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GEROLD ALSMEYER AND MATTHIAS MEINERS

A MIN-TYPE STOCHASTIC FIXED-POINT EQUATION RELATED TO THE SMOOTHING TRANSFORMATION

This paper is devoted to the study of the stochastic fixed-point equation

$$X \stackrel{d}{=} \inf_{i\geq 1: T_i>0} X_i/T_i$$

and the connection with its additive counterpart $X \stackrel{d}{=} \sum_{i\geq 1} T_i X_i$ associated with the smoothing transformation. Here $\stackrel{d}{=}$ means equality in distribution, $T := (T_i)_{i\geq 1}$ is a given sequence of non-negative random variables, and $X, X_1, \ldots$ is a sequence of non-negative i.i.d. random variables independent of T. We draw attention to the question of the existence of non-trivial solutions and, in particular, of special solutions named α-regular solutions ($\alpha > 0$). We give a complete answer to the question of when α-regular solutions exist and prove that they are always mixtures of Weibull distributions or certain periodic variants. We also give a complete characterization of all fixed points of this kind. A disintegration method which leads to the study of certain multiplicative martingales and a pathwise renewal equation after a suitable transform are the key tools for our analysis. Finally, we provide corresponding results for the fixed points of the related additive equation mentioned above. To some extent, these results have been obtained earlier by Iksanov.

1. Introduction

For a given sequence $T := (T_i)_{i\geq 1}$ of non-negative random variables on a probability space $(\Omega, \mathfrak{A}, \mathbb{P})$ with $\sup_{i\geq 1} T_i > 0$ a.s., consider the stochastic fixed-point equation (SFPE)

$$X \stackrel{d}{=} \inf_{i\geq 1} X_i/T_i \tag{1}$$

where $X, X_1, X_2, \ldots$ are i.i.d., non-negative and independent of T, and $X_i/T_i := \infty$ is stipulated on $\{T_i = 0\}$. A distribution F on $[0,\infty)$ is called a solution to (1) if this equation holds true with $X \stackrel{d}{=} F$, and it is called positive if $F(\{0\}) = 0$. Note that $F = \delta_0$, δ_0 the Dirac measure at 0, always provides a trivial solution. The set of all solutions $\neq \delta_0$ will be denoted as $\mathfrak{F}_\wedge$ hereafter, or as $\mathfrak{F}_\wedge(T)$ if we want to emphasize its dependence on T. We will make no notational distinction between a distribution F and its left-continuous distribution function, and we denote, by $\overline{F}$, the associated survival function, *i.e.*, $\overline{F} := 1 - F$. For $F \in \mathfrak{F}_\wedge$, Eq. (1) can be rewritten in terms of $\overline{F}$ as

$$\overline{F}(t) \;=\; \mathbb{E}\prod_{i\geq 1}\overline{F}(tT_i) \tag{2}$$

for all $t \geq 0$. Denote, by $\mathcal{P}, \overline{\mathcal{P}}$, the spaces of probability measures on $[0,\infty)$ and $[0,\infty]$, respectively. Defining the map $M : \mathcal{P} \to \mathcal{P}$ by

$$M(F) \;:=\; \mathbb{P}\left(\inf_{i\geq 1}\frac{X_i}{T_i} \in \cdot\right), \quad X \stackrel{d}{=} F, \tag{3}$$

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we see that, formally speaking, $\mathfrak{F}_\wedge$ is nothing but the set of fixed points $\neq \delta_0$ of M, that is, $\mathfrak{F}_\wedge = \{F \in \mathcal{P} : M(F) = F\} \setminus \{\delta_0\}$.

SFPEs of type (1) or similar with a min- or max-operation involved turn up in various fields of applied probability like the probabilistic combinatorial optimization and the run-time analysis of divide-and-conquer algorithms or branching particle systems, where they typically characterize the asymptotic distribution of some random variable of interest (see *e.g.* [1], [2], and [21]). In particular, the probabilistic worst-case analysis of Hoare's `FIND` algorithm leads to the following fixed-point equation for the distributional limit X of the linearly scaled maximal number of key comparisons of `FIND`.

$$X \stackrel{d}{=} 1 + \max(UX_1, (1-U)X_2),$$

where U is a uniform $[0,1]$ random variable and X_1, X_2 are independent copies of the random variable X, see [12, Theorem 1]. More generally, in the analysis of divide-and-conquer algorithms, equations of the form

$$X \stackrel{d}{=} \max_{i=1,\ldots,K}(A_i X_i + b_i)$$

appear (*cf.* [21] and [22]). Further, from a species competition model, the fixed-point equation

$$X \stackrel{d}{=} \eta + c \max_{i\geq 1} e^{-\xi_i} X_i$$

arises, where $(\xi_i)_{i\geq 1}$ are the points of a Poisson process at rate 1 and η is an Exp(1) variable independent of the process $(\xi_i)_{i\geq 1}$, see [1, Example 38]. By Theorem 4.2 in Rüschendorf [22], under appropriate conditions on the random coefficients of the SFPEs above, there is a one-to-one relationship between the solutions of the max-type equation and the corresponding homogeneous equation. Therefore, it is convenient to study the homogeneous equation

$$X \stackrel{d}{=} \sup_{i\geq 1} A_i X_i \tag{4}$$

which is equivalent to Eq. (1) by an application of the involution $x \mapsto x^{-1}$. (Note that if, in Eq. (4), X has an atom at 0, then this atom becomes an atom at ∞ in the equivalent equation (1). This situation is not explicitly covered by the subsequent analysis but the results of this article remain true (after some minor changes) if an atom in ∞ is permitted.)

A first systematic approach to Eq. (1) was given by Jagers and Rösler [17] who pointed out the connection of (1) with its additive counterpart

$$X \stackrel{d}{=} \sum_{i\geq 1} T_i X_i \tag{5}$$

and the corresponding map $M_\Sigma : \mathcal{P} \to \overline{\mathcal{P}}$, defined by

$$M_\Sigma(F) := \mathbb{P}\left(\sum_{i\geq 1} T_i X_i \in \cdot\right), \quad X \stackrel{d}{=} F, \tag{6}$$

which is usually called the *smoothing transformation* due to Durrett and Liggett [13]. Namely, rewriting (5) in terms of the Laplace transform φ, say, of X, we obtain

$$\varphi(t) = \mathbb{E} \prod_{i\geq 1} \varphi(tT_i) \tag{7}$$

for all $t \geq 0$, which is the direct analog of (2). But since any Laplace transform vanishing at ∞ and thus pertaining to a distribution on $(0,\infty)$ can also be viewed as the survival function of a (continuous) probability distribution on $[0,\infty)$, one has the implication

$$\mathfrak{F}_\Sigma \neq \emptyset \quad \Longrightarrow \quad \mathfrak{F}_\wedge \neq \emptyset, \tag{8}$$

where $\mathfrak{F}_\Sigma$ denotes the set of all positive solutions to (5). Defining $T^{(\alpha)} := (T_i^\alpha)_{i\geq 1}$, one further has

$$\mathfrak{F}_\Sigma(T^{(\alpha)}) \neq \emptyset \text{ for some } \alpha > 0 \quad \Longrightarrow \quad \mathfrak{F}_\wedge \neq \emptyset, \tag{9}$$

owing to the fact that $\mathfrak{F}_\wedge = \{\mathbb{P}(X^{1/\alpha} \in \cdot) : \mathbb{P}(X \in \cdot) \in \mathfrak{F}_\wedge(T^{(\alpha)})\}$. However, Jagers and Rösler also give an example which shows that this implication cannot be reversed. We take up their example (the water cascades example) in Section 8 in a generalized form.

Eq. (1) for the situation where the T_i, $i \geq 1$, are deterministic but not necessarily non-negative is discussed in detail by Alsmeyer and Rösler [6]. Their results concerning the case of non-negative weights can be summarized as follows: Except for simple cases, nontrivial fixed-points exist iff T possesses a *characteristic exponent*, defined as the unique positive number $\alpha > 0$ such that $\sum_{i\geq 1} T_i^\alpha = 1$. In this case, the set of solutions $\mathfrak{F}_\wedge$ can be described as follows: For $\beta > 0$ and $r > 1$, let $\mathfrak{H}(r,\beta)$ be the set of left continuous, multiplicatively r-periodic functions $h : (0,\infty) \to (0,\infty)$ such that $t \mapsto h(t)t^\beta$ is nondecreasing. The distribution F on $(0,\infty)$ with the survival function

$$\overline{F}(t) \ := \ e^{-h(t)t^\beta}, \quad t > 0$$

is then called r-periodic Weibull distribution with parameters h and β, in short r-Weibull(h,β). Put $\mathfrak{W}(r,\beta) := \{r\text{-Weibull}(h,\beta) : h \in \mathfrak{H}(r,\beta)\}$ for $\beta > 0$ and $r > 1$, and let $\mathfrak{W}(1,\beta) := \{\text{Weibull}(c,\beta) : c > 0\}$ denote the set of ordinary Weibull distributions with parameter β, *i.e.*, the set of distributions F having the survival function $\overline{F}(t) = \exp(-ct^\beta)$ $(t \geq 0)$ for some positive constant c. Then $\mathfrak{F}_\wedge = \text{Weibull}(1,\alpha)$ or $\mathfrak{F}_\wedge = \mathfrak{W}(r,\alpha)$, respectively, depending on whether the closed multiplicative subgroup $\subseteq \mathbb{R}^+ = (0,\infty)$ generated by the positive T_i, which we denote by $\mathbb{G}(T)$, equals $\mathbb{R}^+$ or $r^{\mathbb{Z}}$ for some $r > 1$. In view of this result, the result of Alsmeyer and Rösler extends the classical results on extreme value distributions and the problem addressed in this paper, the analysis of Eq. (1) is a further generalization of the analysis of extreme values, namely, of the distributional equation of homogeneous min-stability for, in our situation, the scaling factor is replaced by random coefficients.

One purpose of this paper is to investigate under which conditions similar results hold true in the situation of random weights T_i, $i \geq 1$, *i.e.*, in which cases Weibull distributions or suitable mixtures of them are solutions to (1). This calls for extended definitions of $\mathbb{G}(T)$ and of the characteristic exponent: We define $\mathbb{G}(T)$ as the minimal closed multiplicative subgroup $\mathbb{G} \subseteq \mathbb{R}^+$ such that $\mathbb{P}(T_i \in \mathbb{G} \cup \{0\} \text{ for all } i \in \mathbb{N}) = 1$. We further define $m : [0,\infty) \mapsto [0,\infty]$ by

$$m(\beta) \ := \ \mathbb{E}\sum_{i\geq 1} T_i^\beta \tag{10}$$

and then the characteristic exponent as the minimal positive α such that $m(\alpha) = 1$ if such an α exists. With these generalizations, we obtain a connection between certain Weibull mixtures and *α-regular solutions*, defined as solutions F to (1) such that the ratio $t^{-\alpha}(1 - \overline{F}(t))$ stays bounded away from 0 and ∞ as t approaches 0. Indeed, Theorem 4.2 will show that any α-regular fixed point is a mixture of Weibull distributions with parameter α and particularly *α-elementary*, which means that $t^{-\alpha}(1 - \overline{F}(t))$ converges to a positive constant as $t \downarrow 0$ through a residue class relative to $\mathbb{G}(T)$. Furthermore, Theorem 4.1 will provide an exact characterization of when α-elementary solutions to Eq. (1) exist. Both, the existence of Weibull mixtures as fixed points and the existence of regular fixed points, are related to the existence of the characteristic exponent, which also plays a fundamental role in the analysis of Eq. (5). The further organization of this article is as follows. Section 2 provides a discussion of trivial and simple cases of (1) which will be excluded thereafter. An introduction of the weighted branching model closely related to our SFPE (1) is given in Section 3, followed by the presentation and discussion of the main results in Section 4. Section 5 contains the derivation of a certain pathwise renewal equation related to (1) via disintegration, while Section 6 is devoted to a study of the characteristic exponent. It contains most of the necessary prerequisites to prove Theorem 4.1 and Theorem 4.2 which is done in Section 7. Here the aforementioned pathwise renewal equation will form a key ingredient. As already mentioned, Section 8 provides a discussion of Eq. (1) for a family of examples where the characteristic exponent does not generally exist. Finally, Section 9 contains some results for Eq. (5) which are closely related to ours and can be derived by the same methods. Theorem 9.1 and Theorem 9.2 are extensions of Theorem 2 and Proposition 3 in [16].

2. Basic results and simple cases

This section is devoted to a brief discussion of simple cases and a justification of the following two basic assumptions on T: Put $N := \sum_{i\geq 1} \mathbf{1}_{\{T_i>0\}}$ and consider

$$0 \;<\; \mathbb{P}(N>1) \;\leq\; \mathbb{P}(N\geq 1) \;=\; 1; \tag{A1}$$

$$\mathbb{P}\left(\sup_{i\geq 1} T_i < 1\right) > 0. \tag{A2}$$

By our standing assumption, $\mathbb{P}(N=0) = \mathbb{P}(\sup_{i\geq 1} T_i = 0) = 0$. Hence, if (A1) fails, then $N=1$ a.s. and Eq. (1) reduces to $X \stackrel{d}{=} TX$, where T is independent of X and a.s. positive. But this SFPE can easily be solved, namely $\mathfrak{F}_\wedge \neq \emptyset$ iff $T=1$ a.s., see *e.g.* Liu [18, Lemma 1.1]. Validity of condition (A1) will therefore always be assumed hereafter. As a consequence, the branching process with offspring distribution $\mathbb{P}(N\in\cdot)$ (of simple Galton–Watson type if $N<\infty$ a.s.) explodes with probability 1, a fact that will be used later.

The justification of assumption (A2) is slightly more involved and based upon the following two propositions:

Proposition 2.1. *Suppose that* (A1) *holds true. Then* $\mathfrak{F}_\wedge = \emptyset$ *if*

$$\sup_{i\geq 1} T_i \;\geq\; 1 \text{ a.s.} \quad \text{and} \quad \mathbb{P}\left(\sup_{i\geq 1} T_i > 1\right) \;>\; 0.$$

Proof. Let F be a solution to (1). Then Eq. (2) gives

$$\overline{F}(t) \;=\; \mathbb{E}\prod_{i\geq 1} \overline{F}(tT_i) \;\leq\; \mathbb{E}\overline{F}\left(t\sup_{i\geq 1} T_i\right) \;\leq\; \overline{F}(t)$$

and thus $\overline{F}(t) = \mathbb{E}\overline{F}(t\sup_{i\geq 1} T_i)$ for all $t\geq 0$. Let $(Y_i)_{i\geq 1}$ be a sequence of i.i.d. copies of $\sup_{i\geq 1} T_i$ with associated multiplicative random walk $(\Pi_n)_{n\geq 0}$, *i.e.*, $\Pi_0 := 1$ and $\Pi_n := Y_1 \cdot \ldots \cdot Y_n$ for $n\geq 1$. Then $\overline{F}(t) = \mathbb{E}\overline{F}(t\Pi_n)$ for each n. Our assumptions on $\sup_{i\geq 1} T_i$ ensure $\Pi_n \uparrow \infty$ a.s., whence $\overline{F}(t) = 0$ for all $t>0$, that is, $F = \delta_0$. □

Before proceeding with our second proposition, let us note in passing that any $\sigma(T)$-measurable finite or infinite rearrangement $T_\pi := (T_{\pi(1)}, T_{\pi(2)}, \ldots)$ of T leaves the set of solutions to our SFPE (1) unchanged because the X_i are i.i.d. and independent of T, thus also of T_π. So $\mathfrak{F}_\wedge(T) = \mathfrak{F}_\wedge(T_\pi)$. As a consequence, it is no loss of generality to assume $T_1 = \sup_{i\geq 1} T_i$ whenever the supremum is a.s. attained.

Proposition 2.2. *If* (A1) *holds, the following assertions are equivalent:*

(a) $\sup_{i\geq 1} T_i = 1$ *a.s.*

(b) *There exists* $0 < \gamma \le 1$ *such that* $\mathfrak{F}_\wedge = \{F \in \mathcal{P} : F([\gamma c, c]) = 1$ *for some* $c > 0\}$.
(c) $\delta_c \in \mathfrak{F}_\wedge$ *for all* $c > 0$.
(d) $\delta_c \in \mathfrak{F}_\wedge$ *for some* $c > 0$.

Proof. The implications "(b) ⇒ (c) ⇒ (d) ⇒ (a)" being obvious, we must only prove "(a) ⇒ (b)". Define $\gamma := 1$ if $\mathbb{P}(\exists\, i \in \mathbb{N} : T_i = 1) < 1$, and $\gamma := \operatorname{ess\,sup}_{i\ge 2} T_i$ if $\mathbb{P}(\exists\, i \in \mathbb{N} : T_i = 1) = 1$ and w.l.o.g. $T_1 = \sup_{i\ge 1} T_i$. Note for the latter situation that $\gamma > 0$ by (A1). The two cases $0 < \gamma < 1$ (Case 1) and $\gamma = 1$ (Case 2) will now be discussed separately.

Case 1. Pick any $F \in \mathfrak{F}_\wedge$, and let $X, X_1, X_2, \ldots$ denote a sequence of i.i.d. random variables with common distribution F. As $T_1 = 1$ a.s. the SFPE is

$$X \stackrel{d}{=} X_1 \wedge \inf_{i\ge 2} X_i/T_i \tag{11}$$

and clearly entails $X_1 \le \inf_{i\ge 2} X_i/T_i$ a.s. Now use the independence of the random variables X_1 and $\inf_{i\ge 2} X_i/T_i$ to infer that this can only hold if

$$X_1 \;\le\; c \;\le\; \inf_{i\ge 2} X_i/T_i \quad \text{a.s.}$$

for some c, w.l.o.g. $c := \operatorname{ess\,sup} X_1$ which is positive, for $F \ne \delta_0$. It remains to show that $X \ge \gamma c$ a.s. Assuming the contrary, we also have $\mathbb{P}(X < u\gamma c) > 0$ for some $u \in (0,1)$. Since $\gamma = \operatorname{ess\,sup}_{i\ge 2} T_i$ in the present case, the stopping time $\nu := \inf\{i \ge 2 : T_i > u\gamma\}$ is finite with positive probability, and we have that $\mathbb{P}(X_\nu \in \cdot, \nu < \infty) = \mathbb{P}(X \in \cdot)$ because X_i and T are independent. But this leads to the following contradiction:

$$0 \;=\; \mathbb{P}\left(\inf_{i\ge 2} \frac{X_i}{T_i} < c\right) \;\ge\; \mathbb{P}\left(\nu < \infty, X_\nu < u\gamma c\right) \;\ge\; \mathbb{P}(\nu < \infty)\,\mathbb{P}\left(X < u\gamma c\right) \;>\; 0.$$

So we have proved $F([\gamma c, c]) = 1$. Conversely, if we pick any $F \in \mathcal{P}$ with this property for some $c > 0$, then $F \in \mathfrak{F}_\wedge$ follows immediately from (11), because we have there $X_1 \wedge \inf_{i\ge 2} X_i/T_i = X_1$ a.s.

Case 2. If $\gamma = 1$, we cannot assume $T_1 = \sup_{i\ge 1} T_i$ and then resort to the above argument because, with positive probability, the supremum may not be attained. On the other hand, the claim reduces here to $\mathfrak{F}_\wedge = \{\delta_c : c > 0\}$, and it is easily verified that any δ_c is indeed a solution. For the reverse inclusion, pick any solution F and suppose it is not concentrated at a single point, thus $\overline{F}(s) \in (0,1)$ for some $s > 0$. For $r \in (0,1)$, define a r.v. U_r as follows:

$$U_r \;:=\; \begin{cases} \sup_{i\ne k} T_i & \text{if there is a } k \ge 1 \text{ such that } T_k = 1, \\ T_{\tau_r} & \text{if } T_i < 1 \text{ for all } i \ge 1, \end{cases}$$

where $\tau_r := \inf\{i \ge 1 : r < T_i < 1\}$. Observe that $N = \infty$ in the second case and $U_{r_k} \uparrow 1$ for any choice $r_k \uparrow 1$, so that $\prod_{i\ne\tau_r} \overline{F}(tT_i) \le \lim_{k\to\infty} \overline{F}(tU_{r_k}) = \overline{F}(t)$ by left continuity. For any $t \ge 0$, we now infer

$$\overline{F}(t) \;=\; \mathbb{E}\prod_{i\ge 1} \overline{F}(tT_i) \;\le\; \overline{F}(t)\cdot\mathbb{E}\overline{F}(tU_r)$$

and therefore $\mathbb{E}\overline{F}(tU_r) = 1$ for any t such that $\overline{F}(t) \in (0,1)$. By left continuity, $\overline{F}(rt) < 1$ for any such t and some $r \in (0,1)$. However, $\mathbb{P}(U_r > r) > 0$ then leads to the contradiction

$$1 \;=\; \mathbb{E}\overline{F}(tU_r) \;\le\; \mathbb{P}(U_r \le r) + \overline{F}(rt)\,\mathbb{P}(U_r > r) \;<\; 1.$$

We hence conclude that F must be concentrated at a single point. □

Remark 2.1. As a particular consequence of Proposition 2.2, all solutions to (1) have compact support if (A1) and $\sup_{i\geq 1} T_i = 1$ a.s. hold true. As to a reverse conclusion, let us point out the following:

If (A1) holds true and $\mathbb{P}(N<\infty) = 1$, then the assertions

(a) $\sup_{i\geq 1} T_i = 1$ a.s.,
(b) there exists $F \in \mathfrak{F}_\wedge$ with compact support

are equivalent.

With only "(b) ⇒ (a)" to be proved, let F be an element of $\mathfrak{F}_\wedge$ with compact support and let X be a random variable with distribution F, so $C := \operatorname{ess\,sup} X \in (0,\infty)$. Suppose now there exists $r \in (0,1)$ such that $q := \mathbb{P}(\sup_{i\geq 1} T_i \leq r) > 0$. We can pick r and $t > C$ in such a way that $rt < C < t$ and thus $\overline{F}(rt) > \overline{F}(t) = 0$. Then

$$0 \;=\; \overline{F}(t) \;=\; \mathbb{E}\prod_{i\geq 1}\overline{F}(tT_i) \;\geq\; \mathbb{E}\prod_{i\geq 1}\overline{F}(tT_i)\,\mathbf{1}_{\{\sup T_i\leq r\}} \;\geq\; \mathbb{E}\overline{F}(rt)^N\mathbf{1}_{\{\sup T_i\leq r\}} \;>\; 0,$$

which is a contradiction. Consequently, $\mathbb{P}(\sup_{i\geq 1} T_i \geq 1) = 1$ which in combination with $\mathfrak{F}_\wedge \neq \emptyset$ and Proposition 2.1 proves (a).

We close this section with a lemma that shows that any $F \in \mathfrak{F}_\wedge$ is continuous at 0 and that a search for solutions putting mass on $[0,\infty)$ is actually no restriction.

Lemma 2.1. *Suppose* (A1)*, and let* $F \neq \delta_0$ *be any distribution on* $\mathbb{R}$ *solving Eq.* (1)*. Then* F *is continuous at* 0 *and concentrated either on* $[0,\infty)$ *or* $(-\infty,0]$*. In the latter case, if* $X \stackrel{d}{=} F$ *and* G *denotes the distribution of* $-X^{-1}$ *(*$<\infty$ *a.s.), then* $G \in \mathfrak{F}_\wedge(T^{-1})$*, where* $T^{-1} := (T_i^{-1}\mathbf{1}_{\{T_i>0\}})_{i\geq 1}$.

Proof. Let $X, X_1, X_2, \ldots$ be i.i.d. with distribution F and independent of T. In view of what has been mentioned before Proposition 2.2, we may assume w.l.o.g. that $T_i > 0$ iff $N \geq i$. Then

$$\overline{F}(0) \;=\; \mathbb{P}(X\geq 0) \;=\; \mathbb{P}(X_i \geq 0 \text{ for } 1\leq i\leq N) \;=\; \mathbb{E}\overline{F}(0)^N,$$

whence $\overline{F}(0)$ must be a fixed point of the generating function of N in $[0,1]$. Now use (A1) to infer $\overline{F}(0) \in \{0,1\}$. Next consider $\mathbb{P}(X>0) = \overline{F}(0+)$ and suppose it to be <1. Then we infer, with the help of (1),

$$\overline{F}(0+) \;=\; \lim_{t\downarrow 0}\overline{F}(t) \;\leq\; \lim_{t\downarrow 0}\mathbb{E}\prod_{i=1}^{N\wedge n}\overline{F}(tT_i) \;=\; \mathbb{E}\overline{F}(0+)^{N\wedge n}$$

for each $n \geq 1$ and thereupon $\overline{F}(0+) \leq \mathbb{E}\overline{F}(0+)^N\mathbf{1}_{\{N<\infty\}}$. On the other hand, by another appeal to (A1), $\mathbb{E}s^N \leq s$ for each $s \in [0,1)$ with equality holding iff $s = 0$. Consequently, $\overline{F}(0+) = 0$, which is clearly impossible as $F \neq \delta_0$. We thus conclude $\overline{F}(0) = \overline{F}(0+) = 1$ and thereby $\mathbb{P}(X = 0) = \overline{F}(0) - \overline{F}(0+) = 0$, which proves the continuity of F at 0. As for the final assertion, it suffices to note that $X \stackrel{d}{=} \inf_{i\geq 1} X_i/T_i$ is clearly equivalent to $-X^{-1} \stackrel{d}{=} \inf_{i\geq 1}(-X_i^{-1})T_i$. □

Unless stated otherwise, we will always assume (A1) and (A2) hereafter. As a consequence of (A2), we infer that the closed multiplicative subgroup $\mathbb{G}(T)$ generated by T cannot be $\{1\}$, the trivial subgroup. So we have either $\mathbb{G}(T) = r^{\mathbb{Z}}$ for some $r > 1$ (r-geometric case) or $\mathbb{G}(T) = \mathbb{R}^+$ (continuous case).

3. Connection with weighted branching processes

Let $\mathbb{V}$ be the infinite tree with vertex set $\bigcup_{n\in\mathbb{N}_0}\mathbb{N}^n$, where $\mathbb{N}^0$ contains only the empty tuple $\varnothing$. We abbreviate $v=(v_1,\ldots,v_n)$ by $v_1\ldots v_n$ and write vw for the vertex $(v_1,\ldots,v_n,w_1,\ldots,w_m)$, where $w=(w_1,\ldots,w_m)$. Furthermore, $|v|=n$ and $|v|<n$ will serve as shorthand notation for $v\in\mathbb{N}^n$ and $v\in\mathbb{N}^k$ for some $k<n$, respectively. $|v|\le n$, $|v|\ge n$ and $|v|>n$ are defined similarly. Let $(T(v))_{v\in\mathbb{V}}$ denote a family of i.i.d. copies of T. For the sake of brevity, suppose $T=T(\varnothing)$. Interpret $T_i(v)$ as a weight attached to the edge (v,vi) in the infinite tree $\mathbb{V}$. Then put $L(\varnothing):=1$ and

$$L(v) \ := \ T_{v_1}(\varnothing)\cdot\ldots\cdot T_{v_n}(v_1\ldots v_{n-1})$$

for $v=v_1\ldots v_n\in\mathbb{V}$. So $L(v)$ gives the total multiplicative weight along the unique path from $\varnothing$ to v. For $n\ge 1$, let $\mathcal{A}_n$ denote the σ-algebra generated by the sequences $T(v)$, $|v|<n$, *i.e.*,

$$\mathcal{A}_n \ := \ \sigma\left(T(v):|v|<n\right)$$

Put $\mathcal{A}_0:=\{\emptyset,\Omega\}$ and $\mathcal{A}_\infty:=\sigma(\mathcal{A}_n:\,n\ge 0)=\sigma(T(v):v\in\mathbb{V})$.

Let us further introduce the following bracket operator $[\cdot]_u$ for any $u\in\mathbb{V}$. Given any function $\Psi=\psi((T(v))_{v\in\mathbb{V}})$ of the weight ensemble $(T(v))_{v\in\mathbb{V}}$ pertaining to $\mathbb{V}$, define $[\Psi]_u:=\psi((T(uv))_{v\in\mathbb{V}})$ to be the very same function, but for the weight ensemble pertaining to the subtree rooted at u. Any branch weight $L(v)$ can be viewed as such a function, and we then obtain $[L(v)]_u=T_{v_1}(u)\cdot\ldots\cdot T_{v_n}(uv_1\ldots v_{n-1})$ if $v=v_1\ldots v_n$, and thus $[L(v)]_u=L(uv)/L(u)$ whenever $L(u)>0$.

The weighted branching process (WBP) associated with $(T(v))_{v\in\mathbb{V}}$ is now defined as

$$W_n \ := \ \sum_{|v|=n} L(v), \quad n\ge 0.$$

For any $\alpha\ge 0$, we can replace the $T(v)$ with $T^{(\alpha)}(v):=(T_i(v)^\alpha)_{i\ge 1}$ which leads to the branch weights $L^{(\alpha)}(v):=L(v)^\alpha$ and the associated WBP

$$W_n^{(\alpha)} \ := \ \sum_{|v|=n} L(v)^\alpha, \quad n\ge 0.$$

Note that $T^{(0)}(v)=(\mathbf{1}_{\{T_i(v)>0\}})_{i\ge 1}$, so that $W_n^{(0)}=\sum_{|v|=n}\mathbf{1}_{\{L(v)>0\}}$ counts the positive branch weights in generation n. If $N<\infty$ a.s., then $(W_n^{(0)})_{n\ge 0}$ forms a Galton–Watson process with offspring distribution $\mathbb{P}(N\in\cdot)$, for $W_1^{(0)}\stackrel{d}{=}N$. Suppose there exists an $\alpha>0$ such that $m(\alpha)\le 1$ with m as defined in (10). Then the sequence $(W_n^{(\alpha)})_{n\ge 0}$ constitutes a non-negative supermartingale with respect to $(\mathcal{A}_n)_{n\ge 0}$ and hence converges a.s. to $W^{(\alpha)}:=\liminf_{n\to\infty}W_n^{(\alpha)}$. By Fatou's lemma,

$$0 \ \le \ \mathbb{E}W^{(\alpha)} \ \le \ \liminf_{n\to\infty}\mathbb{E}W_n^{(\alpha)} \ = \ \lim_{n\to\infty} m(\alpha)^n \ \le \ 1,$$

which gives $W^{(\alpha)}=0$ a.s. if $m(\alpha)<1$. In the case $m(\alpha)=1$, we have the dichotomy $\mathbb{E}W^{(\alpha)}=0$ or $\mathbb{E}W^{(\alpha)}=1$ (*cf.* Theorem A.1 in Appendix, or Biggins [7], Lyons [19] and Alsmeyer and Iksanov [3] for details). Henceforth, let Λ_α and φ_α denote the distribution and the Laplace transform, respectively, of $W^{(\alpha)}$.

Remark 3.1. As $m(\alpha)=1$ and $\mathbb{E}W^{(\alpha)}=1$ (or, equivalently, $\mathbb{P}(W^{(\alpha)}>0)>0$) for some $\alpha>0$ will be a frequent assumption hereafter, it is noteworthy that this forces α to be the characteristic exponent of T, that is, the *minimal* $\beta>0$ with $m(\beta)=1$. For a proof using Theorem A.1, we refer to Corollary A.1 in Appendix. Due to our standing assumption (A1), Theorem A.1 further implies that $\mathbb{P}(W^{(\alpha)}>0)>0$ is actually equivalent to the a.s. positivity of $W^{(\alpha)}$.

As explained before Proposition 2.2, the mapping M defined in (3) is invariant under $\sigma(T)$-measurable rearrangements of the T_i, $i \geq 1$. It is therefore stipulated hereafter that $T_i > 0$ if, and only if, $1 \leq i \leq N$.

In order to provide the connection of the previously introduced weighted branching model with the SFPE (1), let $(X(v))_{v\in\mathbb{V}}$ be a family of independent copies of X which is also independent of $(T(v))_{v\in\mathbb{V}}$. If $X \stackrel{d}{=} F$, then the n-fold iteration of (1) yields

$$X \stackrel{d}{=} \inf_{|v|=n} \frac{X(v)}{L(v)} \tag{12}$$

for all $n \geq 0$, which becomes, in terms of the survival function $\overline{F}$,

$$\overline{F}(t) = \mathbb{E} \prod_{|v|=n} \overline{F}(tL(v)), \quad t \geq 0. \tag{13}$$

4. Main results

We continue with the statement of the two main results that will be derived in this article. Theorem 4.1 provides the connection between the existence of certain regular solutions to (1) and the existence of the characteristic exponent of T, while Theorem 4.2 is a representation result which states that any regular solution is a certain Weibull mixture (cf. Definition 4.2 below). The definition of an α-regular fixed point is a part of the following definition.

Definition 4.1. Let $\alpha > 0$ and $F \in \mathfrak{F}_\wedge$. Put $D_\alpha\overline{F}(t) := t^{-\alpha}(1 - \overline{F}(t))$ for $t > 0$. Then F is called

1. *α-bounded*, if $\limsup_{t\downarrow 0} D_\alpha\overline{F}(t) < \infty$.
2. *α-regular*, if $0 < \liminf_{t\downarrow 0} D_\alpha\overline{F}(t) \leq \limsup_{t\downarrow 0} D_\alpha\overline{F}(t) < \infty$.
3. *α-elementary*, if
   — in the case $\mathbb{G}(T) = \mathbb{R}^+$ — there exists some constant $c > 0$ such that $\lim_{t\downarrow 0} D_\alpha\overline{F}(t) = c$.
   — in the case $\mathbb{G}(T) = r^{\mathbb{Z}}$ for some $r > 1$ — for each $s \in [1, r)$, there exists a positive constant $h(s)$ such that $\lim_{n\to\infty} D_\alpha\overline{F}(sr^{-n}) = h(s)$ for each $s \in [1, r)$.

The sets of α-bounded, α-regular, and α-elementary fixed point are denoted by $\mathfrak{F}^\alpha_{\wedge,\mathrm{b}}$, $\mathfrak{F}^\alpha_{\wedge,\mathrm{r}}$, and $\mathfrak{F}^\alpha_{\wedge,\mathrm{e}}$, respectively.

Remark 4.1. (a) The notion of an *α-elementary* fixed point has been introduced by Iksanov [16] in his study of the smoothing transformation M_Σ given in (6) and the associated SFPE (5). His definition is the same as ours for the continuous case when replacing $D_\alpha\overline{F}$ with $D_\alpha\varphi$, where φ denotes the Laplace transform of a solution to (5).

(b) Given the existence of the characteristic exponent α, Guivarc'h [15] and later Liu [18] called a (non-negative) solution to (5) with Laplace transform φ *canonical* if it can be obtained as the stable transformation of a solution to the very same equation for the weight vector $T^{(\alpha)} = (T_i^\alpha)_{i\geq 1}$. If the latter solution has Laplace transform ψ, this means that $\varphi(t) = \psi(t^\alpha)$ for all $t \geq 0$. This definition appears to be more restrictive than that of an α-elementary fixed point, because the latter definition is valid for any $\alpha > 0$. On the other hand, once shown that an α-elementary fixed point actually exists only if α is the characteristic exponent of T (see Theorem 4.1), "α-elementary" (at least in the more restrictive sense of Iksanov) and "canonical" turn out to be just different names for the same objects (see Theorem 2 in [16] and also Theorem 4.2 below).

(c) Note, for the r-geometric case, that $\inf_{s\in(1,r]} h(s)$ must be positive, for $(sr^n)^{-\alpha}(1-\overline{F}(sr^n)) \geq (s/r)^{-\alpha} r^{-\alpha(n+1)}(1-\overline{F}(r^{n+1}))$ for all $s \in (1,r]$ and $n \in \mathbb{Z}$. After this observation, we see that any α-elementary fixed point is also α-regular, and since α-regularity trivially implies α-boundedness, we have that

$$\mathfrak{F}^{\alpha}_{\wedge,\mathrm{e}} \subseteq \mathfrak{F}^{\alpha}_{\wedge,\mathrm{r}} \subseteq \mathfrak{F}^{\alpha}_{\wedge,\mathrm{b}}.$$

Theorem 4.1. *Suppose* (A1) *and* (A2). *Then the following assertions are equivalent for any* $\alpha > 0$*:*

(a) *Eq.* (1) *has an α-elementary solution* ($\mathfrak{F}^{\alpha}_{\wedge,\mathrm{e}} \neq \emptyset$),
(b) *Eq.* (1) *has an α-regular solution* ($\mathfrak{F}^{\alpha}_{\wedge,\mathrm{r}} \neq \emptyset$),
(c) $m(\alpha) = 1$ *and* $\mathbb{P}(W^{(\alpha)} > 0) > 0$,
(d) $m(\alpha) = 1$, *the random walk* $(\overline{S}_{\alpha,n})_{n\geq 0}$ *with increment distribution* $\Sigma_{\alpha,1}(B) := \mathbb{E}\sum_{i\geq 1} T_i^{\alpha}\mathbf{1}_B(T_i)$, $B \in \mathfrak{B}$, *converges to* ∞ *a.s. and*

$$\int_{(1,\infty)} \left[\frac{u\log u}{\mathbb{E}(\overline{S}^{+}_{\alpha,1} \wedge \log u)}\right] \mathbb{P}(W_1^{(\alpha)} \in du) < \infty.$$

The proof of this theorem will be given in Section 7. We note that the equivalence statement of (c) and (d) is a part of Theorem A.1 and only included here for completeness. Let us further point out the connection of our result with a similar one on α-elementary fixed points of the smoothing transformation obtained by Iksanov [16]. By Lemma A.3 in [16], each *continuous* α-elementary solution $\overline{F}$ to (2) is the Laplace transform of a probability measure on $[0,\infty)$ solving (5). Therefore, under the continuity restriction, a part of our theorem could be deduced from Theorem 2 in [16]. On the other hand, the latter result strongly hinges on Proposition 1 in the same reference, the proof of which contains a flaw[1].

In order to state our second theorem, the following definition of certain classes of Weibull mixtures is given, where the definitions of r-Weibull(h,α), Weibull(c,α) and $\mathfrak{H}(r,\alpha)$ should be recalled from Introduction.

Definition 4.2. Let $\alpha > 0$ and Λ be a probability measure on $\mathbb{R}^+$. Then

(a) $\mathfrak{W}_{\Lambda}(1,\alpha)$ denotes the family of Λ-mixtures of Weibull(c,α) distributions F of the form

$$F(\cdot) = \int \mathrm{Weibull}(yc,\alpha)(\cdot)\,\Lambda(dy),$$

where $c > 0$.

(b) $\mathfrak{W}_{\Lambda}(r,\alpha)$ for $r > 1$ denotes the family of Λ-mixtures of r-Weibull(h,α) distributions F of the form

$$F(\cdot) = \int r\text{-}\mathrm{Weibull}(yh,\alpha)(\cdot)\,\Lambda(dy),$$

where $h \in \mathfrak{H}(r,\alpha)$.

The reader should notice that $\mathfrak{W}_{\Lambda}(1,\alpha)$ is always a subclass of $\mathfrak{W}_{\Lambda}(r,\alpha)$ for any $r > 1$.

Theorem 4.2. *Suppose* (A1) *and that* $m(\alpha) = 1$ *and* $\mathbb{E}W^{(\alpha)} = 1$ *for some* $\alpha > 0$. *Recall that* $\Lambda_{\alpha} = \mathbb{P}(W^{(\alpha)} \in \cdot)$. *Then*

$$\mathfrak{F}^{\alpha}_{\wedge,\mathrm{b}} = \mathfrak{F}^{\alpha}_{\wedge,\mathrm{r}} = \mathfrak{F}^{\alpha}_{\wedge,\mathrm{e}} = \mathfrak{W}_{\Lambda_{\alpha}}(d,\alpha),$$

where $d = r > 1$ *in the r-geometric case* ($\mathbb{G}(T) = r^{\mathbb{Z}}$), *and* $d = 1$ *in the continuous case* ($\mathbb{G}(T) = \mathbb{R}^+$).

[1] The flaw occurs in Eq. (14) on p. 36, where it is mistakenly assumed that q in [16] does not depend on v. The author corrected this flaw in a recent preprint named "Elementary fixed points of min-transformations".

As to the proof of Theorem 4.2, let us note that, once $m(\alpha) = 1$ and $\mathbb{E}W^{(\alpha)} = 1$ have been verified, the inclusion $\mathfrak{W}_{\Lambda_\alpha}(d, \alpha) \subseteq \mathfrak{F}^{\alpha}_{\wedge,\mathrm{b}}$ follows upon the direct inspection relying on the well-known fact that $\Lambda_\alpha \in \mathfrak{F}_\Sigma(T^{(\alpha)})$, see Lemma 6.3. So the non-trivial part is the reverse conclusion which will be shown in Section 7.

The reader should further notice that (A2) does not need to be assumed in Theorem 4.2, because it already follows from (A1) and $m(\alpha) = 1$.

Remark 4.2. (a) The two previous theorems can be summarized as follows: The existence of at least one α-regular fixed point is equivalent to α being the characteristic exponent of T with $\mathbb{E}W^{(\alpha)} = 1$, and, in this case, *all* regular solutions are, in fact, Weibull mixtures with mixing distribution Λ_α and particularly α-elementary. Moreover, there are no further solutions in $\mathfrak{F}^{\alpha}_{\wedge,\mathrm{b}} \backslash \mathfrak{F}^{\alpha}_{\wedge,\mathrm{r}}$.

(b) Let us briefly address two natural questions that arise in connection with our results. First, do further non-trivial solutions to (1) exist if $m(\alpha) = 1$ and $\mathbb{E}W^{(\alpha)} = 1$ for some $\alpha > 0$? Clearly, any further $F \in \mathfrak{F}_\wedge$ must satisfy either

$$\lim_{t\downarrow 0} D_\alpha \overline{F}(t) \;=\; \infty$$

or

$$0 \;\le\; \liminf_{t\downarrow 0} D_\alpha \overline{F}(t) \;<\; \limsup_{t\downarrow 0} D_\alpha \overline{F}(t) \;=\; \infty.$$

Lemma 6.5 will show that only the second alternative ($D_\alpha \overline{F}$ oscillating at 0) might be possible. However, whether solutions of that kind really exist in certain instances remains an open question.

Second, one may wonder about the existence of solutions to (1) if T does not possess a characteristic exponent. Although we cannot provide a general answer to this question, it will emerge from our discussion in Section 8 that there are situations, in which there is no characteristic exponent and yet $\mathfrak{F}_\wedge \neq \emptyset$. This was already observed by Jagers and Rösler [17], and the family of examples studied here forms a natural extension of theirs.

(c) There is yet another situation, called the *boundary case* by Biggins and Kyprianou [10], that we have deliberately excluded here from our analysis in order to not overburden the subsequent analysis. It occurs when $m(\alpha) = 1$ and $\mathfrak{F}_\Sigma(T^{(\alpha)})$ contains an element Λ^*_α, say, with infinite mean for some $\alpha > 0$. Then $W^{(\alpha)} = 0$ a.s. and $m'(\alpha) = 0$ provided that $m(\cdot)$ exists in a neighborhood of α. This case is quite different from the one in focus here, where $\mathbb{E}W^{(\alpha)} = 1$, except that $\mathfrak{W}_{\Lambda^*_\alpha}(d, \alpha) \subset \mathfrak{F}_\wedge$ with d as in Theorem 4.2 is easily verified by copying the proof of Lemma 6.3. If φ^*_α denotes the Laplace transform of Λ^*_α, then, under mild conditions (*cf.* [10, Theorem 5]), $1 - \varphi^*_\alpha(t)$ behaves like a constant times $t|\log t|$ as $t \downarrow 0$, and thus $\lim_{t\downarrow 0} D_\alpha \overline{F}(t) = \infty$ for any $F \in \mathfrak{W}_{\Lambda^*_\alpha}(d, \alpha)$. We quote this different behavior as opposed to that in the situation of the results above to argue that the boundary case requires a separate treatment. We refrain from going into further details and refer to a future publication.

5. Disintegration and a pathwise renewal equation

Our further analysis is based on a disintegration of Eq. (13), by which we mean the derivation of a pathwise counterpart (Eq. (15) below) which reproduces (13) upon integration on both sides. We embark on the following known result on the sequence

$$\overline{\mathcal{F}}_n(t) \;:=\; \prod_{|v|=n} \overline{F}(tL(v)), \quad n \ge 0 \tag{14}$$

appearing under the expected value in (13).

Lemma 5.1. *Let $F \in \mathfrak{F}_\wedge$. Then $(\overline{\mathcal{F}}_n(t))_{n\ge 0}$ forms a bounded non-negative martingale with respect to $(\mathcal{F}_n)_{n\ge 0}$ and thus converges a.s. and in mean to a random variable $\overline{\mathcal{F}}(t)$ satisfying*

$$\mathbb{E}\overline{\mathcal{F}}(t) = \overline{F}(t).$$

Source. Biggins and Kyprianou [9, Theorem 3.1]. □

In the situation of Lemma 5.1, we put

$$\mathcal{F}(t) \;:=\; 1 - \liminf_{n\to\infty} \overline{\mathcal{F}}_n(t)$$

and call the stochastic process $\mathcal{F} = (\mathcal{F}(t))_{t\ge 0}$ a *disintegration of* F and also a *disintegrated fixed point*. The announced pathwise fixed-point equation for an arbitrary disintegrated fixed point is next.

Lemma 5.2. *Let $F \in \mathfrak{F}_\wedge$ and $\mathcal{F}$ a disintegration of F. Then*

$$\overline{\mathcal{F}}(t) \;=\; \prod_{|v|=n} [\overline{\mathcal{F}}]_v(tL(v)) \qquad a.s. \tag{15}$$

for each $t \ge 0$ and $n \in \mathbb{N}_0$.

Proof. For any $t \ge 0$,

$$\begin{aligned}
\overline{\mathcal{F}}(t) \;=\; \liminf_{k\to\infty} \prod_{|v|=n+k} \overline{F}(tL(v)) \quad &= \quad \liminf_{k\to\infty} \prod_{|v|=n} \prod_{|w|=k} \overline{F}(tL(v)[L(w)]_v) \\
&\le \quad \liminf_{k\to\infty} \prod_{v\in\{1,\dots,m\}^n} [\overline{\mathcal{F}}_k]_v(tL(v)) \\
&= \quad \prod_{v\in\{1,\dots,m\}^n} [\overline{\mathcal{F}}]_v(tL(v)) \\
&\underset{m\to\infty}{\longrightarrow} \quad \prod_{|v|=n} [\overline{\mathcal{F}}]_v(tL(v)) \qquad (t \ge 0).
\end{aligned}$$

Taking expectations on both sides, this inequality becomes an equality for the $[\overline{\mathcal{F}}]_v(tL(v))$, $|v| = n$ are conditionally independent given $(L(v))_{|v|=n}$ and have conditional expectation $\overline{F}(tL(v))$ by Lemma 5.1. This gives the asserted result. □

Eq. (15) is of essential importance for our purposes. It can be transformed into an additive one by taking logarithms and a change of the variables $t \mapsto e^t$. To this end, fix any $\alpha > 0$ and define

$$\Psi(t) \;:=\; e^{-\alpha t}(-\log \overline{\mathcal{F}}(e^t))$$

for $t \in \mathbb{R}$. Put also $S(v) := -\log L(v)$ for $v \in \mathbb{V}$ with the usual convention $S(v) := \infty$ on $\{L(v) = 0\}$. Then, by (15),

$$\begin{aligned}
\Psi(t) \;&=\; e^{-\alpha t}\left(-\log \prod_{|v|=n} [\overline{\mathcal{F}}]_v(e^t L(v))\right) \\
&=\; \sum_{|v|=n} e^{-\alpha t}\left(-\log[\overline{\mathcal{F}}]_v(e^t L(v))\right) \\
&=\; \sum_{|v|=n} L(v)^\alpha e^{-\alpha(t-S(v))}\left(-\log[\overline{\mathcal{F}}]_v(e^{t-S(v)})\right) \\
&=\; \sum_{|v|=n} L(v)^\alpha [\Psi]_v(t - S(v)) \qquad \text{a.s.,}
\end{aligned}$$

that is, Ψ satisfies the *pathwise renewal equation*

$$\Psi(t) = \sum_{|v|=n} L(v)^{\alpha}[\Psi]_v(t - S(v)) \quad \text{a.s.} \tag{16}$$

for each $t \in \mathbb{R}$. To explain the notion "pathwise renewal equation", we introduce a family of measures related to (16), namely

$$\Sigma_{\alpha,n} \ := \ \mathbb{E} \sum_{|v|=n} L(v)^{\alpha} \delta_{S(v)}, \quad n \in \mathbb{N}_0.$$

If $m(\alpha) = 1$, then $\Sigma_{\alpha,n}$ is a probability distribution on $\mathbb{R}$ and $\Sigma_{\alpha,n} = \Sigma_{\alpha,1}^{*(n)}$ (the n-fold convolution of $\Sigma_{\alpha,1}$) for each $n \geq 0$, see *e.g.* [9, Lemma 4.1]. In the following, we denote, by $(\overline{S}_{\alpha,n})_{n \geq 0}$, a random walk with increment distribution $\Sigma_{\alpha,1}$ if $m(\alpha) = 1$.

Now suppose $m(\alpha) \leq 1$ and define $\psi(t) := \mathbb{E}\Psi(t)$ $(t \in \mathbb{R})$. ψ is well defined due to the fact that $\Psi(t) \geq 0$ a.s. for all $t \in \mathbb{R}$. By taking expectations on both sides of Eq. (16), we obtain

$$\begin{aligned}
\psi(t) &= \mathbb{E} \sum_{|v|=n} L(v)^{\alpha}[\Psi]_v(t - S(v)) \\
&= \mathbb{E}\left(\mathbb{E}\left[\sum_{|v|=n} L(v)^{\alpha}[\Psi]_v(t - S(v)) \Big| \mathcal{A}_n\right]\right) \\
&= \mathbb{E} \sum_{|v|=n} L(v)^{\alpha} \psi(t - S(v)) \\
&= \int \psi(t-s)\, \Sigma_{\alpha,n}(ds),
\end{aligned}$$

having utilized that $[\Psi]_v$ is independent of $\mathcal{A}_n$ for $|v| = n$. Consequently, ψ satisfies the renewal equation

$$\psi(t) \ = \ \int \psi(t-s)\, \Sigma_{\alpha,n}(ds), \quad t \in \mathbb{R}, \tag{17}$$

of which (16) is a disintegrated version. This provides the justification for the notion *"pathwise renewal equation"*. While the uniqueness results for renewal equations of the form (17) are commonly known, the uniqueness results for processes solving a pathwise renewal equation are systematically studied in [20]. The following result is cited from there:

Theorem 5.1. *Suppose that* $\mathbb{E}N > 1$, $\mathbb{P}(T \in \{0,1\}^{\mathbb{N}}) < 1$, *and* $m(\alpha) = 1$ *for some* $\alpha > 0$. *Let* $\Psi : \mathbb{R} \times \Omega \to [0, \infty]$ *denote a* $\mathfrak{B} \otimes \mathcal{A}_{\infty}$*-measurable stochastic process which solves Eq.* (16) *for* $n = 1$. *Then the following assertions hold true:*

(a) *Suppose* $\Sigma_{\alpha,1}$ *is non-arithmetic.*
If, at each $t \in \mathbb{R}$, Ψ *is a.s. left continuous with right-hand limit and locally uniformly integrable and if* $\sup_{t \in \mathbb{R}} \mathbb{E}\Psi(t) < \infty$, *then* Ψ *is a version of* $cW^{(\alpha)}$ *for some* $c \geq 0$,

(b) *Suppose* $\Sigma_{\alpha,1}$ *is d-arithmetic* ($d > 0$).
If $\sup_{n \in \mathbb{Z}} \mathbb{E}\Psi(s+nd) < \infty$ *for all* $s \in [0, d)$, *then there exists a d-periodic function* $p : \mathbb{R} \to [0, \infty)$ *such that* Ψ *is a version of* $pW^{(\alpha)}$.

In order to utilize this theorem in the context of fixed-point equations, we need to check whether the additive transformation Ψ of the disintegrated fixed point $\mathcal{F}$ satisfies the assumptions of the theorem. Clearly, Ψ is product measurable, and the standard arguments also show that it is a.s. left continuous with right-hand limit at any $t \in \mathbb{R}$. Applicability of Theorem 5.1 therefore reduces to a verification of the integrability conditions for Ψ. This is not always possible but works for the subclass of α-bounded fixed points and forms the key ingredient to our proof of Theorem 4.2 in Section 7.

6. The characteristic exponent

The purpose of this section is to provide some results related to the existence of the characteristic exponent α of T. Recall from Remark 3.1 that a sufficient condition for this to be true is that $m(\alpha) = 1$ and $\mathbb{P}(W^{(\alpha)} > 0) > 0$. Section 8 is devoted to a family of examples showing that solutions to our SFPE (1) may exist even if T does not have a characteristic exponent. This will subsequently be used to point out some phenomena which may occur in the situation of Eq. (1) but not in the situation of its additive counterpart, *i.e.*, Eq. (5).

6.1. Necessary conditions for the existence of the characteristic exponent.

Lemma 6.1. *If $m(\alpha) \leq 1$ for some $\alpha > 0$, then*

$$R_n = \sup_{|v|=n} L(v) \longrightarrow 0 \quad a.s. \quad (n \to \infty).$$

Source. A proof of this lemma has been given in [8, Theorem 3]. □

Lemma 6.2. *If the characteristic exponent α exists, then $\overline{F}(t) < 1$ for all $t > 0$ and $F \in \mathfrak{F}_\wedge$.*

Proof. Suppose there exists a $t_0 > 0$ with $\overline{F}(t_0) = 1$. Pick any $t > t_0$ and let $\tau := \inf\{n \geq 0 : R_n \leq t_0/t\}$. Then τ is an a.s. finite stopping time by Lemma 6.1. Hence, a combination of the optional sampling theorem applied to the bounded martingale $(\overline{\mathcal{F}}_n(t))_{n\geq 0}$ and Lemma 5.1 yields

$$\overline{F}(t) \;=\; \mathbb{E}\overline{\mathcal{F}}_\tau(t) \;=\; \mathbb{E}\prod_{|v|=\tau} \overline{F}(tL(v)) \;\geq\; \mathbb{E}\prod_{|v|=\tau} \overline{F}(t_0) \;=\; 1.$$

Since $t > t_0$ was chosen arbitrarily, we have $\overline{F}(t) = 1$ for all $t \geq 0$, which is clearly impossible for any proper distribution on $\mathbb{R}^+$. □

Lemma 6.3. *Suppose that $m(\alpha) = 1$ and $\mathbb{E}W^{(\alpha)} = 1$ for some $\alpha > 0$. Then $\mathfrak{W}_{\Lambda_\alpha}(r, \alpha) \subseteq \mathfrak{F}^\alpha_{\wedge,\mathrm{e}}$ in the r-geometric case ($r > 1$) and $\mathfrak{W}_{\Lambda_\alpha}(1, \alpha) \subseteq \mathfrak{F}^\alpha_{\wedge,e}$ in the continuous case.*

Proof. We restrict ourselves to the r-geometric case. So let $h \in \mathfrak{H}(r, \alpha)$ and then F as in Definition 4.2(a) with $\Lambda = \Lambda_\alpha$. Recall that φ_α denotes the Laplace transform of $W^{(\alpha)}$. Then $\overline{F}(t) = \varphi_\alpha(h(t)t^\alpha)$ ($t \geq 0$). Since h is multiplicatively r-periodic, all T_i take values in $r^{\mathbb{Z}} \cup \{0\}$ a.s., and $\Lambda_\alpha \in \mathfrak{F}_\Sigma(T^{(\alpha)})$, we infer

$$\begin{aligned}
\mathbb{E}\prod_{i\geq 1} \overline{F}(tT_i) &= \mathbb{E}\prod_{i\geq 1} \varphi_\alpha(h(tT_i)(tT_i)^\alpha) \\
&= \mathbb{E}\prod_{i\geq 1} \varphi_\alpha(h(t)t^\alpha T_i^\alpha) \\
&= \varphi_\alpha(h(t)t^\alpha) \;=\; \overline{F}(t).
\end{aligned}$$

Thus $F \in \mathfrak{F}_\wedge$. One can easily check that F is always α-elementary. □

Lemma 6.4. *Let $F \in \mathfrak{F}^\alpha_{\wedge,\mathrm{b}}$, $\mathcal{F} = (\mathcal{F}(t))_{t\geq 0}$ be its disintegration, and let $\alpha > 0$ be such that $m(\alpha) \leq 1$. Then there exists a positive constant $C > 0$ such that*

$$\mathbb{P}\left(-\log \overline{\mathcal{F}}(t) \leq Ct^\alpha W^{(\alpha)} \textit{ for all } t \geq 0\right) \;=\; 1.$$

Proof. By assumption, $1-\overline{F}(t) \leq Ct^{\alpha}/2$ for all sufficiently small $t \geq 0$ and some $C>0$. Using this and $-\log x \leq 2(1-x)$ for all x in a suitable left neighborhood of 1, we infer

$$-\log \overline{F}(t) \leq 2(1-\overline{F}(t)) \leq Ct^{\alpha}$$

for all $0 \leq t \leq \delta$ with $\delta>0$ sufficiently small. By Lemma 6.1, $m(\alpha) \leq 1$ ensures $\mathbb{P}(B)=1$ for $B := \{\sup_{|v|=n} L(v) \to 0\}$. Fixing any $t>0$, we have $tL(v) \leq \delta$ on B for all $|v| \geq n$ and some sufficiently large n. Hence, $-\log \overline{F}(tL(v)) \leq Ct^{\alpha}L(v)^{\alpha}$ on B for all $|v| \geq n$, which implies, in turn,

$$\begin{aligned} -\log \overline{\mathcal{F}}(t) &= \limsup_{n\to\infty} \sum_{|v|=n} -\log \overline{F}(tL(v)) \\ &\leq \limsup_{n\to\infty} Ct^{\alpha} \sum_{|v|=n} L(v)^{\alpha} \\ &= 2Ct^{\alpha}W^{(\alpha)} \quad \text{a.s.} \end{aligned}$$

on the almost certain event B. □

Remark 6.1. (a) Lemma 6.4 allows the following obvious modification: Suppose $m(\alpha) \leq 1$ for some $\alpha>0$ and $\mathbb{G}(T) = r^{\mathbb{Z}}$ for some $r>1$. Let $F \in \mathfrak{F}_{\wedge}$ with disintegration $\mathcal{F}$ be such that $1-\overline{F}(sr^{-n}) \leq C(sr^{-n})^{\alpha}$ for some $s, C>0$ and all sufficiently large $n \in \mathbb{N}$. Then $-\log \overline{\mathcal{F}}(sr^{n}) \leq 2C(sr^{n})^{\alpha}W^{(\alpha)}$ a.s. for all $n \in \mathbb{Z}$.

(b) If $W^{(\alpha)}=0$ a.s., which is always true if $m(\alpha)<1$ and may be true if $m(\alpha)=1$, then the assertion of Lemma 6.4 becomes $\overline{\mathcal{F}}(t)=1$ a.s. for all $t \geq 0$ which implies $\overline{F}(t)=1$ for all $t \geq 0$, which is clearly impossible. Hence, Lemma 6.4 is really about the case where $m(\alpha)=1$ and $\mathbb{E}W^{(\alpha)}=1$.

Lemma 6.5. *Suppose that $m(\alpha)=1$ and $\mathbb{E}W^{(\alpha)}=1$ for some $\alpha>0$. Then the following assertions hold true for any $F \in \mathfrak{F}_{\wedge}$:*

(a) $\limsup_{t\downarrow 0} t^{-\alpha}(1-\overline{F}(t)) > 0$,

(b) $\liminf_{t\downarrow 0} t^{-\alpha}(1-\overline{F}(t)) < \infty$.

Proof. (a) Suppose that $\lim_{t\downarrow 0} t^{-\alpha}(1-\overline{F}(t)) = 0$. Then F clearly satisfies the crucial assumption of Lemma 6.4 for *every* $C>0$, and we conclude for its disintegration $\mathcal{F}$ that $\mathbb{P}(-\log \overline{\mathcal{F}}(t) \leq 2Ct^{\alpha}W^{(\alpha)}$ for all $t \geq 0) = 1$ for every $C>0$. As a consequence, $-\log \overline{\mathcal{F}}(1) = 0$ a.s., which implies, in turn, $\overline{F}(1) = \mathbb{E}\overline{\mathcal{F}}(1) = 1$. But this contradicts Lemma 6.2, and we conclude that $\limsup_{t\downarrow 0} t^{-\alpha}(1-\overline{F}(t)) > 0$, as claimed.

(b) Suppose that $\lim_{t\downarrow 0} t^{-\alpha}(1-\overline{F}(t)) = \infty$. This time, we will produce a contradiction by comparison of $F(t)$ with the class $F_s(t) := 1-\varphi_{\alpha}((t/s)^{\alpha})$, $s>0$, of solutions to (1) (see Lemma 6.3). Since $\lim_{t\downarrow 0} t^{-\alpha}(1-\overline{F}_s(t)) = s^{-\alpha}\mathbb{E}W^{(\alpha)} = s^{-\alpha}$ for any $s>0$, we infer $\overline{F}(t) \leq \overline{F}_s(t)$ for any $s>0$ and $0<t<\varepsilon(s)$ with $\varepsilon(s)$ sufficiently small. Now fix any $t>0$ and consider the bounded martingales $(\overline{\mathcal{F}}_n(t))_{n\geq 0}$ and $(\overline{\mathcal{F}}_{s,n}(t))_{n\geq 0}$ defined by (14) for F and F_s, respectively. By Lemma 6.1, the stopping time $\tau(s) := \inf\{n : tR_n < \varepsilon(s)\}$ is a.s. finite and

$$\overline{\mathcal{F}}_{\tau}(t) \;=\; \prod_{|v|=\tau} \overline{F}(tL(v)) \;\leq\; \prod_{|v|=\tau} \overline{F}_s(tL(v)) \;=\; \overline{\mathcal{F}}_{s,\tau}(t)$$

for any $s>0$. Therefore, by an appeal to the optional sampling theorem,

$$\overline{F}(t) \;=\; \mathbb{E}\overline{\mathcal{F}}_{\tau}(t) \;\leq\; \mathbb{E}\overline{\mathcal{F}}_{s,\tau}(t) \;=\; \overline{F}_s(t)$$

for any $s>0$. Finally, use $\overline{F}_s(t) = \varphi_{\alpha}((t/s)^{\alpha}) \to \varphi_{\alpha}(\infty) = \mathbb{P}(W^{(\alpha)}=0) = 0$ as $s \downarrow 0$ to infer $\overline{F}(t)=0$ and thereupon the contradiction $F=\delta_0$ since $t>0$ was chosen arbitrarily. □

6.2. **A sufficient condition for the existence of the characteristic exponent.**

Lemma 6.6. *Let* $\alpha > 0$ *and* $\mathfrak{F}^{\alpha}_{\wedge,\mathrm{r}} \neq \emptyset$. *Then* $m(\alpha) \leq 1$.

Proof. Let $F \in \mathfrak{F}^{\alpha}_{\wedge,\mathrm{r}}$, and recall that $D_\alpha \overline{F}(t) := t^{-\alpha}(1-\overline{F}(t))$ for $t > 0$, thus $c_1 := \liminf_{t\downarrow 0} D_\alpha \overline{F}(t) > 0$ and $c_2 := \limsup_{t\downarrow 0} D_\alpha \overline{F}(t) < \infty$. It follows from Eq. (2) that (see Eq. (4.1) in [9] and Eq. (10) in [16])

$$1 \;=\; \mathbb{E}\sum_{i=1}^{N} T_i^\alpha \frac{D_\alpha \overline{F}(tT_i)}{D_\alpha \overline{F}(t)} \prod_{j<i} \overline{F}(tT_j).$$

Now use Fatou's lemma to infer

$$\begin{aligned} 1 \;&=\; \mathbb{E}\sum_{i=1}^{N} T_i^\alpha \liminf_{t\downarrow 0} \frac{D_\alpha \overline{F}(tT_i)}{D_\alpha \overline{F}(t)} \prod_{j<i} \overline{F}(tT_j) \\ &\geq\; \frac{c_1}{c_2}\,\mathbb{E}\sum_{i=1}^{N} T_i^\alpha \;=\; \frac{c_1}{c_2}\, m(\alpha). \end{aligned}$$

But the same argument applies to Eq. (13) (that is (2) after n iterations of the SFPE) and gives

$$1 \;\geq\; \frac{c_1}{c_2}\,\mathbb{E}\sum_{|v|=n} L(v)^\alpha \;=\; \frac{c_1}{c_2}\, m(\alpha)^n$$

for each $n \geq 1$ and thereupon the desired conclusion. □

Given two sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ of real numbers, we write $a_n \sim b_n$ hereafter if $\lim_{n\to\infty} b_n^{-1} a_n = 1$ and $a_n \asymp b_n$ if $0 < \liminf_{n\to\infty} b_n^{-1} a_n \leq \limsup_{n\to\infty} b_n^{-1} a_n < \infty$.

Proposition 6.1. *Let* $\alpha > 0$ *and* $\mathfrak{F}^{\alpha}_{\wedge,\mathrm{r}} \neq \emptyset$. *Then* α *is the characteristic exponent of* T *and* $W^{(\alpha)}$ *a.s. positive.*

Proof. Note first that $m(\alpha) \leq 1$ follows by Lemma 6.6 and then $R_n \to 0$ a.s. by Lemma 6.1. Since $F \in \mathfrak{F}^{\alpha}_{\wedge,\mathrm{r}}$ is not a Dirac measure (cf. Proposition 2.2), we have $\mathbb{P}(\overline{\mathcal{F}}(t) < 1) > 0$ for some $t > 0$. By combining these facts with $-\log(1-x) \sim x$ as $x \downarrow 0$ and α-regularity, we infer

$$\begin{aligned} -\log \overline{\mathcal{F}}_n(t) \;=\; \sum_{|v|=n} -\log \overline{F}(tL(v)) \;&\sim\; \sum_{|v|=n} \big(1 - \overline{F}(tL(v))\big) \\ &\asymp\; \sum_{|v|=n} L(v)^\alpha \;=\; W_n^{(\alpha)} \qquad (n\to\infty) \end{aligned}$$

which in combination with $-\log \overline{\mathcal{F}}_n(t) \to -\log \overline{\mathcal{F}}(t)$ a.s. (Lemma 5.1) shows

$$\liminf_{n\to\infty} W_n^{(\alpha)} \;>\; 0 \quad \text{a.s.}$$

on the event $\{\overline{\mathcal{F}}(t) < 1\}$. Hence, $m(\alpha) = 1$, $W^{(\alpha)} = \lim_{n\to\infty} W_n^{(\alpha)} > 0$ a.s. and $\mathbb{E}W^{(\alpha)} = 1$ (by Theorem A.1 and (A1)). From this, it is immediate (see Corollary A.1) that α is the characteristic exponent of T. □

7. Proofs of Theorem 4.1 and Theorem 4.2

7.1. **Proof of Theorem 4.1.** "(a) ⇒ (b)" is trivial as $\mathfrak{F}^{\alpha}_{\wedge,\mathrm{e}} \subset \mathfrak{F}^{\alpha}_{\wedge,\mathrm{r}}$.

"(b) ⇒ (c)" is Proposition 6.1.

"(c) ⇒ (a)". Assuming (c), we have $\mathbb{E}W^{(\alpha)} = 1$ (see Theorem A.1 in Appendix), whence the Laplace transform φ_α of $W^{(\alpha)}$ satisfies $\varphi'_\alpha(0) = -1$. By Lemma 6.3, $F(t) := 1 - \varphi_\alpha(t^\alpha)$, $t \geq 0$, defines a solution to Eq. (1). Furthermore,

$$\lim_{t\downarrow 0} D_\alpha \overline{F}(t) \;=\; \lim_{t\downarrow 0} t^{-\alpha}(1 - \varphi_\alpha(t^\alpha)) \;=\; -\varphi'_\alpha(0) = 1,$$

and thus F constitutes an α-elementary solution to Eq. (1).

"(c) ⇔ (d)". As already pointed out, this is a consequence of Biggins' martingale limit theorem stated as Theorem A.1 in Appendix. □

7.2. **Proof of Theorem 4.2.** In view of Lemma 6.3, it remains to verify that $\mathfrak{F}^\alpha_{\wedge,\mathrm{b}} \subseteq \mathfrak{W}_{\Lambda_\alpha}(d,\alpha)$, where $d = 1$ if $\mathbb{G}(T) = \mathbb{R}^+$, and $d = r$ if $\mathbb{G}(T) = r^{\mathbb{Z}}$. To this end, let F be α-bounded with disintegration $\overline{\mathcal{F}}$. By Lemma 6.4, we have

$$\mathbb{P}\left(-\log \overline{\mathcal{F}}(t) \leq Ct^\alpha W^{(\alpha)} \text{ for all } t \geq 0\right) \;=\; 1 \tag{18}$$

for a suitable $C > 0$. Recall that $\overline{\mathcal{F}}$ satisfies the multiplicative Eq. (15), which, upon logarithmic transformation and setting

$$\Psi(t) \;:=\; e^{-\alpha t}(-\log \overline{\mathcal{F}}(e^t)), \quad t \in \mathbb{R}$$

, becomes the following pathwise renewal equation (see (16)):

$$\Psi(t) \;=\; \sum_{|v|=n} L(v)^\alpha [\Psi]_v(t - S(v)) \quad \text{a.s.}$$

for all $t \in \mathbb{R}$. We want to make use of Theorem 5.1 and must therefore check its conditions as for the random function Ψ. We already mentioned right after Theorem 5.1 that Ψ is product measurable and a.s. left-continuous with right-hand limits at any $t \in \mathbb{R}$. Since, by (18),

$$0 \;\leq\; \Psi(t) \;\leq\; e^{-\alpha t} C e^{\alpha t} W^{(\alpha)} \;=\; CW^{(\alpha)}$$

for all $t \in \mathbb{R}$ on a set of probability one, it follows that $\sup_{t\in\mathbb{R}} \mathbb{E}\Psi(t) < \infty$ and that Ψ is locally uniformly integrable. Hence, Theorem 5.1 applies, and we infer $\Psi(t) = p(t)W^{(\alpha)}$ a.s., where p denotes a measurable $(\log r)$-periodic function in the r-geometric case and a positive constant in the continuous case, respectively. In both cases,

$$\overline{\mathcal{F}}(t) \;=\; e^{-t^\alpha \Psi(\log t)} \;=\; e^{-p(\log t)t^\alpha W^{(\alpha)}} \;=\; e^{-h(t)t^\alpha W^{(\alpha)}} \text{ a.s.}$$

for all $t > 0$, where $h(t) := p(\log t)$. Taking expectations on both sides of this equation provides us with

$$\overline{F}(t) = \varphi_\alpha(h(t)t^\alpha) \qquad (t > 0).$$

Therefore, the proof is complete in the continuous case where h is necessarily constant. For the rest of the proof, suppose we are in the r-geometric case. Then p is $(\log r)$-periodic as mentioned above, and thus h is multiplicatively r-periodic. Furthermore, the left continuity of $\overline{F}$ implies the left continuity of the function $t \mapsto h(t)t^\alpha$ and, therefore, the left continuity of h. Similarly, we conclude that $t \mapsto h(t)t^\alpha$ is non-decreasing. These facts together give $h \in \mathfrak{H}(r,\alpha)$, which finally shows $F \in \mathfrak{W}_{\Lambda_\alpha}(1,\alpha)$. □

A combination of Theorem 4.2 and Lemma 6.5 provides us with a very short proof of the following result about the distribution Λ_α of $W^{(\alpha)}$ as a solution to (5) with $T^{(\alpha)}$ instead of T:

Corollary 7.1. *Suppose* (A1) *and that* $m(\alpha) = 1$ *and* $\mathbb{E}W^{(\alpha)} = 1$ *for some* $\alpha > 0$*. Then* $\mathfrak{F}_\Sigma(T^{(\alpha)}) = \{\Lambda_\alpha(c\,\cdot) : c > 0\}$*, i.e.,* Λ_α *is the unique solution up to scaling to* (5) *for* $T^{(\alpha)}$.

This result (with $\mathbb{E}N > 1$ instead of condition (A1)) has been obtained under slightly stronger conditions on $m(\cdot)$ by Biggins and Kyprianou as Theorem 1.5 in [9] and as Theorem 3 in [10], where the latter result also covers the boundary case briefly discussed in Remark 4.2(c).

Proof. Suppose $\Lambda \in \mathfrak{F}_\Sigma(T^{(\alpha)})$ is another solution to (5) for $T^{(\alpha)}$ with Laplace transform ψ. Regard $\psi(t^\alpha)$ as the survival function of a probability measure G. Then $G \in \mathfrak{F}_\wedge$ and $\lim_{t\downarrow 0} D_\alpha \overline{G}(t) = \lim_{t\downarrow 0} t^{-\alpha}(1-\psi(t^\alpha)) = |\psi'(0)| \in \mathbb{R}^+$, the finiteness following from Lemma 6.5. Consequently, $G \in \mathfrak{F}^\alpha_{\wedge,\mathrm{b}}$ and thus $\psi(t) = \varphi_\alpha(h(t)t)$ for some $h \in \mathfrak{H}(r,1)$ in the r-geometric case or a constant h in the continuous case. In both cases, $|\psi'(0)| = \lim_{t\downarrow 0} t^{-1}(1-\psi(t)) = \lim_{t\downarrow 0} h(t)$ implies $h = |\psi'(0)|$, *i.e.*, $\psi(t) = \varphi_\alpha(t/c)$ with $c := 1/|\psi'(0)| > 0$. This proves $\Lambda = \Lambda_\alpha(c\,\cdot)$. □

8. Beyond α-boundedness: The generalized water cascades example

In the following, we discuss a class of examples which demonstrates that non-trivial solutions to the SFPE (1) may exist even if T does not have a characteristic exponent. This is in contrast to the additive case, *i.e.*, Eq. (5), for which the existence of the characteristic exponent and the existence of non-trivial solutions are equivalent, at least under appropriate conditions on T and N, see [13] and [18].

We fix $N \in \mathbb{N}$, $N \geq 2$, and denote, by $B_1, \ldots, B_N$, independent Bernoulli variables with parameter $\vartheta \in (0,1)$, that is, $\mathbb{P}(B_i = 1) = \vartheta = 1 - \mathbb{P}(B_i = 0)$, $i = 1, \ldots, N$. Put $T_i := \exp(-B_i)$ for $i = 1, \ldots, N$ and $T_i = 0$ for $i > N$. Notice that $\mathbb{G}(T) = e^{\mathbb{Z}}$ (e-geometric case). Then Eq. (1) takes the form

$$X \stackrel{d}{=} \min_{1\leq i\leq N} e^{B_i} X_i. \tag{19}$$

For $N = 2$ and $\vartheta > 1/2$, this example was studied by Jagers and Rösler [17].

Lemma 8.1. *In the situation of Eq.* (19)*, the following assertions are equivalent:*

(a) *The characteristic exponent* $\alpha > 0$ *exists,*
(b) $\vartheta > 1 - 1/N$,
(c) *If* $\tilde{N} := \sum_{i=1}^N \mathbf{1}_{\{B_i=0\}} = \sum_{i=1}^N \mathbf{1}_{\{T_i=1\}}$*, then the Galton–Watson process with offspring distribution* $\mathbb{P}(\tilde{N} \in \cdot)$ *is subcritical.*

Proof. Under stated assumptions, we have

$$m(\alpha) \;=\; \mathbb{E}\sum_{i=1}^N T_i^\alpha \;=\; N(\vartheta e^{-\alpha} + (1-\vartheta)).$$

Therefore, $N(1-\vartheta) < 1$ is necessary and sufficient for the existence of a positive α such that $m(\alpha) = 1$. This proves that (a) and (b) are equivalent. The equivalence of (b) and (c) is obvious as $\mathbb{E}\tilde{N} = N(1-\vartheta)$. □

According to Lemma 8.1, we distinguish three cases:

(1) *Subcritical case*: $\vartheta > 1 - 1/N$,
(2) *Critical case*: $\vartheta = 1 - 1/N$,
(3) *Supercritical case*: $\vartheta < 1 - 1/N$.

Consider the associated WBP as introduced in Section 3. Now define

$$\tilde{L}(v) \;:=\; \mathbf{1}_{\{L(v)=1\}}$$

for $v \in \mathbb{V}_N := \bigcup_{n\geq 0}\{1,\ldots,N\}^n$. By our model assumptions, $\tilde{G}_n := \sum_{|v|=n}\tilde{L}(v)$ $(n\geq 0)$ defines a Galton–Watson process with offspring distribution $\mathbb{P}(\tilde{N}\in\cdot)$. In the supercritical case, $(\tilde{G}_n)_{n\geq 0}$ survives with positive probability, thus

$$\mathbb{P}(\lim_{n\to\infty}\sup_{|v|=n} L(v)=1) \;=\; \mathbb{P}(\tilde{G}_n \text{ survives}) \;>\; 0. \tag{20}$$

The main outcome of the subsequent discussion will be that Eq. (19) has non-trivial solutions in all three possible cases. In view of (20) in the supercritical case, this shows that the existence of a non-trivial solution to (1) does not necessarily entail $\limsup_{n\to\infty}\sup_{|v|=n}L(v)=0$ a.s., as intuition might suggest.

We proceed with the construction of non-trivial solutions to Eq. (19) and start by taking a look at the associated functional equation. Given a solution F, the latter takes the form

$$\overline{F}(t) \;=\; \mathbb{E}\prod_{i=1}^{N}\overline{F}(tT_i) \;=\; \left(\vartheta\overline{F}(te^{-1})+(1-\vartheta)\overline{F}(t)\right)^N, \quad t\geq 0,$$

which, upon solving for $\overline{F}(te^{-1})$, leads to

$$\overline{F}(te^{-1}) \;=\; \vartheta^{-1}\left(\overline{F}(t)^{1/N}-(1-\vartheta)\overline{F}(t)\right) \;=\; g_{N,\vartheta}(\overline{F}(t)), \tag{21}$$

where $g_{N,\vartheta}(u) := \vartheta^{-1}\left(u^{1/N}-(1-\vartheta)u\right)$ for $0\leq u\leq 1$. The following lemma collects some properties of $g_{N,\vartheta}$.

Lemma 8.2. *The following assertions hold true for $g_{N,\vartheta}$:*

(a) $\{u\in[0,1]: g_{N,\vartheta}(u)=u\}=\{0,1\}$,
(b) $g_{N,\vartheta}(u)>u$ *for all* $u\in(0,1)$,
(c) *If $\vartheta\geq 1-1/N$ (subcritical or critical case), then $g_{N,\vartheta}$ is strictly increasing on $(0,1)$, In particular, $g_{N,\vartheta}(u)\in(0,1)$ for all $u\in(0,1)$.*
(d) *If $\vartheta<1-1/N$ (supercritical case), there exists a unique $a_0\in(0,1)$ satisfying $g_{N,\vartheta}(a_0)=1$. $g_{N,\vartheta}$ is strictly increasing on $[0,a_0]$ and >1 on $(a_0,1)$.*

Proof. Obviously, $g_{N,\vartheta}(u)=u$ holds iff $u^{1/N}=u$ and thus iff $u\in\{0,1\}$, for $N\geq 2$. This shows (a). Next, $g_{N,\vartheta}(u)>u$ for all sufficiently small $u>0$ because $g_{N,\vartheta}$ is continuously differentiable on $(0,1]$ with $\lim_{u\downarrow 0}g'_{N,\vartheta}(u)=\infty$. But this gives (b), by the continuity of $g_{N,\vartheta}$ and (a), and we also infer that $g'_{N,\vartheta}$ is positive in a right neighborhood of 0. Furthermore, $g'_{N,\vartheta}(u)=\vartheta^{-1}\left(N^{-1}u^{1/N-1}-(1-\vartheta)\right)$, $u\in(0,1]$, implies that $g'_{N,\vartheta}(u)=0$ for $u>0$ iff $u^{1/N-1}=N(1-\vartheta)$. Hence, $g'_{N,\vartheta}(u)\neq 0$ on $(0,1)$ in the subcritical and critical cases $(N(1-\vartheta)\leq 1)$, and (c) is true. In the supercritical case $(N(1-\vartheta)>1)$, we have $g'_{N,\vartheta}(a)=0$ for a unique $a\in(0,1)$. Consequently, $g_{N,\vartheta}$ is strictly increasing on $(0,a)$ and strictly decreasing on $(a,1)$. Since $g_{N,\vartheta}(1)=1$, (d) must be true (*cf.* also Figure 1). ☐

A. Critical and subcritical cases. Assuming $\vartheta\geq 1-1/N$, we have, by Lemma 8.2, that $g_{N,\vartheta}$ is strictly increasing with unique fixed points 0 and 1 in the unit interval. Therefore, its inverse function, denoted by $g_{N,\vartheta}^{-1}$, exists on $[0,1]$. We can rewrite Eq. (21) in terms of $g_{N,\vartheta}^{-1}$ as

$$\overline{F}(t)=g_{N,\vartheta}^{-1}(t/e), \quad t\geq 0.$$

Equations of this type have been completely solved in Theorem 2.1 of [4], and its application allows us here to provide a full description of $\mathfrak{F}_\wedge$. To this end, let $g_{N,\vartheta}^{n}$ denote the $|n|$-fold composition of $g_{N,\vartheta}$ $(n\geq 1)$ or its inverse $g_{N,\vartheta}^{-1}$ $(n\leq -1)$, and let $g_{N,\vartheta}^{0}$ be the identity function. Although the situation in [4] differs slightly from ours, we adopt the notation from there and write $\mathcal{F}_+$ for the set of non-increasing, left continuous functions

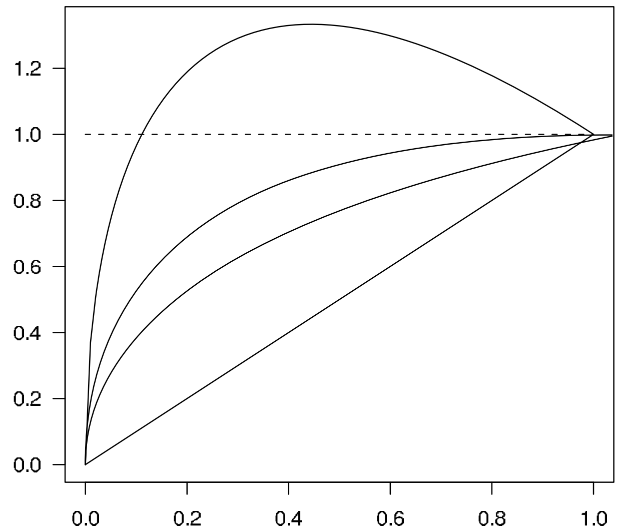

FIGURE 1. $g_{N,\vartheta}$ for $N = 2$ and $\vartheta = 1/4, 1/2, 3/4$ and the identity function (from top to bottom)

$f : (1, e] \to (0, 1)$, satisfying $f(t) \le g_{N,\vartheta}(f(e))$ for all $t \in (1, e]$. Then the following theorem is proved along the same lines as Theorem 2.1 of [4] with only minor changes in obvious places.

Theorem 8.1 (see [4], Theorem 2.1)**.** *If $\vartheta \ge 1 - 1/N$ (subcritical or critical case), there is a one-to-one correspondence $F \leftrightarrow f$ between $\mathfrak{F}_\wedge$ and $\mathcal{F}_+$ established by*

$$\overline{F}(t) = g_{N,\vartheta}^{-n} f(t/e^n), \tag{22}$$

where $n \in \mathbb{Z}$ denotes the unique integer satisfying $1 < t/e^n \le e$.

In the subcritical case where the characteristic exponent exists, we can state the following interesting corollary.

Corollary 8.1. *Suppose $\vartheta > 1 - 1/N$ (subcritical case), and let α be the characteristic exponent, i.e., $m(\alpha) = 1$. Then $\mathfrak{F}_\wedge = \mathfrak{W}_{\Lambda_\alpha}(e, \alpha)$. In particular, any $F \in \mathfrak{F}_\wedge$ can be written as $F(t) = 1 - \varphi_\alpha(h(t)t^\alpha)$, $t > 0$ for a unique $h \in \mathfrak{H}(e, \alpha)$.*

Proof. By Proposition 6.3, $\mathfrak{W}_{\Lambda_\alpha}(e, \alpha) \subseteq \mathfrak{F}_\wedge$. Conversely, fix any $F \in \mathfrak{F}_\wedge$ and put $h(t) := \varphi_\alpha^{-1}(\overline{F}(t))t^{-\alpha}$ $(1 < t \le e)$, where $\varphi_\alpha^{-1} : (0, \infty) \to (0, 1)$ denotes the inverse function of φ_α, the Laplace transform of $W^{(\alpha)}$ (note that $0 < \overline{F}(t) < 1$ for all $t > 0$ by Lemma 6.2 and Remark 2.1). Extend h to a multiplicatively e-periodic function on $(0, \infty)$. Then it can easily be seen that $h \in \mathfrak{H}(e, \alpha)$. Thus, $G(t) := 1 - \varphi_\alpha(h(t)t^\alpha)$ $(t > 0)$ defines an element of $\mathfrak{W}_{\Lambda_\alpha}(e, \alpha)$ and thus an element of $\mathfrak{F}_\wedge$ as well. Moreover, F and G coincide on $(1, e]$ and, as a consequence of Theorem 8.1, F and G are uniquely determined by their values on $(1, e]$. Hence, $F = G \in \mathfrak{W}_{\Lambda_\alpha}(e, \alpha)$. □

B. Supercritical case. Assuming now $\vartheta < 1 - 1/N$, we follow an idea of Rösler and Jagers [17] and construct a particular non-trivial (and discrete) solution F to Eq. (19) which is then shown to be unique up to scaling (Theorem 8.2). The first step is to define a sequence $(a_n)_{n \ge 0}$ such that $\{0, 1 - a_0, 1 - a_1, \ldots\}$ forms the range of $\overline{F}$. Let a_0 be the unique (by Lemma 8.2(d)) value in $(0, 1)$ such that $g_{N,\vartheta}(a_0) = 1$. Lemma 8.2 also ensures that $g_{N,\vartheta}$ is strictly increasing on $[0, a_0]$ and $g_{N,\vartheta}(u) > u$ for $u \in (0, a_0]$. Hence, $g_{N,\vartheta}^{-1} : [0, 1] \to [0, a_0] \subset [0, 1]$ as well as its iterations are well defined, and we can choose $a_n := g_{N,\vartheta}^{-n}(a_0)$ for $n \geq 1$. Evidently, $(a_n)_{n\geq 0}$ constitutes a decreasing sequence of positive numbers and thus $a_\infty := \lim_{n\to\infty} a_n$ exists. Since $g_{N,\vartheta}$ is continuous, we have $g_{N,\vartheta}(a_\infty) = a_\infty$ so that $a_\infty = 0$ by Lemma 8.2(a). We now define our candidate F as

$$F(t) \ := \ \begin{cases} 0 & \text{if } 0 \leq t \leq 1, \\ 1 - a_n & \text{if } e^n < t \leq e^{n+1} \text{ for } n \in \mathbb{N}_0 \end{cases} \tag{23}$$

and note that F is clearly left continuous and increasing with $\lim_{t\to\infty} \overline{F}(t) = 1$ and therefore a distribution function. Moreover, for $n \in \mathbb{N}_0$ and $e^n < t \leq e^{n+1}$,

$$\overline{F}(te^{-1}) \ = \ a_{n-1} = g_{N,\vartheta}(a_n) = g_{N,\vartheta}(\overline{F}(t)),$$

where $a_{-1} := 1$. As $\overline{F}(t) = 1$ for $t \leq 1$, this shows that $\overline{F}$ solves (21), and so $F \in \mathfrak{F}_\wedge$.

Theorem 8.2. *If $\vartheta < 1 - 1/N$ (supercritical case), then, with F given by* (23),

$$\mathfrak{F}_\wedge \ = \ \{G \in \mathcal{P} : G(t) = F(ct) \text{ for all } t \geq 0 \text{ and some } c > 0\}.$$

Proof. We have already proved that $F \in \mathfrak{F}_\wedge$. Also, $\mathfrak{F}_\wedge$ is obviously closed under scaling, *i.e.*, $\mathfrak{F}_\wedge \supseteq \{\overline{F}(c\,\cdot) : c > 0\}$. Conversely, let $G \in \mathfrak{F}_\wedge$ and notice that G cannot be concentrated at a single point by Proposition 2.2. Hence, we can choose $t > 0$ such that $\overline{G}(t) \in (0, 1)$. Now suppose $\overline{G}(t) \notin \{a_n : n \geq 0\}$. Then there exists a unique $n \geq 0$ satisfying $a_n < \overline{G}(t) < a_{n-1}$ (where $a_{-1} := 1$). Then, from the definition of a_n, we obtain

$$\overline{G}(te^{-(n+1)}) = g_{N,\vartheta}^{n+1}(\overline{G}(t)) \in g_{N,\vartheta}((a_0, a_{-1})) \subseteq (1, \infty),$$

which is obviously impossible. Thus, $\overline{G}(t) \in \{1\} \cup \{a_n : n \geq 0\}$ for all $t \geq 0$. Finally, put $c := \sup\{t \geq 0 : \overline{G}(t) = 1\}$, thus $\overline{G}(c) = 1$, and use once again Eq. (21) and the recursive definition of a_n to obtain $G = F(c\,\cdot)$. Further details are omitted. □

9. Related results for the smoothing transformation

We have already pointed out in Introduction that Eqs. (1) and (5), and thus also the associated maps M in (3) and M_Σ in (6), are naturally connected via the functional equations (2) and (7). The connection is even closer owing to the fact that any Laplace transform vanishing at ∞ is also the survival function of a distribution on $[0, \infty)$. It is therefore not surprising that our results stated in Section 4 have corresponding versions for the additive case dealing with the smoothing transform M_Σ and its fixed points. The latter has been studied in a large number of articles, see *e.g.* [13], [18], [11], [16], [10] and [6].

In order to formulate the counterparts of Theorem 4.1 and Theorem 4.2 for M_Σ, we first recall the definition of α-stable laws and their r-periodic extensions from [13] which take here the role of the Weibull distributions in the min-case.

Definition 9.1 (*cf.* [13], p. 280)**.** For $\alpha > 0$ and $r > 1$, let $\mathfrak{P}(r, \alpha)$ be the set of multiplicatively r-periodic functions $p : \mathbb{R}^+ \to \mathbb{R}^+$ such that $p(t)t^\alpha$ has a completely monotone derivative. Then, for $p \in \mathfrak{P}(r, \alpha)$, the *$r$-periodic α-stable law r-$\mathcal{S}(p, \alpha)$* is defined as the distribution on $[0, \infty)$ with the Laplace transform $\varphi(t) = e^{-p(t)t^\alpha}$ $(t > 0)$.

Note that φ defines a Laplace transform by [14, Criterion 2, p. 441], for $t \mapsto p(t)t^\alpha$ is positive with a completely monotone derivative. Further, if $p(t) \equiv c$, then φ is the Laplace transform of a positive stable law with the scale parameter $(c/\cos(\pi\alpha/2))^{1/\alpha}$, shift 0 and the index of stability α, *cf.* [23, Definition 1.1.6 and Proposition 1.2.12]. This distribution, denoted as $\mathcal{S}(c, \alpha)$, does not depend on r (which is thus dropped in the notation). For our convenience, we define r-$\mathcal{S}(0, \alpha) = \mathcal{S}(0, \alpha) := \delta_0$.

We continue with a short proof of the known fact that periodic stable laws as defined above only exist for $\alpha \leq 1$ and a definition which is just the canonical modification of Definition 4.2 for the previously defined generalized stable laws.

Lemma 9.1. *In the situation of Definition 9.1, any element of* $\mathfrak{P}(r,1)$ *is constant, while* $\mathfrak{P}(r,\alpha)=\emptyset$ *for any* $\alpha>1$.

Proof. If $\alpha\ge 1$, we have, for each $s\in(1,r]$,

$$p(s)=\lim_{n\to\infty}\frac{1-e^{-p(sr^{-n})(sr^{-n})^{\alpha}}}{(sr^{-n})^{\alpha}}=\lim_{n\to\infty}\frac{1-e^{-p(sr^{-n})(sr^{-n})^{\alpha}}}{(sr^{-n})}\cdot(sr^{-n})^{1-\alpha}.$$

Since $e^{-p(t)t^{\alpha}}$ is convex and $t^{1-\alpha}$ is non-increasing if $\alpha\ge 1$, we see that p must be non-increasing in s. In fact, p is even strictly decreasing if $\alpha>1$. But, in view of the periodicity of p, the latter is impossible, thus $\mathfrak{P}(r,\alpha)=\emptyset$ for $\alpha>1$, while p must be constant in the case $\alpha=1$. □

Definition 9.2. Let $\alpha\in(0,1]$, and let Λ be a probability measure on $[0,\infty)$. Then

(a) $\mathcal{S}_{\Lambda}(1,\alpha)$ denotes the class of Λ-mixtures of positive α-stable laws F of the form $F(\cdot)=\int\mathcal{S}(yc,\alpha)(\cdot)\,\Lambda(dy)$, where $c>0$.
(b) $\mathcal{S}_{\Lambda}(r,\alpha)$ for $r>1$ denotes the class of Λ-mixtures of r-$\mathcal{S}(p,\alpha)$ distributions F of the form $F(\cdot)=\int r\text{-}\mathcal{S}(yp,\alpha)(\cdot)\,\Lambda(dy)$, where $p\in\mathfrak{P}(r,\alpha)$.

Finally, the notions "α-bounded", "α-regular" and "α-elementary" for fixed points of M_{Σ} are defined exactly as in Definition 4.1 for fixed points of M, when substituting $D_{\alpha}\overline{F}$ with $t^{-\alpha}(1-\varphi(t))$, φ the Laplace transform of F. The respective classes are denoted as $\mathfrak{F}^{\alpha}_{\Sigma,b},\mathfrak{F}^{\alpha}_{\Sigma,r}$, and $\mathfrak{F}^{\alpha}_{\Sigma,e}$.

We are now ready to formulate the results corresponding to our theorems in Section 4 for the smoothing transform. The standing assumptions (A1) and (A2) for the min-case can be replaced here with the weaker ones

(A3)	Supercriticality:	$\mathbb{E}N\ >\ 1;$
(A4)	Nondegeneracy:	$\mathbb{P}(T\in\{0,1\}^{\infty})\ <\ 1.$

Theorem 9.1. *Suppose* (A3) *and* (A4). *Then the following assertions are equivalent for any* $\alpha>0$*:*

(a) *Eq.* (5) *has an* α*-elementary solution* ($\mathfrak{F}^{\alpha}_{\Sigma,e}\neq\emptyset$).
(b) *Eq.* (5) *has an* α*-regular solution* ($\mathfrak{F}^{\alpha}_{\Sigma,r}\neq\emptyset$).
(c) $\alpha\in(0,1]$, $m(\alpha)=1$, *and* $\mathbb{P}(W^{(\alpha)}>0)>0$.
(d) $\alpha\in(0,1]$, $m(\alpha)=1$, *the random walk* $(\overline{S}_{\alpha,n})_{n\ge 0}$ *converges to* ∞ *a.s., and*

$$\int_{(1,\infty)}\left[\frac{u\log u}{\mathbb{E}(\overline{S}^{+}_{\alpha,1}\wedge\log u)}\right]\mathbb{P}(W^{(\alpha)}_1\in du)<\infty.$$

Theorem 9.2. *Suppose* (A3), (A4) *and that* $m(\alpha)=1$ *and* $\mathbb{E}W^{(\alpha)}=1$ *for some* $\alpha\in(0,1]$. *Then*

$$\mathfrak{F}^{\alpha}_{\Sigma,b}=\mathfrak{F}^{\alpha}_{\Sigma,r}=\mathfrak{F}^{\alpha}_{\Sigma,e}=\mathcal{S}_{\Lambda_{\alpha}}(d,\alpha),$$

where $d=r>1$ *in the* r*-geometric case* ($\mathbb{G}(T)=r^{\mathbb{Z}}$) *and* $d=1$ *in the continuous case* ($\mathbb{G}(T)=\mathbb{R}^{+}$).

The proofs of the previous two theorems are essentially the same as those for Theorems 4.1 and 4.2 except for the additional assertion $\alpha\le 1$. But the arguments in the proof of Theorem 4.1 show that the Laplace transform φ of any element of $\mathfrak{F}^{\alpha}_{\Sigma,b}$ can be written as $\varphi(t)=\varphi_{\alpha}(p(t)t^{\alpha})$ ($t>0$), where p is a multiplicatively r-periodic function in the r-geometric case and a constant in the continuous case. Owing to Lemma 9.1, we get $\alpha\le 1$ if we can prove that $p\in\mathfrak{P}(r,\alpha)$ in the r-geometric case. By writing $p(t)=\varphi^{-1}(\varphi_{\alpha}(t))\cdot t^{-\alpha}$ ($t>0$) (where φ^{-1} denotes the inverse function of φ), we see that p

is infinitely often differentiable. It remains to verify that $t \mapsto p(t)t^\alpha$ has a completely monotone derivative. To this end, we observe that

$$r^{\alpha n}(1-\varphi(tr^{-n})) \;=\; \frac{1-\varphi_\alpha(p(tr^{-n})(tr^{-n})^\alpha)}{p(tr^{-n})\cdot(tr^{-n})^\alpha}\cdot p(t)t^\alpha \;\to\; p(t)t^\alpha \quad (n\to\infty),$$

having utilized that $\varphi_\alpha'(0) = -\mathbb{E}W^{(\alpha)} = -1$. Since $t \mapsto r^{\alpha n}(1-\varphi(tr^{-n}))$ has a completely monotone derivative for each $n \in \mathbb{N}$ and since the convergence is uniform on compact sets, $t \mapsto p(t)t^\alpha$ has a completely monotone derivative as well.

Appendix A. Biggins' Theorem

Assumptions (A3) and (A4) are in force throughout. Let $q \in [0,1)$ denote the extinction probability of $W_n^{(0)} = \sum_{|v|=n} \mathbf{1}_{\{L(v)>0\}}$, $n \geq 0$.

The subsequent characterization theorem for martingale limits in branching random walks, which are nothing but limits of WBP's having the martingale property, is a crucial ingredient to our analysis of α-elementary fixed points. In the stated most general form, which imposes no conditions on T beyond $m(\alpha) = 1$, it was recently obtained by Alsmeyer and Iksanov [3], but the first version of the result under additional assumptions on T was obtained more than three decades ago by Biggins [7] and later reproved (under relaxed conditions) by Lyons [19].

Theorem A.1 (*cf.* [3], Theorem 1.3)**.** *Suppose* (A3), (A4), *and* $m(\alpha) = 1$. *Then the following four assertions are equivalent:*

(a) $\mathbb{P}(W^{(\alpha)} > 0) > 0$.
(b) $\mathbb{P}(W^{(\alpha)} > 0) = q$.
(c) $\mathbb{E}W^{(\alpha)} = 1$.
(d) $\lim_{n\to\infty} \overline{S}_{\alpha,n} = \infty$ *a.s. and*

$$\int_{(1,\infty)} \left[\frac{u \log u}{\mathbb{E}(\overline{S}_{\alpha,1}^+ \wedge \log u)}\right] \mathbb{P}(W_1^{(\alpha)} \in du) < \infty.$$

We further state the a corollary which says that $m(\alpha) = 1$ and $\mathbb{P}(W^{(\alpha)} > 0) > 0$ imply that α equals the characteristic exponent of T, that is, the *minimal* value at which m equals 1. This is relevant to be pointed out because m, as a strictly convex function on $\{\beta \geq 0 : m(\beta) < \infty\}$, may equal 1 for two values α_1, α_2.

Corollary A.1. *Suppose* $m(\alpha) = 1$ *and* $\mathbb{P}(W^{(\alpha)} > 0) > 0$. *Then* $m(\beta) > 1$ *for all* $\beta \in [0,\alpha)$ *so that* α *is the characteristic exponent of* T.

Proof. In view of Theorem A.1(d), $m(\beta) \leq 1$ for some $\beta < \alpha$ contradicts the fact that the random walk $(\overline{S}_{\alpha,n})_{n\geq 0}$ drifts to infinity a.s. □

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Universität Münster, Germany
E-mail address: gerolda@math.uni-muenster.de

Universität Münster, Germany
E-mail address: mmeiners@uni-muenster.de