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LARGE DEVIATION PRINCIPLE FOR STOCHASTIC EQUATIONS WITH LOCAL TIME

The large deviation principle for solutions of one-dimensional equations with a local time is proved. The explicit form for the rate function is obtained. We also consider the large deviation principle for solutions of Itô's stochastic equations with discontinuous coefficients.

1. Introduction

In this paper, we are concerned with the large deviation principle (LDP) of one-dimensional stochastic equations with local time and small diffusion

$$\xi_\varepsilon(t) = x + \beta L^{\xi_\varepsilon}(t,0) + \int_0^t b(\xi_\varepsilon(s))ds + \varepsilon \int_0^t \sigma(\xi_\varepsilon(s))dw(s). \tag{1}$$

If $\beta = 0$, the LDP for the family $\{\xi_\varepsilon,\ \varepsilon > 0\}$ is the well-known result for different classes of coefficients in Eq. (1). For smooth coefficients, it is a classic result by Freidlin and Wentzel [4]. Several papers have studied LDP for solutions of Itô stochastic equations with discontinuous coefficients. In [1,2], the $d-$dimensional diffusion with coefficients which are continuous except for a $(d-1)-$dimensional hyperplane $\{x \in E_d : x_1 > 0\}$ was considered (see [2] for a more detailed review).

To study Eq. (1) for $\beta \neq 0$, we use the method offered in [3, Proposition 4.9], [6], [9, Proposition 2.2], [10] which reduces the equation with a local time to the Itô's equation. We note that even when the coefficients in (1) are continuous functions, the coefficients corresponding to Itô's equations are discontinuous functions. For this reason, we apply ideas developed in [1,2]. But the formal use of these results requires the existence of two bounded derivatives of coefficients of the equation. In Section 2, we weaken these conditions up to the Lipschitz conditions.

By $I_A(x)$, we denote an indicator of the set A. Equation (1) has a weak solution if, for given functions $b(x)$ and $\sigma(x)$ and a constant β, there are a probability space $(\Omega, \Im, \Im_t, P)$ with the flow of σ-algebras $\Im_t$, $t \geq 0$, continuous semimartingale $(\xi(t), \Im_t)$, and standard one-dimensional Wiener process $(w(t), \Im_t)$ such that

$$L^\xi(t,0) = \lim_{\delta\to 0} \frac{1}{2\delta} \int_0^t I_{(-\delta,\delta)}(\xi(s))\sigma^2(\xi(s))ds \tag{2}$$

exists almost surely, and (1) is valid almost surely.
For the coefficients of Eq. (1), we introduce the following condition (I).

Condition (I):

I_1. The constant $|\beta| < 1$.

I_2. For the measurable functions $b(x)$ and $\sigma(x)$, $x \in \mathbf{R}$, there are the constants $0 < \lambda \leq \Lambda$ such that

$$|b(x)| \leq \Lambda, \quad \lambda \leq \sigma^2(x) \leq \Lambda.$$

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I_3. For the functions $b(x)$ and $\sigma^2(x)$, the Lipschitz condition holds on the semiaxes $(-\infty, 0]$ and $(0, +\infty)$.

Under conditions I_1 and I_2, there exists a unique weak solution of Eq. (1) [3, theorem 4.33].

Let $\mathbf{R}$ be a one-dimensional space, and let $\mathcal{B}(\mathbf{R})$ be its Borel σ-algebra. By C[0,T], we denote the space of all functions $f(t)$ continuous on the interval [0,T] with the values in $\mathbf{R}$, and C is its Borel σ-algebra. For an absolutely continuous function $f(t)$, we use a standard representation $f(t) = f(a) + \int_a^t \dot{f}(s)ds$. Let AC^+[0,T] denote the set of all absolutely continuous functions on [0,T] starting from 0 with derivatives in [0,1]. The norms in the spaces C[0,T] and AC^+[0,T] are $||x||_T = \sup_{t\in[0,T]} |x(t)|$. We denote $\mathbf{C}_T =$ C[0,T] $\times AC^+$[0,T], and the norm in this space is $||(x, y)||_T = ||x||_T + ||y||_T$.

Let $C[0, \infty)$ be the space of all continuous functions $f(t)$ from $[0, \infty)$ to $\mathbf{R}$; $AC^+[0, \infty)$ denote the set of all absolutely continuous functions on $[0, \infty)$ starting from 0 with derivatives in $[0, 1]$. For the spaces C[0,∞) and $AC^+[0, \infty)$, we use the norm $||x||_\infty = \sum_{k=1}^{\infty} 2^{-k}\Big(\min\Big\{\sup_{t\le k} |x(t)|, 1\Big\}\Big)$. Denote $\mathbf{C}_\infty =$ C[0,∞) $\times AC^+[0, \infty)$ with norm $||(x, y)||_\infty = ||x||_\infty + ||y||_\infty$.

For $f \in$ C[0,T] or $f \in C[0, \infty)$, we write $g \in H^+(f)$ if $g(t)$ is an absolutely continuous function with derivatives $\dot{g}(t)$ such that $\dot{g}(t) = 0$ if $f(t) < 0$, $\dot{g}(t) = 1$ if $f(t) > 0$, and $\dot{g}(t) \in [0, 1]$ if $f(t) = 0$.

Let $(X,\ \mathcal{B}(X))$ be a metric space with metric ρ, and let $I(x) : X \to [0, \infty]$ be a lower semicontinuous functional such that, for any $a > 0$, the set $\{x : I(x) \le a\}$ is compact. Here, we consider several functionals $I(x)$ and prove Lemmas 7-9 concerning their properties.

A family of probability measures μ_ε on X is said to satisfy the LDP with the rate functional $I(x)$ if the following conditions hold:

a) for any open set $G \in \mathcal{B}(X)$,

$$\liminf_{\varepsilon\to 0} \varepsilon^2 \ln \mu_\varepsilon(G) \ge -\inf\{I(x), x \in G\};$$

b) for any closed set $F \in \mathcal{B}(X)$,

$$\limsup_{\varepsilon\to 0} \varepsilon^2 \ln \mu_\varepsilon(F) \le -\inf\{I(x), x \in F\}.$$

We now formulate the contraction principle [4, theorem 5.3.1]. Let the measures μ_ε on X induced by random elements X_ε satisfy the LDP with a rate functional $I(x)$, and let $F(x)$ be a continuous function from X to X'. Then the family of measures μ'_ε on X' induced by random elements $\{F(X_\varepsilon)\}$ satisfies the LDP with the rate functional

$$I'(x) = \inf\{I(y),\ y : F(y) = x\}.$$

By $u_\varepsilon(t) = \int_0^t I_{(0,\infty)}(\xi_\varepsilon(s))ds$, we denote the occupation time of a process $\xi_\varepsilon(t)$ on the positive semiaxis.

Denote $\big(f(x)\big)^+ = \max\big(f(x), 0\big)$; $Leb(A)$ is Lebesgue's measure of the set A.

The paper is organized as follows. The LDP for the Itô's stochastic equation with discontinuous coefficients is considered in Section 2. In Section 3, we will prove the theorem as our main result. The auxiliary results are formulated (some of them are also proved) in Section 4.

Theorem 1.1. *Let Condition (I) be satisfied. Then the measures μ_ε induced by the processes $\xi_\varepsilon(t)$ and $u_\varepsilon(t)$ on $\mathbf{C}_\infty$ satisfy the LDP with the rate function*

$$I_\infty(\phi,\psi) = \int_0^\infty L(\phi(s),\dot\phi(s),\dot\psi(s))ds, \qquad \text{where}$$

$$L(\phi(s),\dot\phi(s),\dot\psi(s)) = \begin{cases} \frac{1}{2}\frac{\left(\dot\phi(s)-b(\phi(s))\right)^2}{\sigma^2(\phi(s))}, & if\ \phi(s)\neq 0, \\ \frac{1}{2}\frac{b^2(0)}{\sigma^2(0)}\frac{\left(1+\beta-2\beta\dot\psi(s)\right)^2}{(1+\beta)^2-4\beta\dot\psi(s)}, & if\ \phi(s)=0 \ and\ \beta b(0)<0, \\ \frac{1}{2}\frac{b^2(0)}{\sigma^2(0)}, & if\ \phi(s)=0\ and\ \beta b(0)\geq 0, \end{cases} \tag{3}$$

for the absolutely continuous ϕ such that $\phi(0)=x$, $\int_0^\infty \left(\dot\phi(s)\right)^2 ds<\infty$, $\psi\in H^+(\phi)$, $\psi(0)=0$, and $I_\infty(\phi,\psi)=\infty$ otherwise.

As a consequence of the contraction principle, we have the following result.

Corollary 1.1. *Let conditions of* **Theorem 1.1** *be satisfied. Then the measures μ_ε induced by the process $\xi_\varepsilon(t)$ on $(\mathrm{C}[0,\infty),C)$ satisfy the LDP with the rate function $I(\phi)=\int_0^\infty L(\phi(s),\dot\phi(s))ds$, where*

$$L(\phi(s),\dot\phi(s)) = \begin{cases} \frac{1}{2}\frac{\left(\dot\phi(s)-b(\phi(s))\right)^2}{\sigma^2(\phi(s))}, & if\ \phi(s)\neq 0, \\ \frac{1}{2}\frac{b^2(0)}{\sigma^2(0)}\left(1-\beta^2\right), & if\ \phi(s)=0\ and\ \beta b(0)<0, \\ \frac{1}{2}\frac{b^2(0)}{\sigma^2(0)}, & if\ \phi(s)=0\ and\ \beta b(0)\geq 0, \end{cases} \tag{4}$$

for the absolutely continuous ϕ such that $\phi(0)=x$ and $\int_0^\infty \left(\dot\phi(s)\right)^2 ds<\infty$ and $\int_0^\infty I_{\{\phi(s)=0\}}I_{\{\beta b(0)<0\}}ds<\infty$ and $I(\phi)=\infty$ for other ϕ.

2. LDP for Itô's equation

Let $B_i(x)$, $i=1,2$, $x\in\mathbf{R}$ be measurable bounded functions, let B_3 be a constant, and we define

$$B(x) = \begin{cases} B_1(x), & \text{if } x<0, \\ B_3, & \text{if } x=0, \\ B_2(x), & \text{if } x>0. \end{cases} \tag{5}$$

Define the process $x_\varepsilon(t)$ as a solution of the stochastic equation

$$x_\varepsilon(t) = x + \int_0^t B(x_\varepsilon(s))ds + \varepsilon w(t), \tag{6}$$

and let $v_\varepsilon(t)=\int_0^t I_{(0,\infty)}(x_\varepsilon(s))ds=\int_0^t \dot v_\varepsilon(s)ds$. We consider the LDP for the process $(x_\varepsilon(t),\, v_\varepsilon(t))$.

Define the function $\tilde B(x)$:

$$\tilde B(x) = \begin{cases} B_1(x), & \text{if } x\leq 0, \\ B_2(x), & \text{if } x>0, \end{cases}$$

and consider the equation

$$\tilde{x}_\varepsilon(t) = x + \int_0^t \tilde{B}(\tilde{x}_\varepsilon(s))ds + \varepsilon w(t). \tag{7}$$

Let $\tilde{v}_\varepsilon(t) = \int_0^t I_{(0,\infty)}(\tilde{x}_\varepsilon(s))ds$.

The LDP on [0,T] for the pair $(\tilde{x}_\varepsilon, \tilde{v}_\varepsilon)$ was proved in [1, theorem 2.1]. Taking into account that both processes x_ε and $\tilde{x}_\varepsilon$ spend zero time at the point 0, it should be noted that the measures induced by them on the space (C[0,T], C) coincide. Conversely, the LDP on [0,T] for $(x_\varepsilon, v_\varepsilon)$ is valid as well. But, as was specified in Introduction, the existence of two bounded derivatives of the functions $B_i(x)$ is required in this theorem. In our case, this condition can be weakened.

Theorem 2.1. *Let the function $B_1(x)$ satisfy the Lipschitz condition for $x \le 0$, and let $B_2(x)$ satisfy the Lipschitz condition for $x \ge 0$. Then the measures μ_ε induced by the processes $(x_\varepsilon(t), v_\varepsilon(t))$ on $\mathbf{C}_\infty$ satisfy the LDP with the rate function*

$$I_\infty^x(\phi,\psi) = \int_0^\infty M(\phi(s),\dot\phi(s),\dot\psi(s))ds, \qquad \textit{where}$$

$$M(\phi(s),\dot\phi(s),\dot\psi(s)) = \begin{cases} \frac{1}{2}\Big(\dot\phi(s) - B(\phi(s))\Big)^2, & if\ \phi(s) \neq 0, \\ \frac{1}{2}\Big(B_2(0)\dot\psi(s) + B_1(0)(1-\dot\psi(s))\Big)^2, & if\ \phi(s) = 0\ and\ B_1(0) > B_2(0), \\ \frac{1}{2}\Big(B_2^2(0)\dot\psi(s) + B_1^2(0)(1-\dot\psi(s))\Big), & if\ \phi(s) = 0\ and\ B_1(0) \le B_2(0), \end{cases} \tag{8}$$

for absolutely continuous ϕ such that $\phi(0) = x$, $\int_0^\infty \big(\dot\phi(s)\big)^2 ds < \infty$, and $\psi \in H^+(\phi)$, $\psi(0) = 0$. For all other pairs $(\phi,\ \psi)$, we set $I_\infty^x(\phi,\psi) = \infty$.

Corollary 2.1. *Let conditions of* **Theorem 2.1** *be satisfied. Then the measures μ_ε induced by the process $x_\varepsilon(t)$ on $(\mathrm{C}[0,\infty), C)$ satisfy the LDP with the rate function*

$$\begin{aligned} I^x(\phi) =& \frac{1}{2}\int_0^\infty I_{\{\phi(s)\neq 0\}}\Big(\dot\phi(s) - B(\phi(s))\Big)^2 ds + \\ &+ \frac{1}{2}\int_0^\infty I_{\{\phi(s)=0\}} ds \min\big(B_1^2(0), B_2^2(0)\big)\Big(1 - I_{\{B_1(0)>0>B_2(0)\}}\Big) \end{aligned} \tag{9}$$

for absolutely continuous ϕ such that $\phi(0) = x$, $\int_0^\infty \big(\dot\phi(s)\big)^2 ds < \infty$, $\int_0^\infty I_{\{\phi(s)=0\}} ds < \infty$. For all other ϕ, we set $I^x(\phi) = \infty$.

Proof of Theorem 2.1.

To prove the theorem, it is sufficient to show that [12, corollary 3.4]

(i) For any $a > 0$, the set $\{(\phi,\psi) : I(\phi,\psi) \le a\}$ is compact.

(ii) For any $R > 0$, there exist a compact set K such that, for any $\delta > 0$, ,

$$P\Big\{\{||x_\varepsilon - \phi||_T < \delta, ||v_\varepsilon - \psi||_T < \delta\} \not\subseteq K\Big\} \le \exp(-\frac{R}{\varepsilon^2}), \text{ if } \varepsilon \text{ is small.}$$

(iii) $$\lim_{\delta\to 0}\liminf_{\varepsilon\to 0} \varepsilon^2 \ln P\{||x_\varepsilon - \phi||_T < \delta, ||v_\varepsilon - \psi||_T < \delta\} =$$

$$= \lim_{\delta\to 0}\limsup_{\varepsilon\to 0} \varepsilon^2 \ln P\{||x_\varepsilon - \phi||_T < \delta, ||v_\varepsilon - \psi||_T < \delta\} = -I_T(\phi,\psi) =$$

$$= -\Bigg(\frac{1}{2}\int_0^T |\dot\phi(s)|^2 ds - \int_0^T B(\phi(s))\dot\phi(s)ds +$$

$$+\frac{1}{2}\int_0^T\Big[B_2^2(\phi(s))-B_1^2(\phi(s))\Big]\dot\psi(s)ds+$$
$$+\frac{1}{2}\int_0^T B_1^2(\phi(s))ds-\frac{((B_1(0)-B_2(0))^+)^2}{2}\int_0^T\dot\psi(s)(1-\dot\psi(s))ds\Bigg). \quad (10)$$

Condition (i) is proved in Lemma 4.7.

At first, we prove (iii) - the local LDP for the measures μ_ε induced by the processes $(x_\varepsilon(t),v_\varepsilon(t))$ on $\mathbf{C}_T$ with the rate function $I_T(\phi,\psi)$ for every T>0.

Denote

$$y_\varepsilon(t)=x+\varepsilon w(t)\,,$$
$$k_\varepsilon(t)=\int_0^t I_{(0,\infty)}(y_\varepsilon(s))ds,$$

μ_{x_ε} are the measures on C[0,T] induced by $x_\varepsilon(t)$, and μ_{y_ε} are the measures on C[0,T] induced by $y_\varepsilon(t)$.

By the Girsanov's theorem,

$$\ln\frac{d\mu_{x_\varepsilon}}{d\mu_{y_\varepsilon}}=\frac{1}{\varepsilon}\int_0^T B(y_\varepsilon(s))dw(s)-\frac{1}{2\varepsilon^2}\int_0^T B^2(y_\varepsilon(s))ds. \quad (11)$$

Let us transform this expression. We get

$$\int_0^T B(y_\varepsilon(s))dw(s)=$$
$$=\int_0^T[B_1(y_\varepsilon(s))I_{(-\infty,0]}(y_\varepsilon(s))+B_2(y_\varepsilon(s))I_{(0,\infty)}(y_\varepsilon(s))+$$
$$+(B_3(0)-B_1(0))I_{\{0\}}(y_\varepsilon(s))]dw(s). \quad (12)$$

As $E\Big(\int_0^T I_{\{0\}}(y(s))dw\Big)^2=\int_0^T P\{x+\varepsilon w(s)=0\}ds=0$ (here and further, E stands for the mathematical expectation), then relation (12) yields

$$\int_0^T B(y_\varepsilon(s))dw(s)=\int_0^T\big[B_1(y_\varepsilon(s))I_{(-\infty,0]}(y_\varepsilon(s))+B_2(y_\varepsilon(s))I_{(0,\infty)}(y_\varepsilon(s))\big]dw(s). \quad (13)$$

Reasoning similarly, we conclude that

$$\int_0^T B^2(y_\varepsilon(s))ds=\int_0^T[B_1^2(y_\varepsilon(s))I_{(-\infty,0]}(y_\varepsilon(s))+B_2^2(y_\varepsilon(s))I_{(0,\infty)}(y_\varepsilon(s))]ds. \quad (14)$$

From (11), (13), and (14), we have

$$\ln\frac{d\mu_{x_\varepsilon}}{d\mu_{y_\varepsilon}}=\frac{1}{\varepsilon}\int_0^T[B_1(y_\varepsilon(s))I_{(-\infty,0]}(y_\varepsilon(s))+B_2(y_\varepsilon(s))I_{(0,\infty)}(y_\varepsilon(s))]dw-$$
$$-\frac{1}{2\varepsilon^2}\int_0^T[B_1^2(y_\varepsilon(s))I_{(-\infty,0]}(y_\varepsilon(s))+B_2^2(y_\varepsilon(s))I_{(0,\infty)}(y_\varepsilon(s))]ds. \quad (15)$$

Let us transform the stochastic integral in (15) by the Tanaka formula. By $Df(x)$, we denote the symmetric derivative of a function $f(x)$:

$$Df(x)=\lim_{\delta\downarrow 0}\frac{f(x+\delta)-f(x-\delta)}{2\delta},$$

and, by $n_f(dx)$ for a function $f(x)$, we denote the signed measure on $(\mathbf{R},\mathcal{B}(\mathbf{R}))$ which is defined from the equality

$$\int\frac{d^2H(x)}{dx^2}f(x)dx=\int H(x)n_f(dx),$$

if it is satisfied for any infinitely differentiable function $H(x)$ with a compact support.

Then, for a function $f(x)$, for which there exist $Df(x)$ and $n_f(dx)$, and for a continuous semimartingale $X(t)$ with the canonical decomposition $X(t)=X(0)+M(t)+A(t)$, where M stands for a continuous local martingale, and A is a continuous process of finite variation, the Tanaka formula has following form [3, formula 4.3]:

$$f(X(t)) = f(X(0)) + \int_0^t Df(X(s))dX(s) + \frac{1}{2}\int L^X(t,y)n_f(dy).$$

We observe that, for an arbitrary measurable function $g(x)$, for which one of the integrals makes sense, the following formula is valid [13, (VI, Corollary 1.6)]:

$$\int_0^t g(X(s))d\langle M\rangle_s = \int g(y)L^X(t,y)dy.$$

Here, $\langle M\rangle$ is a continuous increasing process associated to the local martingale M.

Put $F(x) = \int_0^x B(y)dy$. Then

$$DF(x) = \begin{cases} B_1(x)\ , & \text{if } x<0,\\ \dfrac{B_1(0)+B_2(0)}{2}\ , & \text{if } x=0,\\ B_2(x)\ , & \text{if } x>0. \end{cases}$$

Then, for the infinitely differentiable function $H(x)$ with compact supports in $(-\infty, \infty)$, using the existence of Sobolev's derivatives (it comes from the Lipschitz condition) of the functions $B_i(x)$, we get

$$n_F(dx) = (B_2(0) - B_1(0))\delta_0(x)dx + [\dot B_1(x)I_{(-\infty,0]}(x) + \dot B_2(x)I_{(0,\infty)}(x)]dx$$

with the Dirac-delta function $\delta_0(x)$. According to the Tanaka's formula,

$$\begin{aligned} F(y_\varepsilon(\mathrm{T})) =& F(x) + \varepsilon\int_0^T [B_1(y_\varepsilon(s))I_{(-\infty,0]}(y_\varepsilon(s)) + B_2(y_\varepsilon(s))I_{(0,\infty)}(y_\varepsilon(s))]dw+\\ &+\frac{\varepsilon^2}{2}\int_0^T [\dot B_1(y_\varepsilon(s))I_{(-\infty,0]}(y_\varepsilon(s)) + \dot B_2(y_\varepsilon(s))I_{(0,\infty)}(y_\varepsilon(s))]ds+\\ &+\frac{B_2(0)-B_1(0)}{2}L^{y_\varepsilon}(\mathrm{T},0). \end{aligned} \tag{16}$$

From (15) and (16), it follows that

$$\begin{aligned} \ln\frac{d\mu_{x_\varepsilon}}{d\mu_{y_\varepsilon}} =& \frac{1}{\varepsilon^2}[F(y_\varepsilon(\mathrm{T})) - F(x)] - \frac{B_2(0)-B_1(0)}{2\varepsilon^2}L^{y_\varepsilon}(\mathrm{T},0)-\\ &-\frac{1}{2\varepsilon^2}\int_0^T [B_1^2(y_\varepsilon(s))I_{(-\infty,0]}(y_\varepsilon(s)) + B_2^2(y_\varepsilon(s))I_{(0,\infty)}(y_\varepsilon(s))]ds-\\ &-\frac{1}{2}\int_0^T [\dot B_1(y_\varepsilon(s))I_{(-\infty,0]}(y_\varepsilon(s)) + \dot B_2(y_\varepsilon(s))I_{(0,\infty)}(y_\varepsilon(s))]ds. \end{aligned} \tag{17}$$

In view of (17), taking into account that

$$F(\phi(\mathrm{T})) - F(\phi(0)) = \int_0^T B(\phi(s))\dot\phi(s)ds,$$

we get

$$\begin{aligned} &P\{||x_\varepsilon - \phi||_T < \delta, ||v_\varepsilon - \psi||_T < \delta\} = EI_{\{||y_\varepsilon-\phi||_T<\delta\}}(\omega)I_{\{||k_\varepsilon-\psi||_T<\delta\}}(\omega)\frac{d\mu_{x_\varepsilon}}{d\mu_{y_\varepsilon}} =\\ &= EI_{\{||y_\varepsilon-\phi||_T<\delta\}}(\omega)I_{\{||k_\varepsilon-\psi||_T<\delta\}}(\omega)\exp\Big\{-\frac{B_2(0)-B_1(0)}{2\varepsilon^2}L^{y_\varepsilon}(\mathrm{T},0)+\\ &+\frac{1}{\varepsilon^2}\int_0^T B(\phi(s))\dot\phi(s)ds - \frac{1}{2\varepsilon^2}\int_0^T \big[B_1^2(\phi(s))(1-\dot\psi(s)) + B_2^2(\phi(s))\dot\psi(s)\big]ds\Big\}J_\varepsilon, \end{aligned} \tag{18}$$

where

$$J_\varepsilon = \exp\Bigg\{\frac{1}{\varepsilon^2}[F(y_\varepsilon(\mathrm{T})) - F(\phi(\mathrm{T}))] + \frac{1}{2\varepsilon^2}\int_0^T [B_1^2(\phi(s))(1-\dot\psi(s)) + B_2^2(\phi(s))\dot\psi(s)]ds -$$

$$- \frac{1}{2\varepsilon^2}\int_0^T [B_1^2(y_\varepsilon(s))I_{(-\infty,0]}(y_\varepsilon(s)) + B_2^2(y_\varepsilon(s))I_{(0,\infty)}(y_\varepsilon(s))]ds -$$

$$- \frac{1}{2}\int_0^T [\dot B_1(y_\varepsilon(s))I_{(-\infty,0]}(y_\varepsilon(s)) + \dot B_2(y_\varepsilon(s)))I_{(0,\infty)}(y_\varepsilon(s))]ds\Bigg\}.$$

Consider now J_ε in more details. We denote

$$A_1 = F(y_\varepsilon(\mathrm{T})) - F(\phi(\mathrm{T})),$$

$$A_2 = \int_0^T \Big[B_1^2(\phi(s))(1-\dot\psi(s)) + B_2^2(\phi(s))\dot\psi(s) - B_1^2(y_\varepsilon(s))I_{(-\infty,0]}(y_\varepsilon(s)) -$$

$$- B_2^2(y_\varepsilon(s))I_{(0,\infty)}(y_\varepsilon(s))\Big]ds;$$

$$A_3 = -\int_0^T [\dot B_1(y_\varepsilon(s))I_{(-\infty,0]}(y_\varepsilon(s)) + \dot B_2(y_\varepsilon(s)))I_{(0,\infty)}(y_\varepsilon(s))]ds.$$

It is clear that $J_\varepsilon = \exp\Big\{\frac{1}{\varepsilon^2}A_1 + \frac{1}{2\varepsilon^2}A_2 + \frac{1}{2}A_3\Big\}$.

For any $\gamma > 0$, there exists $\delta > 0$ such that the conditions $||y_\varepsilon - \phi||_T < \delta$, $||k_\varepsilon - \psi||_T < \delta$, and conditions of the theorem yield (with some constant K)

$$|A_1| = |F(y_\varepsilon(\mathrm{T})) - F(\phi(\mathrm{T}))| \le K||y_\varepsilon - \phi||_T \le \gamma.$$

Then, by Lemma 4.1 for any $\gamma > 0$, there exists $\delta > 0$ such that if $||y_\varepsilon - \phi||_T < \delta$, $||k_\varepsilon - \psi||_T < \delta$, we have

$$|A_2| = \Bigg|\int_0^T \Big[B_1^2(\phi(s))(1-\dot\psi(s)) + B_2^2(\phi(s))\dot\psi(s)\Big]ds -$$

$$- \int_0^T \Big[B_1^2(y_\varepsilon(s))I_{(-\infty,0]}(y_\varepsilon(s)) + B_2^2(y_\varepsilon(s))I_{(0,\infty)}(y_\varepsilon(s))\Big]ds\Bigg| =$$

$$= \Bigg|\int_0^T \Big[B_1^2(\phi(s))(1-\dot\psi(s)) + B_2^2(\phi(s))\dot\psi(s)\Big]ds -$$

$$- \int_0^T \Big[B_1^2(y_\varepsilon(s))(1-\dot k_\varepsilon(s)) + B_2^2(y_\varepsilon(s))\dot k_\varepsilon(s)\Big]ds\Bigg| < \gamma.$$

By conditions of the theorem,

$$|A_3| = \Bigg|\int_0^T \Big[\dot B_1(y_\varepsilon(s))I_{(-\infty,0]}(y_\varepsilon(s)) + \dot B_2(y_\varepsilon(s)))I_{(0,\infty)}(y_\varepsilon(s))\Big]ds\Bigg| \le K.$$

Hence, on the set $||y_\varepsilon - \phi||_T < \delta$, $||k_\varepsilon - \psi||_T < \delta$, we obtain

$$\exp\Big\{-\frac{\gamma}{\varepsilon^2} - K\Big\} \le J_\varepsilon \le \exp\Big\{\frac{\gamma}{\varepsilon^2} + K\Big\}. \tag{19}$$

From [1, Lemma 4.5 and formula (5.12)], it follows that

$$\lim_{\delta\to 0}\liminf_{\varepsilon\to 0}\varepsilon^2 \ln EI_{||y_\varepsilon-\phi||_T<\delta}(\omega)I_{||k_\varepsilon-\psi||_T<\delta}(\omega)\exp\Big\{-\frac{B_2(0)-B_1(0)}{2\varepsilon^2}L^{y_\varepsilon}(\mathrm{T},0)\Big\} =$$

$$= \lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\varepsilon^2 \ln EI_{||y_\varepsilon-\phi||_T<\delta}(\omega)I_{||k_\varepsilon-\psi||_T<\delta}(\omega)\exp\Big\{-\frac{B_2(0)-B_1(0)}{2\varepsilon^2}L^{y_\varepsilon}(\mathrm{T},0)\Big\} =$$

$$= -\frac{1}{2}\int_0^T |\dot\phi(s)|^2 ds + \frac{((B_2(0)-B_1(0))^+)^2}{2}\int_0^T \dot\psi(s)(1-\dot\psi(s))ds. \tag{20}$$

From (18)-(20), we get (10).

Second, we prove (ii). It is the condition of *exponential tightness* [11, Chapter 3] (or *strong tightness* [12]) of $(x_\varepsilon, v_\varepsilon)$ on $\mathbf{C}_T$. We recall [12, Theorem 4.1] that the family of processes $(X_\varepsilon(t), u_\varepsilon(t))$ on the space $\mathbf{C}_T$ is exponentially tight (of order ε^2) if, for any $\mathrm{T} > 0$, $\eta > 0$,

$$\lim_{K\to\infty} \limsup_{\varepsilon\to 0} \varepsilon^2 \ln P\Big(||(X_\varepsilon, u_\varepsilon)||_T > K\Big) = -\infty, \tag{21}$$

$$\lim_{\Delta\to 0} \limsup_{\varepsilon\to 0} \varepsilon^2 \ln \sup_{\theta\le \mathrm{T}} P\Big(||\big(X_\varepsilon(\cdot+\theta), u_\varepsilon(\cdot+\theta)\big) - \big(X_\varepsilon(\theta), u_\varepsilon(\theta)\big)||_\Delta > \eta\Big) = -\infty. \tag{22}$$

We will use the Chebyshev inequality

$$P\Big(|\xi(\mathrm{T})| > K\Big) \le \frac{1}{\varphi(K)} E\varphi\big(\xi(\mathrm{T})\big) \tag{23}$$

for the function $\varphi(x) = \exp\Big\{\dfrac{x^2}{2\mathrm{T}\varepsilon^2}\Big\}$.

Let us prove (21). From the inequalities $\sup_{t\le\mathrm{T}} |x_\varepsilon(t)| \le |x| + \int_0^T |B(x_\varepsilon(s))|ds + \sup_{t\le\mathrm{T}} |\varepsilon w(t)|$

and $\sup_{t\le\mathrm{T}} |v_\varepsilon(t)| = \int_0^T I_{(0,\infty)}(x_\varepsilon(s))ds$, we have

$$P\Big(||(x_\varepsilon, v_\varepsilon)||_T > K\Big) \le P\Big(\frac{K}{2} < \sup_{t\le\mathrm{T}} |x_\varepsilon(t)|\Big) + P\Big(\frac{K}{2} < \sup_{t\le\mathrm{T}} |v_\varepsilon(t)|\Big) \le$$

$$\le P\Big(\frac{K}{4} < |x| + \int_0^T |B(x_\varepsilon(s))|ds\Big) + P\Big(\frac{K}{4} < \sup_{t\le\mathrm{T}} \big|\varepsilon w(t)\big|\Big) + P\Big(\frac{K}{4} < \int_0^T I_{(0,\infty)}(x_\varepsilon(s))ds\Big).$$

Denote last three items by P_1, P_2, and P_3 respectively. Then, by (23) and Condition (I), we have

$$P_1 \le \exp\{-\frac{K^2}{32\mathrm{T}\varepsilon^2}\} E\exp\left\{\frac{\Big(|x| + \int_0^T |B(x_\varepsilon(s))|ds\Big)^2}{2\mathrm{T}\varepsilon^2}\right\} \le$$

$$\le \exp\left\{\frac{16\Big(|x| + \Lambda\mathrm{T}\Big)^2 - K^2}{32\mathrm{T}\varepsilon^2}\right\};$$

$$P_3 \le \exp\{-\frac{K^2}{32\mathrm{T}\varepsilon^2}\} E\exp\left\{\frac{\Big(\int_0^T I_{(0,\infty)}(x_\varepsilon(s))ds\Big)^2}{2\mathrm{T}\varepsilon^2}\right\} \le \exp\left\{\frac{16\mathrm{T}^2 - K^2}{32\mathrm{T}\varepsilon^2}\right\}.$$

From [5, Cor. of Theorem 5, p.173], we have

$$P_2 = P\Big(\frac{K}{4} < \varepsilon\sqrt{\mathrm{T}} \sup_{t\le 1} \big|w(t)\big|\Big) = P\Big(\frac{K}{4\sqrt{\mathrm{T}}} < \varepsilon \sup_{t\le 1} \big|w(t)\big|\Big) \le 2\exp\{-\frac{K^2}{32\mathrm{T}\varepsilon^2}\}.$$

Then

$$P\Big(||(x_\varepsilon, v_\varepsilon)||_T > K\Big) \le \exp\Big\{-\frac{K^2}{32\mathrm{T}\varepsilon^2}\Big\}\Big(\exp\Big\{\frac{16\big(|x| + \Lambda\mathrm{T}\big)^2}{32\mathrm{T}\varepsilon^2}\Big\} + \exp\{\frac{16\mathrm{T}^2}{32\mathrm{T}\varepsilon^2}\} + 2\Big);$$

and

$$\lim_{K\to\infty} \limsup_{\varepsilon\to 0} \varepsilon^2 \ln P\Big(||(x_\varepsilon, v_\varepsilon)||_T > K\Big) \le \lim_{K\to\infty} \Big(-\frac{K^2}{32\mathrm{T}} + \frac{\big(|x| + \Lambda\mathrm{T}\big)^2}{2\mathrm{T}}\Big) = -\infty.$$

So (21) is proved.

Let us prove (22). Similarly, we have

$$P\Big(||\big(x_\varepsilon(\cdot+\theta),v_\varepsilon(\cdot+\theta)\big)-\big(x_\varepsilon(\theta),v_\varepsilon(\theta)\big)||_\Delta>\eta\Big)\le$$

$$\le P\Big(\sup_{t\le\Delta}\Big|\int_\theta^{t+\theta}B(x_\varepsilon(s))ds+\varepsilon\big(w(t+\theta)-w(\theta)\big)\Big|>\frac{\eta}{2}\Big)+$$

$$+P\Big(\sup_{t\le\Delta}\Big|\int_\theta^{t+\theta}I_{(0,\infty)}(x_\varepsilon(s))ds\Big|>\frac{\eta}{2}\Big)\le$$

$$\le P\Big(\int_\theta^{\Delta+\theta}\Big|B(x_\varepsilon(s))\Big|ds>\frac{\eta}{4}\Big)+P\Big(\sup_{t\le\Delta}\Big|\varepsilon\int_\theta^{t+\theta}dw(s)\Big|>\frac{\eta}{4}\Big)+$$

$$+P\Big(\int_\theta^{\Delta+\theta}I_{(0,\infty)}(x_\varepsilon(s))ds>\frac{\eta}{4}\Big).$$

By the analogy to the proof of (21), we denote last three items by P_4, P_5, and P_6, respectively. Then we have

$$P_4\le\exp\big\{\frac{16\Lambda^2\Delta^2-\eta^2}{32\Delta\varepsilon^2}\big\};$$

$$P_6\le\exp\big\{\frac{16\Delta^2-\eta^2}{32\Delta\varepsilon^2}\big\};$$

From the property of the $\Im_t$-stopping time and from [5, Cor. of Theorem 5, p.173], we have

$$P_5=P\Big(\sup_{t\le\Delta}\Big|\varepsilon\int_0^t dw(s)\Big|>\frac{\eta}{4}\Big)=P\Big(\varepsilon\sqrt{\Delta}\sup_{t\le1}\Big|\int_0^t dw(s)\Big|>\frac{\eta}{4}\Big)\le2\exp\big\{-\frac{\eta^2}{32\Delta\varepsilon^2}\big\}.$$

Further, we obtain

$$P\Big(||\big(x_\varepsilon(\cdot+\theta),v_\varepsilon(\cdot+\theta)\big)-\big(x_\varepsilon(\theta),v_\varepsilon(\theta)\big)||_\Delta>\eta\Big)\le$$

$$\le\exp\big\{-\frac{\eta^2}{32\Delta\varepsilon^2}\big\}\Big(\exp\Big\{\frac{\Lambda^2\Delta}{2\varepsilon^2}\Big\}+\exp\big\{\frac{\Delta}{2\varepsilon^2}\big\}+2\Big)$$

and
$$\limsup_{\varepsilon\to0}\varepsilon^2\ln\sup_{\theta\le\mathrm{T}}P\Big(||(x_\varepsilon(\cdot+\theta),v_\varepsilon(\cdot+\theta))-(x_\varepsilon(\theta),v_\varepsilon(\theta))||_\Delta>\eta\Big)\le-\frac{\eta^2}{32\Delta}+\frac{\Delta}{2};$$

$$\lim_{\Delta\to0}\limsup_{\varepsilon\to0}\varepsilon^2\ln\sup_{\theta\le\mathrm{T}}P\Big(||\big(x_\varepsilon(\cdot+\theta),v_\varepsilon(\cdot+\theta)\big)-\big(x_\varepsilon(\theta),v_\varepsilon(\theta)\big)||_\Delta>\eta\Big)\le$$

$$\le\lim_{\Delta\to0}\Big(-\frac{\eta^2}{32\Delta}+\frac{\Delta}{2}\Big)=-\infty.$$

So, (22) is proved, and we prove the exponential tightness of $(x_\varepsilon, v_\varepsilon)$.

So, we have established the local LDP on $\mathbf{C}_T$ for every T> 0 (Local LDP) and exponential tightness for $(x_\varepsilon(t), v_\varepsilon(t))$. Using [12, Theorem 4.5],

$$\text{LDP}\Longleftrightarrow\begin{cases}\text{Exponential tightness}\\ \text{Local LDP}\end{cases}$$

we obtain the rate function in such a form: $I^x_\infty(\phi,\psi)=\sup_{\mathrm{T}} I_T(\phi,\psi)$.

Hence, the measures μ_{x_ε} induced by the processes $(x_\varepsilon(t), v_\varepsilon(t))$ satisfy the LDP on $\mathrm{C}[0,\infty)\times AC^+[0,\infty)$ with the rate function $I^x_\infty(\phi,\psi)=\sup_{\mathrm{T}} I_T(\phi,\psi)$, where $I_T(\phi,\psi)$ is defined by (10). Consider now three cases: $\phi(s)\neq0$; $\phi(s)=0$ *and* $B_1(0)>B_2(0)$; $\phi(s)=0$ *and* $B_1(0)\le B_2(0)$. Using the property $Leb\Big(s\in[0,\mathrm{T}]:\phi(s)=0,\ \dot\phi(s)\neq0\Big)=0$, we get (8).

Theorem 2.1 is proved.

Proof of Corollary 2.1. From (8) by the contraction principle, we have

$$I^x(\phi) = \frac{1}{2}\int_0^\infty I_{\{\phi(s)\neq 0\}}\Big(\dot\phi(s) - B(\phi(s))\Big)^2 ds +$$

$$+ \inf_{\psi\in H^+(\phi)}\left(\frac{1}{2}\int_0^\infty I_{\{\phi(s)=0\}}I_{\{B_1(0)>B_2(0)\}}\Big(B_2(0)\dot\psi(s) + B_1(0)(1-\dot\psi(s))\Big)^2 ds\right) +$$

$$+ \inf_{\psi\in H^+(\phi)}\left(\frac{1}{2}\int_0^\infty I_{\{\phi(s)=0\}}I_{\{B_1(0)\leq B_2(0)\}}\Big(B_2^2(0)\dot\psi(s) + B_1^2(0)(1-\dot\psi(s))\Big) ds\right).$$

Let us consider two last items in more details. Using Lemma 4.6, we have

$$\inf_{\psi\in H^+(\phi)}\left(\frac{1}{2}\int_0^\infty I_{\{\phi(s)=0\}}I_{\{B_1(0)>B_2(0)\}}\Big(B_2(0)\dot\psi(s) + B_1(0)(1-\dot\psi(s))\Big)^2 ds\right) =$$

$$= \frac{1}{2}\inf_{a\in[0,1]}\Big(B_2(0)a + B_1(0)(1-a)\Big)^2\int_0^\infty I_{\{\phi(s)=0\}}I_{\{B_1(0)>B_2(0)\}}ds =$$

$$= \frac{1}{2}\inf_{a\in[0,1]}\Big((B_2(0)a + B_1(0)(1-a))^2 I_{\{B_1(0)\leq 0\}}\Big)\int_0^\infty I_{\{\phi(s)=0\}}I_{\{B_1(0)>B_2(0)\}}ds +$$

$$+ \frac{1}{2}\inf_{a\in[0,1]}\Big((B_2(0)a + B_1(0)(1-a))^2 I_{\{B_2(0)\geq 0\}}\Big)\int_0^\infty I_{\{\phi(s)=0\}}I_{\{B_1(0)>B_2(0)\}}ds +$$

$$+ \frac{1}{2}\inf_{a\in[0,1]}\Big((B_2(0)a + B_1(0)(1-a)) I_{\{B_1(0)>0>B_2(0)\}}\Big)^2\int_0^\infty I_{\{\phi(s)=0\}}I_{\{B_1(0)>B_2(0)\}}ds.$$

In the first and second cases, we reach inf at $a = 0$ and $a = 1$, respectively. These two cases can be united in such a way: $\inf_{a\in[0,1]}\big(B_2(0)a + B_1(0)(1-a)\big)^2 = \min(B_1^2(0), B_2^2(0))$. In the third case, we reach inf at $a = \frac{-B_1(0)}{B_2(0)-B_1(0)} \in [0,1]$, and it equals 0. By analogy,

$$\inf_{\psi\in H^+(\phi)}\left(\frac{1}{2}\int_0^\infty I_{\{\phi(s)=0\}}I_{\{B_1(0)\leq B_2(0)\}}\Big(B_2^2(0)\dot\psi(s) + B_1^2(0)(1-\dot\psi(s))\Big) ds\right) =$$

$$= \frac{1}{2}\inf_{a\in[0,1]}\Big(\Big(B_2^2(0)a + B_1^2(0)(1-a)\Big) I_{\{B_2(0)\leq 0\}}\Big)\int_0^\infty I_{\{\phi(s)=0\}}I_{\{B_1(0)\leq B_2(0)\}}ds +$$

$$+ \frac{1}{2}\inf_{a\in[0,1]}\Big(\Big(B_2^2(0)a + B_1^2(0)(1-a)\Big) I_{\{B_1(0)\geq 0\}}\Big)\int_0^\infty I_{\{\phi(s)=0\}}I_{\{B_1(0)\leq B_2(0)\}}ds +$$

$$+ \frac{1}{2}\inf_{a\in[0,1]}\Big(\Big(B_2^2(0)a + B_1^2(0)(1-a)\Big) I_{\{B_1(0)<0<B_2(0)\}}\Big)\int_0^\infty I_{\{\phi(s)=0\}}I_{\{B_1(0)\leq B_2(0)\}}ds.$$

Here, we reach inf in the first case at $a = 1$, in the second one at $a = 0$, and in the third case either at $a = 1$ (if $B_2^2(0) \leq B_1^2(0)$) or $a = 0$ (else). All three cases can be united in such a way: $\inf_{a\in[0,1]}\Big(B_2^2(0)a + B_1^2(0)(1-a)\Big) = \min(B_1^2(0), B_2^2(0))$.

So,

$$I^x(\phi) = \frac{1}{2}\int_0^\infty I_{\{\phi(s)\neq 0\}}\Big(\dot\phi(s) - B(\phi(s))\Big)^2 ds +$$
$$+ \frac{1}{2}\int_0^\infty I_{\{\phi(s)=0\}}\Big(1 - I_{\{B_1(0)>0>B_2(0)\}}\Big)\min\big(B_1^2(0), B_2^2(0)\big) ds.$$

After simple transformations, we get (9). *Corollary 2.1 is proved.*

Now let us consider the solution of the equation

$$\zeta_\varepsilon(t) = x + \int_0^t B(\zeta_\varepsilon(s))ds + \varepsilon \int_0^t r(\zeta_\varepsilon(s))w(t), \tag{24}$$

where the function $B(x)$ has form (5), and the function $r(x)$ has the same construction, i.e.

$$r(x) = \begin{cases} r_1(x), & \text{if } x < 0, \\ r_3, & \text{if } x = 0, \\ r_2(x), & \text{if } x > 0, \end{cases} \tag{25}$$

where $r_i(x), i = 1, 2,\ x \in \mathbf{R}$, are measurable bounded functions, and r_3 is a constant. By analogy with [2, Section 4], we use the random change of time. Denote $z_\varepsilon(t) = \int_0^t I_{(0,\infty)}(\zeta_\varepsilon(s))ds$. Let's prove the next theorem.

Theorem 2.2. *In Eq. (24), let $r^2(x) \geq \lambda > 0$; for functions $B_i(x)$ and $r_i^2(x)$, $i = 1, 2$, the Lipschitz condition holds on $(-\infty, 0]$ and $(0, +\infty)$, respectively. Then the measures μ_ε induced by the processes $(\zeta_\varepsilon, z_\varepsilon)$ on $\mathbf{C}_\infty$ satisfy the large deviation principle with the rate function $I_\infty^\zeta(\phi, \psi) = \int_0^\infty N(\phi(s), \dot\phi(s), \dot\psi(s))ds$, where*

$$N(\phi(s), \dot\phi(s), \dot\psi(s)) = \begin{cases} \dfrac{1}{2} \dfrac{\Big(\dot\phi(s) - B(\phi(s))\Big)^2}{r^2(\phi(s))}, & \textit{if } \phi(s) \neq 0, \\ \dfrac{1}{2} \dfrac{\Big(B_2(0)\dot\psi(s) + B_1(0)\big(1 - \dot\psi(s)\big)\Big)^2}{r_2^2(0)\dot\psi(s) + r_1^2(0)(1 - \dot\psi(s))}, & \textit{if } \phi(s) = 0 \textit{ and } \dfrac{B_1(0)}{r_1^2(0)} > \dfrac{B_2(0)}{r_2^2(0)}, \\ \dfrac{1}{2} \Big(\dfrac{B_2^2(0)}{r_2^2(0)} \dot\psi(s) + \dfrac{B_1^2(0)}{r_1^2(0)} \big(1 - \dot\psi(s)\big)\Big), & \textit{if } \phi(s) = 0 \textit{ and } \dfrac{B_1(0)}{r_1^2(0)} \leq \dfrac{B_2(0)}{r_2^2(0)}, \end{cases} \tag{26}$$

for an absolutely continuous ϕ such that $\phi(0) = x$, $\int_0^\infty \dot\phi^2(s)ds < \infty$, and $\psi \in H^+(\phi)$ such that $\psi(0) = 0$. For all other pairs $(\phi,\ \psi)$, we set $I_\infty^\zeta(\phi, \psi) = \infty$.

Proof

For this rate function, Condition (i) is proved in Lemma 4.8.

Further, we set

$$\begin{aligned} \beta_\varepsilon(t) &= \int_0^t r^2(\zeta_\varepsilon(s))ds, \\ \beta(t) &= \int_0^t \Big(r_2^2(\phi(s))\dot\psi(s) + r_1^2(\phi(s))(1 - \dot\psi(s))\Big)ds, \end{aligned} \tag{27}$$

and $\gamma_\varepsilon(t)$ and $\gamma(t)$ are their inverses, respectively (they exist because $\beta_\varepsilon(t)$ and $\beta(t)$ monotonously increase).

For $\zeta_\varepsilon(t)$, let us make a random change of time. We have $\zeta_\varepsilon(\gamma_\varepsilon(t)) = \pi_\varepsilon(t)$ or $\zeta_\varepsilon(t) = \pi_\varepsilon(\beta_\varepsilon(t))$, where

$$\pi_\varepsilon(t) = x + \int_0^t \frac{B(\pi_\varepsilon(s))}{r^2(\pi_\varepsilon(s))}ds + \varepsilon \hat{w}(s), \tag{28}$$

where $\hat{w}(s)$ is another Wiener process. We now denote $z_\varepsilon^\pi(t) = \int_0^t I_{(0,\infty)}(\pi_\varepsilon(s))ds$.

By the analogy with [2, Section 4], we set

$$\begin{aligned}
&\phi^\pi(t) = \phi(\gamma(t)) \ ;\\
&\dot\phi^\pi(t) = \dot\phi(\gamma(t))\dot\gamma(t) = \frac{\dot\phi(\gamma(t))}{\dot\beta(\gamma(t))} \ ;\\
&\psi^\pi(t) = \int_0^{\gamma(t)} \dot\psi(s) r_2^2\big(\phi(s)\big)ds = \int_0^t \dot\psi(\gamma(s)) r_2^2\Big(\phi(\gamma(s))\Big)\dot\gamma(s)ds \ ;\\
&\dot\psi^\pi(t) = \dot\psi(\gamma(t)) r_2^2\Big(\phi(\gamma(t))\Big)\dot\gamma(t) = \dot\psi(\gamma(t))\frac{r_2^2\Big(\phi(\gamma(t))\Big)}{\dot\beta(\gamma(t))} \ .
\end{aligned} \tag{29}$$

We now mark $D(x) = \dfrac{B(x)}{r^2(x)}$. Then

$$D(x) = \begin{cases} D_1(x) = \dfrac{B_1(x)}{r_1^2(x)}, & \text{if } x < 0,\\ D_3 = \dfrac{B_3}{r_3^2}\ , & \text{if } x = 0,\\ D_2(x) = \dfrac{B_2(x)}{r_2^2(x)}, & \text{if } x > 0. \end{cases}$$

Using (8), we can establish the LDP on $\mathbf{C}_\infty$ for the measures μ_ε^π induced by the processes $(\pi_\varepsilon, z_\varepsilon^\pi)$ and obtain its rate function $I_\infty^\pi(\phi^\pi, \psi^\pi)$:

$$\begin{aligned}
&I_\infty^\pi(\phi^\pi,\psi^\pi) = \int_0^\infty N_s^\pi(\phi^\pi, \dot\phi^\pi, \dot\psi^\pi)ds =\\
=&\frac{1}{2}\int_0^\infty I_{\{\phi^\pi(s)\neq 0\}}\Big(\dot\phi^\pi(s) - D(\phi^\pi(s))\Big)^2 ds +\\
+&\frac{1}{2}\int_0^\infty I_{\{\phi^\pi(s)=0\}} I_{\{D_1(0)>D_2(0)\}}\Big(D_2(0)\dot\psi^\pi(s) + D_1(0)(1-\dot\psi^\pi(s))\Big)^2 ds +\\
+&\frac{1}{2}\int_0^\infty I_{\{\phi^\pi(s)=0\}} I_{\{D_1(0)\le D_2(0)\}}\Big(D_2^2(0)\dot\psi^\pi(s) + D_1^2(0)(1-\dot\psi^\pi(s))\Big)ds.
\end{aligned} \tag{30}$$

In view of Lemma 4.3, we have, for every $T>0$,

$$I_T^\zeta(\phi,\psi) = \lim_{\delta\to 0}\liminf_{\varepsilon\to 0}\varepsilon^2 \ln P\{||\zeta_\varepsilon - \phi||_T < \delta, ||z_\varepsilon - \psi||_T < \delta\} =$$

$$= \lim_{\delta\to 0}\liminf_{\varepsilon\to 0}\varepsilon^2 \ln P\{||\pi_\varepsilon - \phi^\pi||_{\beta(T)} < \delta, ||z_\varepsilon^\pi - \psi^\pi||_{\beta(T)} < \delta\} = I_{\beta(T)}^\pi(\phi^\pi,\psi^\pi).$$

Using this result and Lemmas 4.10 and 4.11, we get, by Theorem 4.5 [12],

$$I_\infty^\zeta(\phi,\psi) = \sup_{\mathrm{T}} I_T^\zeta(\phi,\psi) = \lim_{\delta\to 0}\liminf_{\varepsilon\to 0}\varepsilon^2 \ln P\{||\zeta_\varepsilon - \phi||_\infty < \delta, ||z_\varepsilon - \psi||_\infty < \delta\} =$$

$$= \lim_{\delta\to 0}\liminf_{\varepsilon\to 0}\varepsilon^2 \ln P\{||\pi_\varepsilon - \phi^\pi||_\infty < \delta, ||z_\varepsilon^\pi - \psi^\pi||_\infty < \delta\} =$$

$$= \sup_{\beta(\mathrm{T})} I_{\beta(T)}^\pi(\phi^\pi,\psi^\pi) = I_\infty^\pi(\phi^\pi,\psi^\pi).$$

Hence, the measures μ_ε induced by the processes $(\zeta_\varepsilon(t), z_\varepsilon(t))$ satisfy the LDP on $\mathrm{C}[0,\infty)\times AC^+[0,\infty)$ with the rate function

$$I_\infty^\zeta(\phi,\psi) = I_\infty^\pi(\phi^\pi,\psi^\pi).$$

Using formulae (29) and (30) and making a change in the integrals, we obtain the explicit form of $I_\infty(\phi,\psi)$:

$$\begin{aligned} I_\infty^\zeta(\phi,\psi) &= \frac{1}{2}\int_0^\infty I_{\{\phi(s)\neq 0\}}\frac{\Big(\dot\phi(s)-B(\phi(s))\Big)^2}{r^2(\phi(s))}ds+ \\ &+ \frac{1}{2}\int_0^\infty I_{\{\phi(s)=0\}}I_{\left\{\frac{B_1(0)}{r_1^2(0)}>\frac{B_2(0)}{r_2^2(0)}\right\}}\Bigg(\frac{B_2(0)}{r_2^2(0)}\dot\psi(s)\frac{r_2^2(0)}{\dot\beta(0)}+ \\ &+ \frac{B_1(0)}{r_1^2(0)}\left(1-\dot\psi(s)\frac{r_2^2(0)}{\dot\beta(0)}\right)\Bigg)^2\dot\beta(0)ds+ \\ &+ \frac{1}{2}\int_0^\infty I_{\{\phi(s)=0\}}I_{\left\{\frac{B_1(0)}{r_1^2(0)}\le\frac{B_2(0)}{r_2^2(0)}\right\}}\Bigg(\frac{B_2^2(0)}{r_2^4(0)}\dot\psi(s)\frac{r_2^2(0)}{\dot\beta(0)}+ \\ &+ \frac{B_1^2(0)}{r_1^4(0)}\left(1-\dot\psi(s)\frac{r_2^2(0)}{\dot\beta(0)}\right)\Bigg)\dot\beta(0)ds. \end{aligned} \tag{31}$$

Using now (27), we obtain (26).

Theorem 2.2 is proved.

3. Proof of Theorem 1.1.

Condition (i) for this rate function is proved in Lemma 4.9.

Let us consider Eq. (1).

Following [10], we introduce a function

$$\kappa(x)=\begin{cases}(1-\beta)x, & x\le 0\\ (1+\beta)x, & x\ge 0.\end{cases}$$

Let $\varphi(x)$ be the inverse function to $\kappa(x)$. We put

$$\begin{aligned} \widetilde{b}(x) &= \frac{b(\kappa(x))}{1+\beta\mathrm{sgn}x}, \\ \widetilde{\sigma}(x) &= \frac{\sigma(\kappa(x))}{1+\beta\mathrm{sgn}x}, \end{aligned} \tag{32}$$

where

$$\mathrm{sgn}x=\begin{cases}1, & if\;\; x>0,\\ 0, & if\;\; x=0,\\ -1, & if\;\; x<0,\end{cases}$$

and consider now Itô's stochastic equation

$$\eta_\varepsilon(t)=\varphi(x)+\int_0^t\widetilde{b}(\eta_\varepsilon(s))ds+\varepsilon\int_0^t\widetilde{\sigma}(\eta_\varepsilon(s))dw(s). \tag{33}$$

Equation (33) has a weak solution by [14], and the process

$$\xi_\varepsilon(t)=\kappa(\eta_\varepsilon(t))$$

is a solution of Eq. (1) [8, Lemma 2], [10, lemma 1]. It is necessary to note that the functions $\tilde b(x)$ and $\tilde\sigma(x)$ have the form of the functions $B(x)$ from (5) and $r(x)$ from (25).

For the process $\eta_\varepsilon(t)$, we denote its occupation time on the positive semiaxis by

$$v_\varepsilon(t)=\int_0^t I_{(0,\infty)}(\eta_\varepsilon(s))ds.$$

Then it is possible to apply Theorem 2.2. Using formula (26), we can establish the LDP on $\mathbf{C}_\infty$ for the measures μ_ε^η induced by the processes $(\eta_\varepsilon,v_\varepsilon)$ and obtain its rate function $I^{\eta}_{\infty}(\phi^{\eta},\psi^{\eta})=\int_0^{\infty}L_2(\phi^{\eta},\dot{\phi}^{\eta},\dot{\psi}^{\eta})ds$, where

$$L_2(\phi^{\eta},\dot{\phi}^{\eta},\dot{\psi}^{\eta})=\begin{cases}\dfrac{1}{2}\dfrac{\left(\dot{\phi}^{\eta}(s)-\tilde{b}(\phi^{\eta}(s))\right)^2}{\tilde{\sigma}^2(\phi^{\eta}(s))}, & if\ \phi^{\eta}(s)\neq 0,\\[2ex] \dfrac{1}{2}\dfrac{\left(\tilde{b}_2(0)\dot{\psi}^{\eta}(s)+\tilde{b}_1(0)(1-\dot{\psi}^{\eta}(s))\right)^2}{\tilde{\sigma}_2^2(0)\dot{\psi}^{\eta}(s)+\tilde{\sigma}_1^2(0)(1-\dot{\psi}^{\eta}(s))}, & \begin{array}{l} if\ \phi^{\eta}(s)=0\ and\\ \dfrac{\tilde{b}_1(0)}{\tilde{\sigma}_1^2(0)}>\dfrac{\tilde{b}_2(0)}{\tilde{\sigma}_2^2(0)},\end{array}\\[2ex] \dfrac{1}{2}\left[\dfrac{\tilde{b}_2^2(0)}{\tilde{\sigma}_2^2(0)}\dot{\psi}^{\eta}(s)+\dfrac{\tilde{b}_1^2(0)}{\tilde{\sigma}_1^2(0)}(1-\dot{\psi}^{\eta}(s))\right], & \begin{array}{l} if\ \phi^{\eta}(s)=0\ and\\ \dfrac{\tilde{b}_1(0)}{\tilde{\sigma}_1^2(0)}\leq\dfrac{\tilde{b}_2(0)}{\tilde{\sigma}_2^2(0)},\end{array}\end{cases} \tag{34}$$

Since $\xi_{\varepsilon}(t)=\kappa(\eta_{\varepsilon}(t))$ or $\eta_{\varepsilon}(t)=\varphi(\xi_{\varepsilon}(t))$ and $\psi^{\eta}(t)=\int_0^t I_{(0,\infty)}(\phi^{\eta}(s))ds=\int_0^t I_{(0,\infty)}(\varphi(\phi(s)))ds=\int_0^t I_{(0,\infty)}(\phi(s))ds=\psi(t)$, the contraction principle yields

$$I_{\infty}(\phi,\psi)=\min\left\{I^{\eta}_{\infty}(\phi^{\eta},\psi^{\eta}):\ \phi^{\eta}=\varphi(\phi),\psi^{\eta}=\psi\right\}=I^{\eta}_{\infty}(\varphi(\phi^{\eta}),\psi).$$

(Here, $I_{\infty}(\phi,\psi)$ is a rate function for the measures μ_{ε} induced by the processes $(\xi_{\varepsilon},u_{\varepsilon})$).

From formulae (32) and (34) after a cancellation, we get the statement of the theorem. *Theorem 1.1 is proved.*

Proof of Corollary 1.1. From (3) by the contraction principle, we have

$$I(\phi)=\frac{1}{2}\int_0^{\infty}I_{\{\phi(s)\neq 0\}}\frac{\left(\dot{\phi}(s)-b(\phi(s))\right)^2}{\sigma^2(\phi(s))}ds+$$

$$+\inf_{\psi\in H^+(\phi)}\left(\frac{1}{2}\int_0^{\infty}I_{\{\phi(s)=0\}}I_{\{\beta b(0)<0\}}\frac{b^2(0)}{\sigma^2(0)}\frac{\left(1+\beta-2\beta\dot{\psi}(s)\right)^2}{(1+\beta)^2-4\beta\dot{\psi}(s)}ds\right)+$$

$$+\frac{1}{2}\int_0^{\infty}I_{\{\phi(s)=0\}}I_{\{\beta b(0)\geq 0\}}\frac{b^2(0)}{\sigma^2(0)}ds.$$

Consider now the second item in more details. Using (35), we get

$$\inf_{\psi\in H^+(\phi)}\frac{1}{2}\int_0^{\infty}I_{\{\phi(s)=0\}}I_{\{\beta b(0)<0\}}\frac{b^2(0)}{\sigma^2(0)}\frac{\left(1+\beta-2\beta\dot{\psi}(s)\right)^2}{(1+\beta)^2-4\beta\dot{\psi}(s)}ds=$$

$$=\frac{1}{2}\inf_{0\leq a\leq 1}\frac{\left(1+\beta-2\beta a\right)^2}{(1+\beta)^2-4\beta a}\int_0^{\infty}I_{\{\phi(s)=0\}}I_{\{\beta b(0)<0\}}\frac{b^2(0)}{\sigma^2(0)}ds.$$

We reach inf at $a=\dfrac{1+\beta}{2\beta}\in[0,1]$ for $|\beta|<1$. Then, after simple transformations, we get (4). *Corollary 1.1 is proved.*

4. Auxiliary results.

Lemma 4.1. (Lemma 6.7 from [1]).

Let $f(x)$ be a function such as $f(x)=\begin{cases}f_1(x), & if\ \ x\leq 0,\\ f_2(x), & if\ \ x>0,\end{cases}$ *where $f_1(x)$ and $f_2(x)$ are bounded and continuous. Then the function*

$$(\varphi,\psi)\to\int_0^T\left(f_2(\varphi(t))\dot{\psi}(t)+f_1(\varphi(t))\left(1-\dot{\psi}(t)\right)\right)dt$$

is continuous on the set $\{(\varphi,\psi),\ \varphi\in \mathrm{C[0,T]};\ \psi\in H^+(\varphi)\}$.

Four next lemmas are fulfilled under the Lipschitz condition by analogy with Lemmas 4.1-4.4 from [2].

Lemma 4.2. *For any $\gamma > 0$, there exists a δ_0 such that if $||\zeta_\varepsilon - \phi||_T < \delta_0, ||z_\varepsilon - \psi||_T < \delta_0$, then $|\beta_\varepsilon(t) - \beta(t)| < \gamma$ for all $t \in [0, T]$ and $|\gamma_\varepsilon(t) - \gamma(t)| < \gamma$ for all $t \in [0, \beta(\mathrm{T}) - \gamma]$ and $\varepsilon > 0$.*

Proof. By Lemma 4.1, for any $\gamma > 0$, there exists a $\delta > 0$ such that if $||\zeta_\varepsilon - \phi||_T < \delta_0$ and $||z_\varepsilon - \psi||_T < \delta_0$, we have

$$|\beta_\varepsilon(t) - \beta(t)| =$$

$$= \Big| \int_0^t \Big(r_2^2(\zeta_\varepsilon(s))\dot{z}_\varepsilon(s) + r_1^2(\zeta_\varepsilon(s))(1 - \dot{z}_\varepsilon(s)) \Big) ds -$$

$$- \int_0^t \Big(r_2^2(\phi(s))\dot{\psi}(s) + r_1^2(\phi(s))(1 - \dot{\psi}(s)) \Big) ds \Big| < \gamma.$$

The statement for $\gamma_\varepsilon(t)$ and $\gamma(t)$ can be proved the same way.

By Lemmas 4.2, 4.4, and 4.5, we have

Lemma 4.3.

$$N^\pi(\phi^\pi(s), \dot{\phi}^\pi(s), \dot{\psi}^\pi(s)) = N\Big(\phi(\gamma(t)), \dot{\phi}(\gamma(t)), \dot{\psi}(\gamma(t))\Big)\dot{\gamma}(t)$$

and

$$I^\pi(\phi^\pi, \psi^\pi) = I^\zeta(\phi, \psi).$$

Lemma 4.4. *For any $T > 0$ and any $\gamma > 0$, there exists δ_0 such that $||z_\varepsilon^\pi - \psi^\pi||_{\beta(T)-\gamma} \le \gamma$ for all ε if $||\zeta_\varepsilon - \phi||_T < \delta, ||z_\varepsilon - \psi||_T < \delta$, $\delta < \delta_0$.*

Lemma 4.5. *For any $T > 0$ and any $\gamma > 0$, there exists θ_0 such that, for $\theta < \theta_0$, $||\zeta_\varepsilon - \phi||_{T-\delta} \le \delta$ and $||z_\varepsilon - \psi||_{T-\delta} \le \delta$ if $||\pi_\varepsilon - \phi^\pi||_{\beta(T)-\theta} < \theta, ||z_\varepsilon^\pi - \psi^\pi||_{\beta(T)-\theta} < \theta$.*

Lemma 4.6. *For bounded functions $f(s) \ge 0$ and $g(s) \ge 0$ such as $\int_0^\infty f(s)ds < \infty$, we have*

$$\inf_{\psi \in AC^+} \int_0^\infty f(s) g(\dot{\psi}(s)) ds = \inf_{a \in [0,1]} g(a) \int_0^\infty f(s) ds. \tag{35}$$

Proof. On the one hand,

$$\inf_{a \in [0,1]} g(a) \le g(\dot{\psi}(s)),$$

$$\int_0^\infty f(s) \inf_{a \in [0,1]} g(a) ds \le \int_0^\infty f(s) g(\dot{\psi}(s)) ds. \tag{36}$$

Denote $\overline{g} = \inf_{a \in [0,1]} g(a)$. Then, for every $\varepsilon > 0$, there exists $\dot{\psi}_\varepsilon(t) \in [0, 1]$ such that

$$g(\dot{\psi}_\varepsilon(s)) \le \overline{g} + \varepsilon$$

and

$$\int_0^\infty f(s) g(\dot{\psi}_\varepsilon(s)) ds \le \int_0^\infty f(s) \overline{g} ds + \varepsilon \int_0^\infty f(s) ds.$$

Thus, since $\varepsilon > 0$ is arbitrary,

$$\int_0^\infty f(s) g(\dot{\psi}_\varepsilon(s)) ds \le \int_0^\infty f(s) \overline{g} ds = \inf_{a \in [0,1]} g(a) \int_0^\infty f(s) ds. \tag{37}$$

From (36) and (37), we get (35).

Lemma 4.7. *Let $I(\phi, \psi) = \int_0^\infty M(\phi(s), \dot{\phi}(s), \dot{\psi}(s)) ds$ from Theorem 2.1. Then, for any $a > 0$, the set $A(a) = \{(\phi, \psi) : I(\phi, \psi) \le a\}$ is compact.*

Proof. Since the space $\mathbf{C}$ is complete, it is sufficiently to prove

1) lower semicontinuity of $I(\phi,\psi)$ on $\mathbf{C}$;

2) relative compactness of $A(a)$.

In our case:

1) follows from Lemma 6.3 [1];

2) follows from the Ascoli–Arzela theorem [7; Theorem VI.1.5]: $A(a)$ is relatively compact if and only if

(i) $\sup\limits_{(x,y)\in A(a)} |x(0),y(0)| < \infty$;

(ii) $\lim\limits_{\delta\to 0} \sup\limits_{(x,y)\in A(a)} \sup\limits_{s,t\in[0,T];|s-t|<\delta} |(x(t),y(t))-(x(s),y(s))|=0$ for all $T>0$.

It is easy to show that both these conditions are satisfied.

Lemma 4.8. *Let $I(\phi,\psi)=\int_0^\infty N(\phi(s),\dot\phi(s),\dot\psi(s))ds$ from Theorem 2.2. Then, for any $a>0$, the set $A(a)=\{(\phi,\psi): I(\phi,\psi)\le a\}$ is compact.*

Proof. Similarly to the previous lemma:

1) follows from Lemma 6.3 [1] and Lemma 3.9 [2];

2) analogously to the proof of such an item in Lemma 4.7.

Lemma 4.9. *Let $I(\phi,\psi)=\int_0^\infty L(\phi(s),\dot\phi(s),\dot\psi(s))ds$ from Theorem 1.1. Then, for any $a>0$, the set $A(a)=\{(\phi,\psi): I(\phi,\psi)\le a\}$ is compact.*

Proof. It follows from Lemma 4.8 and the contraction principle [4, Theorem 5.3.1].

The proofs of the following lemmas are similar to the proof of (21) and (22).

Lemma 4.10. *$(\zeta_\varepsilon, z_\varepsilon)$ is exponentially tight on the space $\mathbf{C}_T$ for every $T>0$.*

Lemma 4.11. *$(\pi_\varepsilon, z_\varepsilon^\pi)$ is exponentially tight on the space $\mathbf{C}_T$ for every $T>0$.*

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