Theory of Stochastic Processes
Vol. 15 (31), no. 2, 2009, pp. 42–53

ROVSHAN ALIYEV, TAHIR KHANIYEV, AND NURGUL OKUR BEKAR

WEAK CONVERGENCE THEOREM FOR THE ERGODIC DISTRIBUTION OF THE RENEWAL-REWARD PROCESS WITH A GAMMA DISTRIBUTED INTERFERENCE OF CHANCE

In this study, a renewal-reward process with a discrete interference of chance ($X(t)$) is investigated. The ergodic distribution of this process is expressed by a renewal function. We assume that the random variables $\{\zeta_n\}$, $n \geq 1$ which describe the discrete interference of chance form an ergodic Markov chain with the stationary gamma distribution with parameters (α, λ), $\alpha > 0$, $\lambda > 0$. Under this assumption, an asymptotic expansion for the ergodic distribution of the stochastic process $W_\lambda(t) = \lambda(X(t) - s)$ is obtained, as $\lambda \to 0$. Moreover, the weak convergence theorem for the process $W_\lambda(t)$ is proved, and the exact expression of the limit distribution is derived. Finally, the accuracy of the approximation formula is tested by the Monte-Carlo simulation method.

1. Introduction

It is known that numerous interesting problems of queuing, inventory and reliability theories, mathematical insurance, financial mathematics, mathematical biology, physics, etc. are expressed by means of stochastic processes with a discrete interference of chance, especially by means of the random walk and the renewal-reward processes. There are many interesting studies on these topics in the literature (see, for example, [1]-[10]).

There are also many studies on the asymptotic behavior of characteristics of the renewal-reward processes. For instance, in [7], Jewell studied the fluctuations of a renewal-reward process embedded in the renewal process. In [2], Brown and Solomon considered the following renewal-reward process with absolutely continuous component:

$$C(t) = \begin{cases} 0 & , \quad t < X_0 \\ \sum_{k=0}^{N(t)-1} Y_k \, , & t \geq X_0 \end{cases},$$

where $N(t) = min\{k : S_k > t\}$, $S_n = \sum_{i=0}^{n} X_i$, $n = 0, 1, 2, \ldots$; $\{X_i,\ i = 0, 1, 2, \ldots\}$ is a renewal sequence, and $\{(X_i, Y_i),\ i = 0, 1, 2, \ldots\}$ is a sequence of independent and identically distributed random vectors. In [2], Brown and Solomon obtained the second-order asymptotic expansions for the first and second moments of the renewal-reward process $C(t)$. Note that the renewal-reward process $C(t)$ occurs in various stochastic optimization models, particularly in Markov and semi-Markov decision models. In these models, Y_i represents the reward or cost associated with a given policy over the renewal interval $(S_{i-1}, S_i]$.

Another important problem in this area is considered in [1] by Alsmeyer. He considered the extended renewal process $\{(S_n, U_n\}_{n\geq 0}$, where

$$S_n = \sum_{i=0}^{n} X_i, \qquad U_n = \sum_{i=0}^{n} Y_i.$$

2000 *Mathematics Subject Classification.* Primary 60G50; Secondary 60K15; 60F99.

Key words and phrases. Renewal-reward process; discrete interference of chance; Laplace transform; asymptotic expansion; weak convergence; Monte-Carlo method.

Under appropriate conditions on two-dimensional random vectors (X_i, Y_i), $i \geq 0$, the asymptotic expansions for $EU_{T(t)}$, $VarU_{T(t)}$ and $Cov(U_{T(t)}, T(t))$ are obtained, as $t \to \infty$, where $T(t) = inf\{n \geq 0 : S_n > t\}$. The corresponding results for $EU_{N(t)}$, $VarU_{N(t)}$ and $Cov(U_{N(t)}, N(t))$ are obtained, when X_0, X_1 are both almost surely non-negative and

$$N(t) = \sup\{n \geq 0 : S_n \leq t\}.$$

One of the most used application areas of the aforementioned renewal-reward process is insurance theory. In collective risk problems, the random variables X_1, X_2,... are interpreted as the time between claims; Y_1, Y_2,... are interpreted as the corresponding claim amounts; $N(t)$, $t \geq 0$ denotes the number of claims up to time t, and $U_{N(t)}$ denotes the total value of claims made till the time t by the insurance company (see, for example, [10]).

In [3], Csenki derived an asymptotic representation for the expected value of renewal-reward processes with retrospective reward structure.

Recently, the heavy-tailed distributions have been used in the renewal-reward processes. For example, in [6], Levy and Taqqu investigated renewal-reward processes with heavy-tailed inter-renewal times and heavy-tailed rewards. In their paper, both the inter-renewal times and the rewards were allowed to have infinite variance.

In the studies mentioned above, generally, the first two moments of the renewal-reward processes were obtained. But, in [8], Khaniyev obtained the third-order asymptotic expansions as $t \to \infty$ for the first four moments of the generalized renewal-reward process.

Many of the authors mentioned earlier derived asymptotic expansions for the moments of a renewal-reward process. But, in order to solve some practical and theoretical problems, the ergodic distribution of the renewal-reward process with a discrete interference of chance is needed. The basic aim of this study is to prove the weak convergence theorem for the ergodic distribution of this process.

Let us consider the following inventory model before expressing the problem mathematically.

The model. Assume that the stock level in a depot at the initial time ($t = 0$) is equal to $X(0) \equiv X_0 \equiv s + v$, where $0 < s < \infty$ represents the stock control level and $v > 0$. In addition, it is assumed that, at random times T_1, T_2,..., T_n,..., the stock level ($X(t)$) in the depot decreases by η_1, η_2,..., η_n,..., respectively, until the stock level $X(t)$ falls below the predetermined control level s. Thus, the stock level in the depot changes as follows:

$$X(T_1) \equiv X_1 = s + v - \eta_1, \quad X(T_2) \equiv X_2 = s + v - (\eta_1 + \eta_2), \ldots,$$

$$X(T_n) \equiv X_n = s + v - \sum_{i=1}^{n} \eta_i, \tag{1}$$

where η_n represents the quantity of the n^{th} demand, $n = 1, 2, 3, \ldots$.

In other words, demands are inserted to the system at the random times

$$T_n = \sum_{i=1}^{n} \xi_i,$$

where ξ_n represents the time between $(n-1)^{th}$ and n^{th} demand, $n = 1, 2, 3, \ldots$. The system passes from one state to another by jumping at time T_n, according to the quantities of demand $\{\eta_n\}$, $n \geqslant 1$ as shown in Eq. (1). This variation of the system continues up to a certain random time τ_1, where τ_1 is the first time that the stock level $X(t)$ drops below the control level $s > 0$. When this occurs, the system is immediately brought to the state $s + \zeta_1$, where $\zeta_1 \in [0, \infty)$. Thus, the first period completes, and the second one starts. Afterwards, the process $X(t)$ continues to change from the new initial state $s + \zeta_1$, similar to the way it changed in the first period. When the stock level $X(t)$ falls below s, for the second time, by an interference to the system, the stock level is brought to the random level $s+\zeta_2$, similar to the preceding period, and so on. Here, ζ_1, ζ_2,... are independent and identically distributed positive-valued random variables.

In this study, a stochastic process which describes the model mentioned above is constructed and investigated. In this case, the distribution of interference will be chosen from a narrow but important class of distributions. Namely, we will assume that the random variables $\{\zeta_n\}$, $n \geq 1$ which describe the discrete interference of chance form an ergodic Markov chain with the stationary gamma distribution with parameters (α, λ), $\alpha > 0$, and $\lambda > 0$. Under this assumption, we aim to investigate the asymptotic behavior of the process $W_\lambda(t) \equiv \lambda(X(t) - s)$, as $\lambda \to 0$. Moreover, we will prove the weak convergence theorem for the ergodic distribution of the process $W_\lambda(t)$. Finally, the accuracy of the approximation formula obtained in this study is tested by the Monte-Carlo simulation method.

2. Mathematical construction of the process $X(t)$

Let $\{\xi_n\}$ and $\{\eta_n\}$, $n \geq 1$ be two independent sequences of random variables defined on any probability space $(\Omega, \Im, P)$, such that the variables in each sequence are independent and identically distributed. We introduce also the sequence of random variables $\{\zeta_n\}$, $n \geq 1$ which describes the discrete interference of chance and form an ergodic Markov chain with the stationary gamma distribution with parameters (α, λ), $\alpha > 0$, $\lambda > 0$. Suppose that ξ_i's and η_i's take only positive values. We denote their distribution functions by $\Phi(t)$ and $F(x)$ respectively. So,

$$\Phi(t) = P\{\xi_1 \leq t\},\ t > 0; \quad F(x) = P\{\eta_1 \leq x\},\ x > 0.$$

Using the initial sequences of the random variables $\{\xi_n\}$ and $\{\eta_n\}$, we define the renewal sequences $\{T_n\}$, $\{S_n\}$ and their distribution functions as follows:

$$T_n = \sum_{i=1}^{n} \xi_i, \quad S_n = \sum_{i=1}^{n} \eta_i, \quad T_0 = S_0 = 0,\ n \geq 1,$$

$$\Phi_n(t) = P\{T_n \leq t\}, \quad F_n(x) = P\{S_n \leq x\},$$

Moreover, we define the sequence of integer-valued random variables $\{N_n\}$, $n \geq 0$, as follows:

$$N_0 = 0, \quad N_1 = N(v) = inf\{k \geq 1 : S_k > v\},\ v \geq 0;$$

$$N_{n+1} \equiv N_{n+1}(\zeta_n) = inf\{k \geq N_n + 1 : s + \zeta_n - S_k + S_{N_n} < s\}$$

$$= inf\{k \geq N_n + 1 : S_k - S_{N_n} > \zeta_n\},\ n \geq 1,$$

where $inf(\emptyset) = +\infty$ is stipulated.

Put

$$\tau_0 = 0; \quad \tau_1 = T_{N_1} = \sum_{i=1}^{N(v)} \xi_i; \quad \tau_{n+1} \equiv \tau_{n+1}(\zeta_n) = T_{N_{n+1}} = \sum_{i=1}^{N_{n+1}} \xi_i,\ n \geq 1,$$

and define $\nu(t)$ as

$$\nu(t) = max\{n \geq 0 : T_n \leq t\},\ t > 0.$$

We can now construct the desired stochastic process $X(t)$ as

$$X(t) = s + \zeta_n - S_{\nu(t)} + S_{N_n}, \quad if\ \tau_n \leq t < \tau_{n+1},\ n \geq 0,$$

where $\zeta_0 = s + v$ and $S_{\nu(\tau_n+0)} = S_{N_n}$.

We call the process $X(t)$ as the "renewal-reward process with a gamma distributed interference of chance". Figure 1 gives a trajectory of the process X(t).

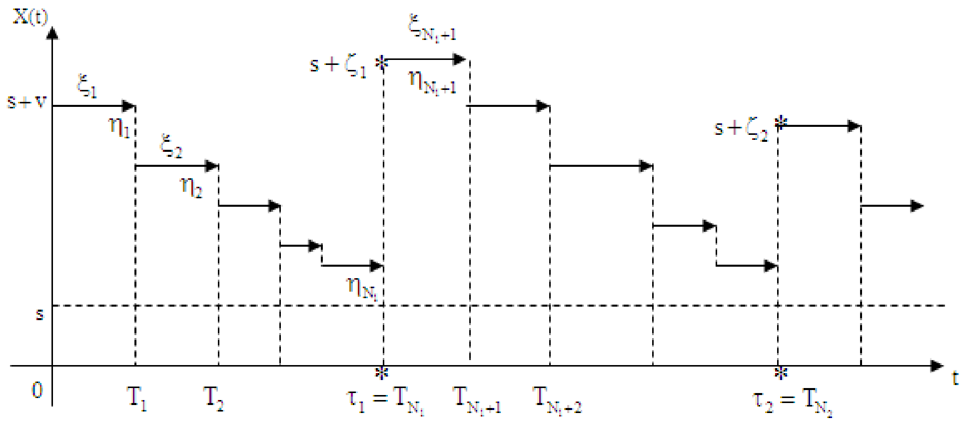

Figure 1: A trajectory of the process $X(t)$.

3. Preliminary Discussions

To investigate the stationary characteristics of the considered process $X(t)$, it is necessary to prove that $X(t)$ is ergodic. This property can be expressed by the following proposition.

Proposition 3.1. *Let the initial sequences of the random variables* $\{\xi_n\}$, $\{\eta_n\}$ *and* $\{\zeta_n\}$, $n \geq 1$, *satisfy the following additional conditions:*

1) $E\xi_1 < \infty$*;*

2) $m_2 = E(\eta_1^2) < +\infty$*;*

3) η_1 *is a non-arithmetic random variable;*

4) the sequence of the random variables $\{\zeta_n\}$, $n \geq 1$ *, which describes the discrete interference of chance forms an ergodic Markov chain having the gamma distribution with the parameters* (α, λ), $\alpha > 0$, $\lambda > 0$ *as the stationary distribution of a chain.*

Then the process $X(t)$ *is ergodic.*

Proof. The process $X(t)$ belongs to a wide class of processes which is known in the literature as "the class of semi-Markov processes with a discrete interference of chance". The ergodic theorem of the type of Smith's "key renewal theorem" exists in the literature for this wide class (see, [5], p. 243). By this theorem, it is necessary and sufficient to verify the following assumptions:

Assumption 3.1. *It is required to choose a sequence of ascending random times such that the values of the process* $X(t)$ *at these times form an imbedded Markov chain which is ergodic and has a stationary distribution.*

For this aim it suffices to consider the sequence of the random times $\{\tau_n\}$, $n \geq 0$, which is defined in the section 2. On the other hand, the values of the process $X(t)$ at these times $\zeta_n = X(\tau_n + 0)$, $n \geq 1$ form an imbedded Markov chain. Since, ζ_n, $n \geq 1$ are identically distributed random variables then imbedded Markov chain

$$\{X(\tau_n + 0)\}, \qquad n \geq 1$$

is ergodic with a stationary distribution

$$d\pi(v) = \frac{\lambda^\alpha}{\Gamma(\alpha)} v^{\alpha-1} e^{-\lambda v} dv, \qquad v \geq 0.$$

Therefore, the first assumption of the general ergodic theorem is satisfied.

Assumption 3.2. *The mathematical expectation of a time interval between successive Markov moments* $\{\tau_n\}, n = 1, 2, 3, \ldots$, *must be finite, i.e., for every* $n = 1, 2, 3, \ldots$,

$$E(\tau_n - \tau_{n-1}) < \infty. \tag{2}$$

Since

$$\tau_n - \tau_{n-1}, \qquad n = 2, 3, \dots,$$

are independent and identically distributed random variables, condition (2) holds if the integral

$$E(\tau_1) \equiv \int_0^\infty E_v(\tau_1) d\pi(v) \tag{3}$$

is finite. On the other hand, by using Wald's identity (see, [4], p.601), we have

$$E_v(\tau_1) = E(\sum_{i=1}^{N(v)} \xi_i) = E(\xi_1)E(N(v)) = E(\xi_1)U_\eta(v). \tag{4}$$

Here, $U_\eta(v)$ is a renewal function generated by the sequence of random variables $\{\eta_n\}$. Therefore,

$$E(\tau_1) = E(\xi_1) \int_0^\infty U_\eta(v) d\pi(v). \tag{5}$$

Remember that $0 < E(\xi_1) < \infty$. In this case, condition (2) holds, if the integral

$$\int_0^\infty U_\eta(v) d\pi(v) \tag{6}$$

is finite.

It is known that the renewal function $U_\eta(v)$ is finite for every $0 < v < +\infty$ (see, [4]). But, in our case, it is not sufficient that $U_\eta(v) < +\infty$ for every $0 < v < +\infty$. In addition, it is necessary to show that the relation

$$EU_\eta(\zeta_1) = \int_0^\infty U_\eta(v) d\pi(v) < +\infty \tag{7}$$

is valid. In other words, we must show

$$EU_\eta(\zeta_1) = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty U_\eta(v) v^{\alpha-1} e^{-\lambda v} dv < +\infty. \tag{8}$$

By the conditions of Proposition 3.1, the second moment of the random variable η_1 is finite, i.e., $E(\eta_1^2) < \infty$. In this case, by a sharper form of the renewal theorem (see, [4]) as $v \to \infty$, we have

$$U_\eta(v) = \frac{v}{m_1} + \frac{m_2}{2m_1^2} + g(v). \tag{9}$$

Here, the function $g(v)$ tends to zero as $v \to \infty$, i.e.,

$$\lim_{v \to \infty} g(v) = 0.$$

For this reason, for every $\varepsilon > 0$, it is possible to find the number $b \equiv b(\varepsilon)$ such that $0 < b(\varepsilon) < +\infty$, and, for every $v \geq b(\varepsilon)$,

$$|g(v)| < \frac{\varepsilon}{2}. \tag{10}$$

The expression in (8) can be written as follows:

$$\begin{aligned} EU_\eta(\zeta_1) = {} & \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{b(\varepsilon)} U_\eta(v) v^{\alpha-1} e^{-\lambda v} dv \\ & + \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_{b(\varepsilon)}^\infty U_\eta(v) v^{\alpha-1} e^{-\lambda v} dv \equiv J_1(\varepsilon) + J_2(\varepsilon). \end{aligned} \tag{11}$$

Since the function $U_\eta(v)$ is monotone non-decreasing, the inequality

$$U_\eta(v) \leq U_\eta(b(\varepsilon)) < +\infty$$

is true for every $v \leq b(\varepsilon)$. Therefore,

$$J_1(\varepsilon) \equiv \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{b(\varepsilon)} U_\eta(v) v^{\alpha-1} e^{-\lambda v} dv$$

$$\le U_\eta(b(\varepsilon))\frac{\lambda^\alpha}{\Gamma(\alpha)}\int_0^{b(\varepsilon)} v^{\alpha-1}e^{-\lambda v}dv \le U_\eta(b(\varepsilon)). \tag{12}$$

On the other hand, from the definition of the number $b(\varepsilon)$, we have

$$U_\eta(b(\varepsilon)) \le \frac{\mathrm{b}(\varepsilon)}{m_1} + \frac{m_2}{2m_1^2} + \frac{\varepsilon}{2}. \tag{13}$$

Hence, relations (12) and (13) yield the following inequality:

$$J_1(\varepsilon) \le \frac{\mathrm{b}(\varepsilon)}{m_1} + \frac{m_2}{2m_1^2} + \frac{\varepsilon}{2}. \tag{14}$$

Now, we estimate the second term in (11):

$$J_2(\varepsilon) \equiv \frac{\lambda^\alpha}{\Gamma(\alpha)}\int_{b(\varepsilon)}^\infty U_\eta(v)v^{\alpha-1}e^{-\lambda v}dv \le \frac{\lambda^\alpha}{m_1\Gamma(\alpha)}\int_{b(\varepsilon)}^\infty v^{\alpha-1}e^{-\lambda v}dv$$

$$+(\frac{m_2}{2m_1^2}+\frac{\varepsilon}{2})\frac{\lambda^\alpha}{\Gamma(\alpha)}\int_{b(\varepsilon)}^\infty v^{\alpha-1}e^{-\lambda v}dv \le \frac{\alpha}{\lambda m_1} + \frac{m_2}{2m_1^2} + \frac{\varepsilon}{2}\,, \tag{15}$$

where $\Gamma(\alpha)=\int_0^\infty t^{\alpha-1}e^{-t}dt$ is the Euler's gamma function.

In view of (14) and (15), relation (11) yields

$$EU_\eta(\zeta_1) \equiv J_1(\varepsilon)+J_2(\varepsilon) \le \frac{\alpha}{\lambda m_1} + \frac{b(\varepsilon)}{m_1} + \frac{m_2}{2m_1^2} + \varepsilon. \tag{16}$$

Under the condition of Proposition 3.1, $m_1 > 0$, and $m_2 \equiv E(\eta_1)^2 < \infty$ holds. On the other hand, $0 < \alpha < \infty$, $0 < \lambda < \infty$, and, for each $\varepsilon > 0$, the number $b(\varepsilon)$ is finite, i.e., $b(\varepsilon) < \infty$. Therefore, from (16), we have

$$EU_\eta(\zeta_1) < \infty. \tag{17}$$

So, $E(\tau_1) < \infty$ is proved. It is shown that Assumption 2 is satisfied.

In this case, under the conditions of Proposition 3.1, the conditions of the general ergodic theorem are satisfied. Therefore, the process $X(t)$ is ergodic.

This completes the proof of Proposition 3.1. □

Note. According to the general ergodic theorem, when the conditions of Proposition 3.1 are satisfied, the time averaging of the process $X(t)$ converges to the phase averaging with probability 1 as $t \to \infty$ (see, for example, [5], p. 243). This property of the process $X(t)$ can be given by the following proposition.

Proposition 3.2. *Under the conditions of Proposition 3.1, the following relation is correct with probability 1 for each measurable bounded function $f(x)$ ($f : [s, +\infty) \to R$):*

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t f(X(u))du = S_f \equiv \frac{\int_s^\infty\int_0^\infty f(x)\left[U_\eta(v)-U_\eta(s+v-x)\right]d\pi(v)dx}{\int_0^\infty U_\eta(v)d\pi(v)}, \tag{18}$$

where $U_\eta(x) = \sum_{n=0}^\infty F_n(x)$ is the renewal function generated by the sequence $\{\eta_n\}$, $n \ge 1$, and $\pi(v)$ is a stationary distribution of the Markov chain $\{\zeta_n\}$, $n \ge 1$.

Proof. According to the general ergodic theorem for the semi-Markov processes with a discrete interference of chance (see, [5], p.243), the following expression is correct with probability 1 for any bounded measurable function $f(x)$, when the conditions of Proposition 3.1 are satisfied:

$$S_f = \frac{1}{E(\tau_1)}\int_0^\infty\int_0^\infty\int_s^\infty f(x)P_{s+v}\left\{\tau_1 > t;\ X(t)\in dx\right\}dtd\pi(v). \tag{19}$$

We now introduce the notation $G(t,x,v) \equiv P_{s+v}\{\tau_1 > t; X(t) \le x\}$ for the sake of brevity. In this case, the following equality can be written:

$$G(t,x,v) \equiv P_{s+v}\left\{\tau_1 > t;\ X(t) \le x\right\} = \sum_{n=0}^\infty \Delta\Phi_n(t)\left[F_n(v)-F_n(s+v-x)\right]. \tag{20}$$

Here, $\Delta\Phi_n(t) = \Phi_n - \Phi_{n+1}$; $\Phi_n(t) = P\{T_n \le t\}$, $F_n(x) = P\{S_n \le x\}$.

First, applying the Laplace transformation to (20) with respect to the parameter t and taking the limit of both sides of Eq. (20) with respect to the parameter λ, as $\lambda \to 0$, we obtain

$$\lim_{\lambda\to 0} \tilde{G}(\lambda, x, v) = E\xi_1 \left[U_\eta(v) - U_\eta(s + v - x)\right], \tag{21}$$

where $\tilde{G}(\lambda, x, v)$ denotes the Laplace transform of the function $G(t, x, v)$, and $U_\eta(x)$ is a renewal function generated by the distribution of the random variable η_1, i.e., $U_\eta(x) = \sum_{n=0}^{\infty} F_n(x)$, where the function $F_n(x)$ represents the n^{th} convolution multiplication of $F(x)$.

On the other hand, since $\tau_{1v} = \sum_{i=1}^{N(v)} \xi_i$, the following identity can be written:

$$E(\tau_{1v}) = E(\xi_1)E(N(v)) = E(\xi_1)U_\eta(v).$$

Then

$$E(\tau_1) = E(\xi_1)\int_0^\infty U_\eta(v)d\pi(v) = E(\xi_1)E(U_\eta(\zeta_1)). \tag{22}$$

Therefore, if expressions (21) and (22) are taken into account in equality (9), result (8) can be obtained. This completes the proof of Proposition 3.2. $\square$

Note. If the indicator function is used instead of a function $f(x)$ in Proposition 3.2, the following exact formula can be written for the ergodic distribution function $(Q_X(x))$ after the corresponding calculations:

$$Q_X(x) = 1 - \frac{EU_\eta(\zeta_1 + s - x)}{EU_\eta(\zeta_1)},\ \ x \in [s, \infty). \tag{23}$$

For a detailed study, we now introduce the notation $\bar{X}(t) \equiv X(t) - s$. We can write the ergodic distribution function $(Q_{\bar{X}}(x))$ of the process $\bar{X}(t)$ as follows:

$$Q_{\bar{X}}(x) = 1 - \frac{EU_\eta(\zeta_1 - x)}{EU_\eta(\zeta_1)}. \tag{24}$$

Using this formula, it is possible to obtain an exact expression for the ergodic distribution of the process $\bar{X}(t)$, when the random variable η_1 has a certain well-known distribution (for example, exponential, Erlang, etc.).

Example 3.1. Let the conditions of Proposition 3.1 be satisfied. Moreover, we assume that the random variable η_1 has an exponential distribution with parameter $\mu > 0$. Then the ergodic distribution function $(Q_{\bar{X}}(x))$ of the process $\bar{X}(t)$ can be expressed as follows:

$$Q_{\bar{X}}(x) = 1 - \frac{\mu x}{\lambda + \alpha\mu}\, g_{\alpha,\lambda}(x) + \left(1 - \frac{\lambda\mu x}{\lambda + \alpha\mu}\right)\left(1 - G_{\alpha,\lambda}(x)\right), \tag{25}$$

where $g_{\alpha,\lambda}(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1}e^{-\lambda x}$, $G_{\alpha,\lambda}(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)}\int_0^x v^{\alpha-1}e^{-\lambda v}dv$,

$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$ *is the Euler's gamma function.*

Example 3.2. Let the conditions of Proposition 3.1 be satisfied. Moreover, we assume that the random variable η_1 has the second-order Erlang distribution with parameter $\mu > 0$. Then the ergodic distribution function $(Q_{\bar{X}}(x))$ of the process $\bar{X}(t)$ can be expressed as follows:

$$\begin{aligned} Q_{\bar{X}}(x) = 1 - c\Big[&\frac{\mu x}{2\lambda} g_{\alpha,\lambda}(x) + \left(\frac{\mu\alpha}{2\lambda} + \frac{3}{4} - \frac{\mu x}{2}\right)\left(1 - G_{\alpha,\lambda}(x)\right) \\ &+ \frac{e^{2\mu x}}{4}\left(\frac{\lambda}{\lambda + 2\mu}\right)^\alpha \left(1 - G_{\alpha,\lambda+2\mu}(x)\right)\Big], \end{aligned} \tag{26}$$

where $G_{\alpha,\lambda+2\mu}(x) = \frac{(\lambda+2\mu)^\alpha}{\Gamma(\alpha)}\int_0^x v^{\alpha-1}e^{-(\lambda+2\mu)v}dv$, $c = \frac{4\lambda(\lambda+2\mu)^\alpha}{(\lambda+2\mu)^\alpha(3\lambda+2\mu\alpha)+\lambda^{\alpha+1}}$.

Indeed,

$$EU_\eta(\zeta_1 - x) = \frac{\mu}{2}\frac{\lambda^{\alpha-1}}{\Gamma(\alpha)}x^\alpha e^{-\lambda x} + \Big(\frac{\mu}{2}\frac{\alpha}{\lambda} + \frac{3}{4} - \frac{\mu}{2}x\Big)(1 - G_{\alpha,\lambda}(x))$$

$$+\frac{e^{-2\mu x}}{4}\Big(\frac{\lambda}{\lambda+2\mu}\Big)^\alpha (1 - G_{\alpha,\lambda+2\mu}(x))\,. \tag{27}$$

Using a similar method, we have

$$EU_\eta(\zeta_1) = \frac{\mu}{2}\frac{\alpha}{\lambda} + \frac{3}{4} + \frac{1}{4}\Big(\frac{\lambda}{\lambda+2\mu}\Big)^\alpha. \tag{28}$$

Finally, substituting expressions (27) and (28) in (24) and carrying out the corresponding calculations, we obtain (26). □

Remark 3.1. As seen from Examples 3.1 and 3.2, the exact expression of the ergodic distribution of the process can be extremely complex even in simple cases. In addition, when the distribution of the random variable η_1 is different from the exponential or Erlang distribution, to find the clear form of the renewal function $U_\eta(x)$ becomes a very complex problem. Thus, it is advisable to derive an approximate formula for the ergodic distribution which is simpler than the exact formula. Therefore, in the following section, we will try to obtain an asymptotic expansion for the ergodic distribution.

4. Asymptotic expansion and the weak convergence theorem for the ergodic distribution of the process $W_\lambda(t)$

In this section, our aim is to obtain an asymptotic expansion and to prove the weak convergence theorem for the ergodic distribution of the process $W_\lambda(t) \equiv \lambda(X(t)-s)$, as $\lambda \to 0$. For this purpose, we prove the following auxiliary lemma:

Lemma 4.1. *Let $g(x)$ $(g : R^+ \to R)$ be the bounded function, and $\lim_{x\to\infty} g(x) = 0$. Then, for all $\alpha > 0$, the following relation is true:*

$$\lim_{\lambda\to 0}\int_0^\infty t^{\alpha-1}e^{-t}g(\frac{t}{\lambda})dt = 0. \tag{29}$$

Proof. Under the conditions of Lemma 4.1, for each $\varepsilon > 0$, there exists $m(\varepsilon) > 0$ so that, for every $x > m(\varepsilon)$, the inequality $|g(x)| < \varepsilon$ holds. Choose $b > 0$ such that $\int_0^b t^{\alpha-1}e^{-t}dt < \varepsilon$. The function $g(x)$ is bounded. Therefore, for each $\lambda < \frac{b}{m(\varepsilon)}$, we have

$$\begin{aligned}\left|\int_0^\infty t^{\alpha-1}e^{-t}g(\frac{t}{\lambda})dt\right| &\le \int_0^b t^{\alpha-1}e^{-t}\left|g(\frac{t}{\lambda})\right|dt + \int_b^\infty t^{\alpha-1}e^{-t}\left|g(\frac{t}{\lambda})\right|dt \\ &\le \max_{x\ge 0}|g(x)|\int_0^b t^{\alpha-1}e^{-t}dt + \varepsilon\int_b^\infty t^{\alpha-1}e^{-t}dt \\ &\le \varepsilon M + \varepsilon\int_0^\infty t^{\alpha-1}e^{-t}dt = \varepsilon(M+\Gamma(\alpha)),\end{aligned}$$

where $M = \max_{x\ge 0}|g(x)|$, and $\Gamma(\alpha)$ is the Euler's gamma function.

Since M and $\Gamma(\alpha)$ are finite, and $\varepsilon > 0$ is an arbitrary positive number, the proof of Lemma 4.1 is completed. □

We now investigate the asymptotic behavior of the ergodic distribution function $(Q_{W_\lambda}(x))$ of the process $W_\lambda(t) \equiv \lambda(X(t)-s)$, as $\lambda \to 0$.

Theorem 4.1. *Let the conditions of Proposition 3.1 be satisfied. Then, for each $\alpha > 0$ and $x \ge 0$, the following asymptotic expansion can be written for the ergodic distribution function $(Q_{W_\lambda}(x))$ of the process $W_\lambda(t)$, as $\lambda \to 0$:*

$$Q_{W_\lambda}(x) = R_\alpha(x) + \frac{m_2}{2m_1\alpha}\left(G_\alpha(x) - R_\alpha(x)\right)\lambda + o(\lambda), \tag{30}$$

where $m_k = E(\eta_1^k)$, $k = 1, 2$; $R_\alpha(x) = \frac{1}{\alpha}\int_0^x (1 - G_\alpha(t))dt$; $G_\alpha(x) = \frac{1}{\Gamma(\alpha)}\int_0^x t^{\alpha-1}e^{-t}dt.$

Proof. Taking the definition of the process $W_\lambda(t) \equiv \lambda(X(t) - s)$ into account in (8), we have

$$Q_{W_\lambda}(x) \equiv \lim_{t\to\infty} P\left\{W_\lambda(t) \le x\right\} = 1 - EU_\eta(\zeta_1 - \frac{x}{\lambda})\left(EU_\eta(\zeta_1)\right)^{-1}. \tag{31}$$

If the condition $m_2 = E(\eta_1^2) < +\infty$ is satisfied, then the following asymptotic expansion is correct (see [4], p.366):

$$U_\eta(v) = \frac{v}{m_1} + \frac{m_2}{2m_1^2} + g(v), \;\; as \;\; v \to \infty. \tag{32}$$

Here, $\lim_{v\to\infty} g(v) = 0$.

Using formula (32), we obtain the following expression, as $\lambda \to 0$:

$$EU_\eta(\zeta_1) = \frac{\alpha}{\lambda m_1} + \frac{m_2}{2m_1^2} + J_1(\alpha, \lambda), \tag{33}$$

where

$$J_1(\alpha, \lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)}\int_0^\infty v^{\alpha-1}e^{-\lambda v}g(v)dv = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-t}g(\frac{t}{\lambda})dt. \tag{34}$$

Using Lemma 4.1, for each $\alpha > 0$, relation (34) yields

$$\lim_{\lambda\to 0} J_1(\alpha, \lambda) = 0. \tag{35}$$

Then, substituting (35) into (33), we can obtain the following asymptotic expansion, as $\lambda \to 0$:

$$EU_\eta(\zeta_1) = \frac{\alpha}{\lambda m_1} + \frac{m_2}{2m_1^2} + o(1). \tag{36}$$

Then, as $\lambda \to 0$, we can rewrite (36) as follows:

$$(EU_\eta(\zeta_1))^{-1} = \frac{\lambda m_1}{\alpha}\Big(1 - \frac{m_2\lambda}{2m_1\alpha} + o(\lambda)\Big). \tag{37}$$

Similarly, we obtain

$$\begin{aligned} EU_\eta(\zeta_1 - \frac{x}{\lambda}) &= \left(\frac{x^\alpha e^{-x}}{m_1\Gamma(\alpha)} + \frac{\alpha - x}{m_1}(1 - G_\alpha(x))\right)\frac{1}{\lambda} \\ &+ \frac{m_2}{2m_1^2}(1 - G_\alpha(x)) + J_2(\alpha, \lambda, x), \end{aligned} \tag{38}$$

where

$$J_2(\alpha, \lambda, x) = \frac{\lambda^\alpha}{\Gamma(\alpha)}\int_{\frac{x}{\lambda}}^\infty v^{\alpha-1}e^{-\lambda v}g(v)dv.$$

For each $x \ge 0$,

$$J_2(\alpha, \lambda, x) = \frac{1}{\Gamma(\alpha)}\int_x^\infty t^{\alpha-1}e^{-t}g(\frac{t}{\lambda})dt \le \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-t}g(\frac{t}{\lambda})dt. \tag{39}$$

Using Lemma 4.1, for each $\alpha > 0$ and $x \ge 0$, relation (39) yields

$$\lim_{\lambda\to 0} J_2(\alpha, \lambda) = 0. \tag{40}$$

Therefore, from (39) and (40), we get, as $\lambda \to 0$:

$$EU_\eta(\zeta_1 - \frac{x}{\lambda}) = \left(\frac{x^\alpha e^{-x}}{m_1\Gamma(\alpha)} + \frac{\alpha - x}{m_1}(1 - G_\alpha(x))\right)\frac{1}{\lambda} + \frac{m_2}{2m_1^2}(1 - G_\alpha(x)) + o(1). \tag{41}$$

Substituting (37) and (41) into (31) and carrying out the corresponding calculations, as $\lambda \to 0$, we obtain

$$\begin{aligned} Q_{W_\lambda}(x) &= \frac{x}{\alpha}(1 - G_\alpha(x)) - \frac{x^\alpha e^{-x}}{\alpha\Gamma(\alpha)} + G_\alpha(x) \\ &+ \frac{m_2}{2m_1\alpha}\left(\frac{x^\alpha e^{-x}}{\alpha\Gamma(\alpha)} - \frac{x}{\alpha}(1 - G_\alpha(x))\right)\lambda + o(\lambda). \end{aligned} \tag{42}$$

As $\lambda \to 0$, we can rewrite (42) as

$$Q_{W_\lambda}(x) = R_\alpha(x) + \frac{m_2}{2m_1\alpha}\left(G_\alpha(x) - R_\alpha(x)\right)\lambda + o(\lambda), \tag{43}$$

where

$$R_\alpha(x) = \frac{x}{\alpha}\left(1 - G_\alpha(x)\right) - \frac{x^\alpha e^{-x}}{\alpha\Gamma(\alpha)} + G_\alpha(x).$$

Let's simplify $R_\alpha(x)$:

$$\begin{aligned} R_\alpha(x) &= \frac{x}{\alpha}\left(1 - G_\alpha(x)\right) - \frac{x^\alpha e^{-x}}{\alpha\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)}\int_0^x t^{\alpha-1}e^{-t}dt \\ &= \frac{x}{\alpha}\left(1 - G_\alpha(x)\right) + \frac{1}{\alpha\Gamma(\alpha)}\int_0^x t^\alpha e^{-t}dt = \frac{1}{\alpha}\int_0^x \left(1 - G_\alpha(t)\right)dt. \end{aligned}$$

Consequently, we have

$$R_\alpha(x) = \frac{1}{\alpha}\int_0^x \left(1 - G_\alpha(t)\right)dt. \tag{44}$$

Finally, substituting (44) into (43) yields (30).

This completes the proof of Theorem 4.1. □

Now, we can prove the weak convergence theorem for the ergodic distribution function $(Q_{W_\lambda}(x))$ of the process $W_\lambda(t) \equiv \lambda(X(t) - s)$, as $\lambda \to 0$.

Theorem 4.2. *(Weak convergence theorem). Under the conditions of Theorem 4.1, for each $x \geq 0$ and $\alpha > 0$, we have*

$$\lim_{\lambda \to 0} Q_{W_\lambda}(x) = R_\alpha(x) \equiv \frac{1}{\alpha}\int_0^x \left(1 - G_\alpha(t)\right)dt. \tag{45}$$

Here, $G_\alpha(x) = \frac{1}{\Gamma(\alpha)}\int_0^x t^{\alpha-1}e^{-t}dt$.

Proof. Since $R_\alpha(x)$ and $G_\alpha(x)$ are distribution functions, we have $0 \leq R_\alpha(x) \leq 1$ and $0 \leq G_\alpha(x) \leq 1$ for each $x \geq 0$, $\alpha > 0$. Therefore, for each $\alpha > 0$, we have

$$\max_x \left|G_\alpha(x) - R_\alpha(x)\right| \leq 1. \tag{46}$$

Since $m_2 = E(\eta_1^2) < \infty$, we have

$$\frac{m_2}{2m_1\alpha}\left|G_\alpha(x) - R_\alpha(x)\right| \leq \frac{m_2}{2m_1\alpha} < \infty \tag{47}$$

for each $\alpha > 0$, by using (46). Then the second term in the asymptotic expansion (30) tends to zero, as $\lambda \to 0$.

In other words, the ergodic distribution of the process $W_\lambda(t)$ weakly converges to the limit distribution $R_\alpha(x)$, as $\lambda \to 0$, i.e., for each $x \geq 0$ and $\alpha > 0$,

$$Q_{W_\lambda}(x) \to R_\alpha(x) \equiv \frac{1}{\alpha}\int_0^x \left(1 - G_\alpha(t)\right)dt.$$

This completes the proof of Theorem 4.2. □

Remark 4.1. So, we have obtained the asymptotic expansion for the ergodic distribution function of $W_\lambda(t)$. It is an important mathematical problem to see how close these expansions are to the exact expressions. In this study, we do not consider this problem in detail but investigate a special case to determine how much the obtained expansions could be close to the exact expressions, by using the Monte-Carlo simulation method.

5. SIMULATION RESULTS

In the previous sections, the main aim of this study has been attained. However, it is advisable to test the adequateness of approximate formulas to the exact ones. For this purpose, using the Monte-Carlo experiments, we can give the following simulation results. First, let's suppose that the random variable ζ_1 has the gamma distribution with the parameters $(\alpha = 5; \lambda = 1)$ and $(\alpha = 5; \lambda = 0.5)$, and the random variable η_1 has the second-order Erlang distribution with the parameter $\mu = 10$. In addition, we assume that $\hat{Q}_{W_\lambda}(x)$ denotes the value of the ergodic distribution function of the process $W_\lambda(t)$ which is calculated by using the Monte-Carlo simulation method and $\tilde{Q}_{W_\lambda}(x)$ denotes the value of the first two terms of the asymptotic expansion given by Theorem 4.1. We define

$$\Delta_k = \left|\hat{Q}_{W_\lambda}(x) - \tilde{Q}_{W_\lambda}(x)\right|; \quad \delta_k = \frac{\Delta_k}{\hat{Q}_{W_\lambda}(x)} 100\%; \quad AP_k = 100 - \delta_k, \ k = 1, 2.$$

In other words, the numbers Δ_k, δ_k, and AP_k denote the absolute error, relative error, and accuracy percentage between the simulation and asymptotic results for the ergodic distribution function of the process $W_\lambda(t)$, respectively. So we can generate Tables 1 and 2.

Table 1. $\lambda = 1$

x	$\hat{Q}_{W_\lambda}(x)$	$\tilde{Q}_{W_\lambda}(x)$	Δ_1	δ_1 (%)	AP_1 (%)
0.2	0.03890	0.038800063	0.0000999370	0.256907457	99.74309254
0.4	0.07770	0.077601225	0.0000987749	0.127123456	99.87287654
0.6	0.11650	0.116404990	0.0000950100	0.081553624	99.91844638
0.8	0.15530	0.155206876	0.0000931239	0.059963884	99.94003612
1.0	0.19410	0.193988040	0.0001119600	0.057681748	99.94231825
1.2	0.23280	0.232708602	0.0000913983	0.039260430	99.96073957
1.4	0.27140	0.271304393	0.0000956067	0.035227236	99.96477276
1.6	0.30975	0.309687006	0.0000629945	0.020337200	99.97966280
1.8	0.34780	0.347746650	0.0000533505	0.015339410	99.98466059
2.0	0.38535	0.385356902	0.0000690164	0.001791006	99.99820899

Table 2. $\lambda = 0.5$

x	$\hat{Q}_{W_\lambda}(x)$	$\tilde{Q}_{W_\lambda}(x)$	Δ_2	δ_2 (%)	AP_2 (%)
0.2	0.03936000	0.039400029	0.0000400289	0.10169942600	99.89830057
0.4	0.07874000	0.078800294	0.0000602936	0.07657299600	99.92342700
0.6	0.11815000	0.118198945	0.0000489455	0.04142657200	99.95857343
0.8	0.15754000	0.157585084	0.0000450844	0.02861776100	99.97138224
1.0	0.19692000	0.196931075	0.0000110752	0.00562422100	99.99437578
1.2	0.23618000	0.236187033	0.0000703335	0.00297796300	99.99702204
1.4	0.27528000	0.275278747	0.0000125294	0.00045515200	99.99954485
1.6	0.31411000	0.314108700	0.0000129951	0.00041371100	99.99958629
1.8	0.35255960	0.352559624	0.0000237329	0.00000673161	99.99999327
2.0	0.39049964	0.390499642	0.0000166158	0.00000425500	99.99999957

6. Conclusion

In this study, a renewal-reward process $(X(t))$ with a discrete interference of chance is investigated. The weak convergence theorem is proved for the ergodic distribution of the process $W_\lambda(t) \equiv \lambda(X(t) - s)$, as $\lambda \to 0$. Moreover, an exact expression for the limit distribution is derived, when the random variable ζ_1, which describes the discrete interference of chance, has a gamma distribution with parameters (α, λ). Finally, the accuracy of the approximation formula is tested by the Monte Carlo simulation method. For the calculation of each value of $\hat{Q}_{W_\lambda}(x)$ in Tables 1 and 2, we taken 10^8 realizations of the process $X(t)$. As seen from Tables 1 and 2, the approximate formulas provide a high accuracy even for the values of the parameter λ which are not very small. For example, as seen from the tables, the accuracy percentages (AP_k) are greater than %99, for all values of the parameter x from the interval $[0.2, 2.0]$, when $\lambda = 1$ and $\lambda = 0.5$. This indicates that the asymptotic expansion obtained can safely be applied to different problems of inventory or queuing models, even for not small values of the parameter λ.

Note that it is of interest to obtain the similar results for others types of discrete interference of chance by using the methods introduced in this paper.

Acknowledgement: The authors express their gratitude to Academician of the National Academy of Sciences of Ukraine, Professor of Michigan State University A.V. Skorohod for his support and encouragement which have led them to the investigations of the stochastic processes with a discrete interference of chance and their applications to inventory theory.

References

1. G. Alsmeyer, *Second-order approximations for certain stopped sums in extended renewal theory*, Advances in Applied Probability, **20** (1988), 391-410.
2. M. Brown, H. Solomon, *A second-order approximation for the variance of a renewal reward process*, Stochastic Processes and Applications, **3** (1975), 301-314.
3. A. Csenki, *Asymptotics for renewal-reward processes with retrospective reward structure*, Operation Research and Letters, **26** (2000), 201-209.
4. W. Feller, *An Introduction to Probability Theory and Its Applications II.*, Wiley, New York, 1971.
5. I.I. Gikhman, A.V. Skorohod, *Theory of Stochastic Processes II*, Springer, Berlin, 1975.
6. J.B. Levy, M.S. Taqqu, *Renewal reward processes with heavy-tailed inter-renewal times and heavy-tailed rewards*, Bernoulli, **6** (1) (2000), 23-44.
7. W.S. Jewell, *Fluctuation of a Renewal-Reward Process*, Journal of Mathematical Analysis and Applications, **19** (1967), 309-329.
8. T.A. Khaniyev, *About the moments of a generalized renewal process*, Transactions of NAS of Azerbaijan, Series of Physical Technical and Mathematical Sciences, **25** (1) (2005), 95-100.
9. T.A. Khaniyev, T. Kesemen, R.T. Aliyev, A. Kokangul, *Asymptotic expansions for the moments of a semi-Markovian random walk with an exponential distributed interference of chance*, Statistics and Probability Letters, **78** (6) (2008), 785-793.
10. S.M. Ross, *Stochastic Processes*, John Wiley & Sons, New York, 1996.

Baku State University, Faculty of Applied Mathematics and Cybernetics, Department of Probability Theory and Mathematical Statistics, Z. Khalilov 23, Az 1148, Baku, Azerbaijan

Current address: Karadeniz Technical University, Faculty of Arts and Sciences, Department of Statistics and Computer Sciences, 61080, Trabzon, Turkey

E-mail address: aliyevrovshan@yahoo.com

Institute of Cybernetics of Azerbaijan National Academy of Sciences, F. Agayev str.9, Baku, Az 1141, Azerbaijan

Current address: TOBB University of Economics and Technology, Faculty of Engineering, Department of Industrial Engineering, 06560, Sogutozu, Ankara, Turkey

E-mail address: tahirkhaniyev@etu.edu.tr, khaniyevtahir@yahoo.com

Karadeniz Technical University, Faculty of Arts and Sciences, Department of Mathematics, 61080, Trabzon, Turkey

E-mail address: nrgokur@gmail.com