A MIN-TYPE STOCHASTIC FIXED-POINT EQUATION RELATED TO THE SMOOTHING TRANSFORMATION

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Abstract. This paper is devoted to the study of the stochastic fixed-point equation

$$X \stackrel{d}{=} \inf_{i\geq 1: T_i>0} X_i/T_i$$

and the connection with its additive counterpart $X \stackrel{d}{=} \sum_{i\geq 1} T_i X_i$ associated with the smoothing transformation. Here $\stackrel{d}{=}$ means equality in distribution, $T := (T_i)_{i\geq 1}$ is a given sequence of non-negative random variables, and $X, X_1, \ldots$ is a sequence of non-negative i.i.d. random variables independent of T. We draw attention to the question of the existence of non-trivial solutions and, in particular, of special solutions named α-regular solutions ($\alpha > 0$). We give a complete answer to the question of when α-regular solutions exist and prove that they are always mixtures of Weibull distributions or certain periodic variants. We also give a complete characterization of all fixed points of this kind. A disintegration method which leads to the study of certain multiplicative martingales and a pathwise renewal equation after a suitable transform are the key tools for our analysis. Finally, we provide corresponding results for the fixed points of the related additive equation mentioned above. To some extent, these results have been obtained earlier by Iksanov.

1. Introduction

For a given sequence $T := (T_i)_{i\geq 1}$ of non-negative random variables on a probability space $(\Omega, \mathfrak{A}, \mathbb{P})$ with $\sup_{i\geq 1} T_i > 0$ a.s., consider the stochastic fixed-point equation (SFPE)

$$X \stackrel{d}{=} \inf_{i\geq 1} X_i/T_i \tag{1}$$

where $X, X_1, X_2, \ldots$ are i.i.d., non-negative and independent of T, and $X_i/T_i := \infty$ is stipulated on $\{T_i = 0\}$. A distribution F on $[0,\infty)$ is called a solution to (1) if this equation holds true with $X \stackrel{d}{=} F$, and it is called positive if $F(\{0\}) = 0$. Note that $F = \delta_0$, δ_0 the Dirac measure at 0, always provides a trivial solution. The set of all solutions $\neq \delta_0$ will be denoted as $\mathfrak{F}_\wedge$ hereafter, or as $\mathfrak{F}_\wedge(T)$ if we want to emphasize its dependence on T. We will make no notational distinction between a distribution F and its left-continuous distribution function, and we denote, by $\overline{F}$, the associated survival function, *i.e.*, $\overline{F} := 1 - F$. For $F \in \mathfrak{F}_\wedge$, Eq. (1) can be rewritten in terms of $\overline{F}$ as

$$\overline{F}(t) = \mathbb{E} \prod_{i\geq 1} \overline{F}(tT_i) \tag{2}$$

for all $t \geq 0$. Denote, by $\mathcal{P}, \overline{\mathcal{P}}$, the spaces of probability measures on $[0,\infty)$ and $[0,\infty]$, respectively. Defining the map $M : \mathcal{P} \to \mathcal{P}$ by

$$M(F) := \mathbb{P}\left(\inf_{i\geq 1} \frac{X_i}{T_i} \in \cdot\right), \quad X \stackrel{d}{=} F, \tag{3}$$

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