})\ge v(x')-\varepsilon,\ \ x'\in\mathsf{E}',$$

i.e.

$$M_{s,x}g\big(s+\tau'_\varepsilon(s,\omega),X_{s+\tau'_\varepsilon(s,\omega)}(\omega)\big)\geq v(s,x)-\varepsilon.$$

But it is obvious that $s+\tau'_\varepsilon(s,\omega)=\tau_\varepsilon(\omega)$; hence,

$$M_{s,x}g(\tau_\varepsilon,X_{\tau_\varepsilon})\geq v(s,x)-\varepsilon.$$

Assume now the upper semicontinuity of the function $g(x')$. Then, from the same theorem, we get again that the stopping time

$$\tau'_0=\inf\big\{t\geq 0:\ v(X'_t)=g(x'_t)\big\}$$

is optimal:

$$M'_{x'}g(X_{\tau'_0})=v(x').$$

From this, similarly to the previous reasoning, we get the optimality of the stopping time $\tau_0(\omega)$. Theorem 5 is proved.

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