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ON STRONG EXISTENCE AND CONTINUOUS DEPENDENCE FOR SOLUTIONS OF ONE-DIMENSIONAL STOCHASTIC EQUATIONS WITH ADDITIVE LÉVY NOISE

One-dimensional stochastic differential equations (SDEs) with additive Lévy noise are considered. Conditions for strong existence and uniqueness of a solution are obtained. In particular, if the noise is a Lévy symmetric stable process with $\alpha \in (1;2)$, then the measurability and the boundedness of a drift term is sufficient for the existence of a strong solution. We also study the continuous dependence of the strong solution on the initial value and the drift.

Introduction

Consider the SDE

$$\xi(t) = x + \int_0^t a(\xi(s))ds + Z(t),\ t \geq 0, \tag{1}$$

where $a : \mathbb{R} \to \mathbb{R}$ is a measurable function, and Z is a Lévy process. We study a question of the strong existence and the uniqueness for a solution of (1) and its continuous dependence on the initial value x and a function a.

At first, we obtain a few general results and then apply them to the case where Z is a symmetric stable process with $\alpha \in (1, 2)$. In particular, the strong solution exists in this case and is unique if a is bounded. Moreover, let $\{\xi_n, n \geq 1\}$ be a sequence that satisfies (1) with initial values $\{x_n, n \geq 1\}$ and drift functions $\{a_n, n \geq 1\}$. We prove that if x_n converges to x, a_n converges to a almost surely with respect to the Lebesgue measure, and a sequence of functions $\{a_n, n \geq 1\}$ is uniformly bounded, then we have the uniform convergence of solutions in probability:

$$\forall\ T > 0 : \sup_{t\in[0,T]} |\xi_n(t) - \xi(t)| \xrightarrow{P} 0,\ n \to \infty.$$

A lot of ideas and methods of investigation are quite standard. We use the Yamada–Watanabe theorem and prove that the minimum of two solutions is a solution. In addition, we use the Skorokhod theorem on a joint probability space. However, we cannot find the direct reference to a general result, which can be applied to SDEs with Lévy noise, in the literature.

The case when a is continuous was considered by Tanaka, Tsuchiya and Watanabe [1]. A remarkable result on pathwise uniqueness in multidimensional case for Holder a was obtained by Priola [2].

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1. Pathwise uniqueness

In this section, we prove that a weak uniqueness of (1) yields a pathwise uniqueness. If we suppose also a weak existence, then the reasoning of the Yamada–Watanabe theorem and some minor technical assumptions will yield the strong existence and the uniqueness.

We need the following simple statement about solutions of non-random integral equations.

Lemma 1.1. *Let $a:\mathbb{R}\to\mathbb{R}$, $Z:[0,\infty)\to\mathbb{R}$ be measurable (non-random) functions. Assume that measurable functions $\xi_i:[0,\infty)\to\mathbb{R}, i=1,2,$ satisfy the equation*

$$\xi_i(t)=x+\int_0^t a(\xi_i(s))ds+Z(t),\ t\ge 0. \tag{2}$$

Then $\xi_-(t)=\xi_1(t)\wedge\xi_2(t)$ and $\xi_+(t)=\xi_1(t)\vee\xi_2(t)$ are also solutions of (2).

Remark. We assume that, for any $T>0$,

$$\int_0^T |a(\xi_i(s))|ds<\infty,\ i=1,2.$$

Proof. At first, let us observe that

$$\int_0^T |a(\xi_\pm(s))|ds\le\int_0^T (|a(\xi_1(s))|+|a(\xi_2(s))|)ds<\infty,$$

so the integrals $\int_0^T a(\xi_\pm(s))ds$ are well-defined.

Let us show that $\xi(t)=\xi_-(t)$ is a solution of (2). The reasoning for $\xi_+(t)$ is the same. Since the function $\xi_1(t)-\xi_2(t)=\int_0^t(a(\xi_1(s))-a(\xi_2(s)))ds$ is continuous, the set

$$U=\{t\ge 0:\xi_1(t)\ne\xi_2(t)\}$$

is open.

Let $U=\cup_k(a_k,b_k)$, where $(a_k,b_k)\cap(a_j,b_j)=\varnothing$ for $k\ne j$ (possibly $b_k=\infty$ for some k).

For any k, only one of the equalities is satisfied, either $\xi(t)=\xi_1(t), t\in(a_k,b_k)$, or $\xi(t)=\xi_2(t), t\in(a_k,b_k)$. Moreover,

$$\forall\ k:\ \xi_1(a_k)=\xi_2(a_k),\ \xi_1(b_k)=\xi_2(b_k).$$

This yields

$$\begin{aligned}\int_{a_k}^{b_k} a(\xi_1(s))ds=\int_{a_k}^{b_k} a(\xi_2(s))ds=\int_{a_k}^{b_k} a(\xi(s))ds=\\ =-Z(b_k)+\xi_1(b_k)+Z(a_k)-\xi_1(a_k).\end{aligned} \tag{3}$$

Let $t\in(a_n,b_n)\subset U$. Assume that $\xi_1(t)<\xi_2(t)$. Then

$$\int_0^t a(\xi(s))ds=\left(\int_{[0,t]\setminus U}+\sum_{(a_k,b_k)\subset[0,t]}\int_{a_k}^{b_k}+\int_{a_n}^t\right)a(\xi(s))ds.$$

For any $s\notin U, \xi(s)=\xi_1(s)=\xi_2(s)$. So the first integral equals

$$\int_{[0,t]\setminus U} a(\xi_1((s))ds.$$

Due to (3), we have that the second integral is equal to

$$\sum_{(a_k,b_k)\subset[0,t)}\int_{a_k}^{b_k} a(\xi_1(s))ds.$$

For any $s \in (a_n, b_n), \xi_1(s) < \xi_2(s)$. So

$$\xi(s) = \xi_1(s) \wedge \xi_2(s) = \xi_1(s),\ s \in (a_n, t).$$

Thus, the third integral equals

$$\int_{a_n}^{t} a(\xi_1(s))ds,$$

i.e.,

$$\int_0^t a(\xi(s))ds = \int_0^t a(\xi_1(s))ds,$$

$$\xi(t) = \xi_1(t) = x + \int_0^t a(\xi_1(s))ds + Z(t) = x + \int_0^t a(\xi(s))ds + Z(t).$$

The case $t \notin U$ can be considered analogously. Lemma 1.1 is proved. □

Let now $Z(t), t \geq 0$, be a Lévy process defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. In this case, we will consider only $(\mathcal{F}_t)$-adapted solutions of (1).

Lemma 1.1 and the weak uniqueness of a solution of (1) imply the pathwise uniqueness. For the corresponding definitions, see, for example, [3], Ch.IX § 1.

Corollary 1.1. *Assume that* (1) *satisfies the weak uniqueness property. Then we have the pathwise uniqueness for a solutions of* (1).

Really, let $\xi_1(t)$ and $\xi_2(t)$ be solutions of (1) defined on the same filtered probability space. Then $\xi_-(t) = \xi_1(t) \wedge \xi_2(t)$ and $\xi_+(t) = \xi_1(t) \vee \xi_2(t)$ are also solutions of (1). If

$$P(\exists\ t \geq 0 : \xi_1(t) \neq \xi_2(t)) > 0,$$

then

$$\exists\ t \geq 0, t \in Q :\ P(\xi_1(t) \neq \xi_2(t)) > 0$$

because the trajectories of ξ_1 and ξ_2 are cádlág. So

$$\exists\ t \geq 0,\ P(\xi_-(t) < \xi_+(t)) > 0.$$

Since $\xi_-(t) \leq \xi_+(t)$, the distributions of $\xi_-(t)$ and $\xi_+(t)$ cannot coincide. This contradicts the weak uniqueness. Thus,

$$P(\forall\ t \geq 0 : \xi_1(t) = \xi_2(t)) = 1.$$

Applying the Yamada–Watanabe theorem and Corollary 1.1, we obtain the following statement on strong existence (the formulation of the Yamada–Watanabe theorem is different, but the proof can be applied to our situation almost without changes).

Corollary 1.2. *Assume that there exists a unique weak solution of* (1). *Then there exists a unique strong solution of* (1).

As an application of Corollary 1.2, let us consider a case where $Z(t), t \geq 0$, is a symmetric stable process, i.e., $Z(t), t \geq 0$, is a cádlág process with stationary independent increments and

$$\exists \alpha \in (0, 2]\ \exists c > 0\ \forall \lambda \in \mathbb{R}\ \forall t \geq 0 :\ E \exp\{i\lambda Z(t)\} = \exp\{-ct|\lambda|^\alpha\}.$$

We need the following result on the existence, uniqueness, and properties of a weak solution of (1) with symmetric stable noise.

Theorem 1.1. *Assume that $Z(t), t \geq 0$, is a symmetric stable process with $\alpha \in (1, 2)$.*

1) If $a \in L_\infty(\mathbb{R})$, then there exists a unique weak solution to (1).

2) If $a \in L_p(\mathbb{R}), p \in \left(\frac{1}{\alpha-1}; +\infty\right]$, then there exists a weak solution of (1) *such that*

a) ξ is a Markov process with a continuous transition density $p(t, x, y), t > 0,\ x \in \mathbb{R},\ y \in \mathbb{R}$;

b) for any $T > 0$, there exists a constant $N = N(T, \|a\|_{L_p})$ such that

$$\forall\ t \in (0, T]\ \forall x, y \in \mathbb{R}\ \forall\ k \in \{0; 1\} : \ \left| \frac{\partial^k p(t, x, y)}{\partial x^k} \right| \leq \frac{Nt}{(t + |x - y|)^{\alpha+k+1}}.$$

For the proof of the first item, see [4], the second one can be found in [5, 6].

Corollary 1.3. *Let $Z(t), t \geq 0$, be a symmetric stable process with $\alpha \in (1, 2)$ and $a \in L_\infty(\mathbb{R})$. Then there exists a unique strong solution to* (1).

Remark. Using a localization technique, it is not difficult to prove that the unique solution to (1) exists if a measurable function a has a linear growth.

2. Continuous dependence on the initial condition and coefficients of the equation

Assume that $\{\xi_n(t), t \geq 0\}, n \geq 0$, are solutions of the equations

$$\xi_n(t) = x_n + \int_0^t a_n(\xi_n(s))ds + Z(t), t \geq 0, \tag{4}$$

where $\{Z(t), t \geq 0\}$ is a Lévy process defined on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. As in the previous section, we also require $\mathcal{F}_t$-measurability of $\xi_n(t)$.

The main result of this Section is the Theorem and the Corollary below.

Theorem 2.1. *Assume that*

1) $\lim_{n\to\infty} x_n = x_0$;

2) $\sup_{n\geq 0} \sup_x |a_n(x)| < \infty$;

3) there exists a finite measure μ on $\mathcal{B}(\mathbb{R})$ such that, for any $n \geq 1$ and λ-a.a. $t \geq 0$, the distribution of $\xi_n(t)$ has a density $p_n(x, t)$ w.r.t. $\mu(dx)$;

4) $a_n \to a_0, n \to \infty$, in measure μ;

5) for λ-a.a. $t \geq 0$ (λ is the Lebesgue measure), a sequence $\{p_n(\cdot, t), n \geq 1\}$ is uniformly integrable w.r.t. μ;

6) there exists a unique solution to Eq. (4) *where $n = 0$.*

Then, for any $T > 0$,

$$\sup_{t\in[0,T]} |\xi_n(t) - \xi_0(t)| \xrightarrow{P} 0,\ n \to \infty. \tag{5}$$

This theorem, Corollary 1.2, and Theorem 1.1 imply the following result on the continuous dependence on a parameter for the solution of (4) with a stable noise.

Corollary 2.1. *Let $\{Z(t), t \geq 0\}$ be a symmetric stable process with $\alpha \in (1, 2)$. Assume that items 1), 2), and 4) of Theorem 2.1 are satisfied, where $\mu(dx) = (1 + |x|)^{\alpha+1}dx$. Then* (4) *has a unique strong solution for any $n \geq 0$, and* (5) *holds true.*

Remark. The convergence of a sequence of functions in the measure μ is equivalent to the convergence in any absolute continuous finite measure with positive density.

Proof of Theorem 2.1. We use the Skorokhod idea of using a joint probability space [7], Ch.1 §6, Ch.3 §3. Consider the sequence of processes

$$X_n(\cdot) = (\xi_n(\cdot), \xi_0(\cdot), Z(\cdot), x_n + \int_0^\cdot a_n(\xi_n(s))ds, x_0 + \int_0^\cdot a_0(\xi_0(s))ds), n \geq 1$$

as a sequence with values in

$$(D([0, T]))^3 \times (C([0, T]))^2.$$

It easily follows from assumptions 1) and 2) of the theorem that this sequence is tight. So there exists a weakly convergent subsequence $\{X_{n_k}\}$. Without loss of generality, we assume that $\{X_n\}$ itself is weakly convergent.

By the Skorokhod theorem [7], Ch.1 §6, there exist a new probability space and a sequence $\{\widetilde{X}_n, n \geq 1\}$ such that $\widetilde{X}_n \stackrel{d}{=} X_n, n \geq 1$, and $\{\widetilde{X}_n, n \geq 1\}$ converges in probability to some random element $\widetilde{X}_0$. Denote the three first coordinates of $\{\widetilde{X}_n, n \geq 1\}$ by $\widetilde{\xi}_n(\cdot), \widehat{\xi}_n(\cdot), Z_n(\cdot)$. Note that the fourth and fifth coordinates of $\{\widetilde{X}_n, n \geq 1\}$, are measurable functions of the first and second ones. So they are equal to $x_n + \int_0^\cdot a_n(\widetilde{\xi}_n(s))ds$, $x_0 + \int_0^\cdot a_0(\widehat{\xi}_n(s))ds$, respectively.

Let

$$\widetilde{X}_0 = (\widetilde{\xi}_0(\cdot), \widehat{\xi}_0(\cdot), Z_0(\cdot), \alpha(\cdot), \beta(\cdot)),$$

where $\alpha(t), \beta(t), t \in [0,T]$, are continuous processes. We don't know yet that

$$\alpha(t) = x_0 + \int_0^t a_0(\widetilde{\xi}_0(s))ds, \ \beta(t) = x_0 + \int_0^t a_0(\widehat{\xi}_0(s))ds.$$

Note that, for any $t \in [0,T]$, the random variables $\widetilde{\xi}_0(t)$ and $\widehat{\xi}_0(t)$ are independent of $\sigma(Z_0(t+s) - Z_0(t), s \geq 0)$.

Let us verify that $\widetilde{\xi}_0$ is a solution of the equation

$$\widetilde{\xi}_0(t) = x_0 + \int_0^t a_0(\widetilde{\xi}_0(s))ds + Z_0(t), t \in [0,T]. \tag{6}$$

To prove this, it is sufficient to prove that, for λ-a.a. $t \in [0,T]$,

$$x_0 + \int_0^t a_0(\widetilde{\xi}_0(s))ds = \alpha(t) \text{ a.s.} \tag{7}$$

It follows from the convergence in probability in $D([0,T])$ that, for all $t \in [0,T]$, except for a possibly countable set, the convergence in probability

$$\widetilde{\xi}_n(t) \xrightarrow{P} \widetilde{\xi}_0(t) \tag{8}$$

holds.

Lemma 2.1. *Let $\{\eta_n, n \geq 0\}$ be a sequence of random variables. Assume that, for any $n \geq 1$, the distribution of η_n is absolutely continuous w.r.t. a probability measure μ. Denote the corresponding density by p_n. Let $\{a_n, n \geq 0\}$ be a sequence of measurable functions on $\mathbb{R}$. Suppose that the following conditions are satisfied:*

1) $\eta_n \xrightarrow{P} \eta_0, \ n \to \infty$;

2) $a_n \xrightarrow{\mu} a_0, \ n \to \infty$;

3) a sequence of densities $\{p_n, n \geq 1\}$ is uniformly integrable w.r.t. μ.

Then

$$a_n(\eta_n) \xrightarrow{P} a_0(\eta_0), \ n \to \infty.$$

The proof is similar to [8], Lemma 2, where a sequence of random elements with values in a Polish space was considered. Note that all functions $\{a_n\}$ may be discontinuous.

It follows from Lemma 2.1, assumptions of the theorem, and (8) that, for λ-a.a. $s \in [0,T]$,

$$a_n(\widetilde{\xi}_n(s)) \xrightarrow{P} a_0(\widetilde{\xi}_0(s)), \ n \to \infty.$$

So

$$\begin{aligned} &E \sup_{t \in [0,T]} \left| \int_0^t a_n(\widetilde{\xi}_n(s))ds - \int_0^t a_0(\widetilde{\xi}_0(s))ds \right| \leq \\ &\leq E \int_0^t |a_n(\widetilde{\xi}_n(s)) - a_0(\widetilde{\xi}_0(s))|ds \to 0, n \to \infty, \end{aligned} \tag{9}$$

by the Lebesgue theorem on dominated convergence. Thus, (7) is satisfied, and, hence, $\widetilde{\xi}_0$ is a solution of (6).

Similarly, it can be proved that $\widehat{\xi}_0$ satisfies the same equation

$$\widehat{\xi}_0(t) = x_0 + \int_0^t a_0(\widehat{\xi}_0(s))ds + Z_0(t), t \in [0,T], \text{ a.s.}$$

Since this equation has a unique solution, we have the equality

$$\widehat{\xi}_0(t) = \widetilde{\xi}_0(t), t \in [0,T], \text{ a.s.}$$

Let us return to the initial probability space. Let $\varepsilon > 0$ be fixed. Then

$$P(\sup_{t\in[0,T]} |\xi_n(t) - \xi_0(t)| > \varepsilon) =$$

$$= P(\sup_{t\in[0,T]} |\widetilde{\xi}_n(t) - \widehat{\xi}_n(t)| > \varepsilon) =$$

$$= P(|x_n - x_0| + \sup_{t\in[0,T]} |\int_0^t a_n(\widetilde{\xi}_n(s))ds - \int_0^t a_n(\widehat{\xi}_n(s))ds| > \varepsilon) \leq$$

$$\leq P(|x_n - x_0| + \sup_{t\in[0,T]} |\int_0^t a_n(\widetilde{\xi}_n(s))ds - \int_0^t a_0(\widetilde{\xi}_0(s))ds| > \frac{\varepsilon}{2})+$$

$$+P(\sup_{t\in[0,T]} |\int_0^t a_n(\widehat{\xi}_n(s))ds - \int_0^t a_0(\widehat{\xi}_0(s))ds| > \frac{\varepsilon}{2}).$$

The items on the r.h.s. converge to zero by (9) and a similar statement for $\widehat{\xi}_n$. The theorem is proved. □

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