Theory of Stochastic Processes
Vol. 18 (34), no. 2, 2012, pp. 54–58

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STOCHASTIC DIFFERENTIAL EQUATIONS WITH INTERACTION AND THE LAW OF ITERATED LOGARITHM

We consider a one-dimensional stochastic differential equation with interaction with no drift part. For single trajectories, we obtain the result similar to the law of iterated logarithm for a Wiener process.

1. Introduction

The main object of this paper is the one-dimensional stochastic differential equation (SDE) with interaction with no drift part

$$\begin{cases} dx(u,t) = \sigma(x(u,t), \mu_t)dw(t) \\ x(u,0) = u, \ u \in \mathbb{R} \\ \mu_t = \mu_0 \circ x(\cdot, t)^{-1}, \ t \geq 0. \end{cases} \tag{1}$$

Such equations and stochastic flows driven by them were introduced by A.A. Dorogovtsev [1]. It occurs that the properties of the solutions to such equations differ from those of the solutions to usual SDEs [5]. So it is of interest to compare their asymptotic behavior with the behavior of the diffusion processes. We want to obtain some kind of the law of iterated logarithm for trajectories of (1). For the stochastic differential equations without interaction, such results are known (see, e.g., [2], [3]). The difficulty of our case is that the behavior of a single trajectory $\{x(u,t), t \geq 0\}$ depends on all other trajectories through the measure-valued process $\{\mu_t, t \geq 0\}$.

2. Main results

Let $\mathfrak{M}_1$ be the space of all probability measures on $\mathbb{R}$ having the first moment with the Wasserstein metric [1]. Consider Eq. (1) with $\sigma : \mathbb{R} \times \mathfrak{M}_1 \to \mathbb{R}$ being global Lipschitz, and $\mu_0 \in \mathfrak{M}_1$. Under such conditions, there exists a unique strong solution to (1) such that x is a flow of homeomorphisms ([1]). The following result shows that, for the equation without drift (1), the trajectories of different particles remain not far from one another at infinity.

Lemma 1. *For all $u, v \in \mathbb{R}$, there exists*

$$\lim_{t \to \infty} (x(u,t) - x(v,t)) \ \textit{a.e.}$$

Proof. Fix $u > v \in \mathbb{R}$. Then, by the random time change [4],

$$x(u,t) - x(v,t) = u - v + \int_0^t (\sigma(x(u,s), \mu_s) - \sigma(x(v,s), \mu_s))dw(s) =$$

$$= u - v + w_{u,v}\Big(\int_0^t (\sigma(x(u,s), \mu_s) - \sigma(x(v,s), \mu_s))^2 ds \Big)$$

2010 *Mathematics Subject Classification.* Primary 60H10; Secondary 60G17.

Key words and phrases. Law of iterated logarithm, stochastic flow, stochastic differential equation with interaction, measure-valued process.

for some Wiener process $w_{u,v}$. Since $x(u,t) - x(v,t) > 0$ for all $t \geq 0$ and

$$\int_0^t (\sigma(x(u,s),\mu_s) - \sigma(x(v,s),\mu_s))^2 ds$$

is a continuous random process,

$$\int_0^\infty (\sigma(x(u,s),\mu_s) - \sigma(x(v,s),\mu_s))^2 ds \leq \tau_{u,v} < +\infty.$$

Here, $\tau_{u,v}$ is the time of the first strike of $-(u-v)$ by $w_{u,v}$. So with probability 1,

$$\lim_{t\to\infty}(x(u,t) - x(v,t)) = w_{u,v}\Big(\int_0^\infty (\sigma(x(u,s),\mu_s) - \sigma(x(v,s),\mu_s))^2 ds\Big).$$

The proof is completed. □

Of course, the difference $x(u,t) - x(v,t)$ is a martingale, which does not change the sign. So, it has a limit at infinity by the general theory [4]. But we present a direct proof, since we need the precise formula for the characteristics of $x(u,t) - x(v,t)$.

The corollary of this lemma is that the law of iterated logarithm holds (or does not hold) for all $u \in \mathbb{R}$ simultaneously. Indeed, since $x(u,\cdot) - x(v,\cdot)$ is bounded,

$$\lim_{t\to\infty}\Big(\frac{x(u,t)}{\sqrt{2t\,\mathrm{lln}\,t}} - \frac{x(v,t)}{\sqrt{2t\,\mathrm{lln}\,t}}\Big) = 0 \text{ a.e.}$$

So, we can obtain the law of iterated logarithm only for some fixed u, for example, $u = 0$. Denote $z(u,t) = x(u,t) - x(0,t)$. We need the following result.

Lemma 2. *Let $\beta > 0$ be a positive number. Then*

$$\overline{\lim_{|u|\to\infty}} \frac{\sup_{t\geq 0}|z(u,t)|}{1+|u|^{1+\beta}} < \infty \text{ a.e.} \tag{2}$$

Proof. We will consider only $u > 0$, the proof for $u < 0$ being similar. Consider some sequence $u_n \uparrow +\infty$. We have

$$z(u_n,t) = u_n + w_{u_n,0}\Big(\int_0^t \sigma^2(x(u,s),\mu_s)ds\Big).$$

Hence,

$$P\{\sup_{t\geq 0} z(u_n,t) > u_n^{1+\beta}\} \leq P\{\sup_{0\leq s\leq \tau_{u_n,0}} w_{u_n,0}(s) > u_n^{1+\beta} - u_n\} =$$

$$= \frac{u_n}{u_n^{1+\beta}} = u_n^{-\beta}.$$

Put $u_n = n^{2/\beta}$. Then we have

$$\sum_{n=1}^\infty P\{\sup_{t\geq 0} z(u_n,t) > u_n^{1+\beta}\} < +\infty,$$

and, by the Borel–Cantelli lemma with probability 1, there exists $N \in \mathbb{N}$ such that

$$\sup_{t\geq 0} z(u_n,t) < u_n^{1+\beta} \text{ for all } n \geq N.$$

Since x is a homeomorphism, for $u > u_N, u \in [u_n, u_{n+1}]$,

$$\sup_{t\geq 0} z(u,t) < \sup_{t\geq 0} z(u_{n+1},t) < 2^{\frac{2}{\beta}+2}u^{1+\beta},$$

so (2) holds. The proof is completed. □

Consider now Eq. (1) with $\sigma(u,\mu) = \int_{\mathbb{R}} b(u-v)\mu(dv)$ for some real function b. Then Eq. (1) takes the form

$$\begin{cases} dx(u,t) = \int_{\mathbb{R}} b(x(u,t) - x(v,t))\mu_0(dv)dw(t) \\ x(u,0) = u. \end{cases} \tag{3}$$

Now, the interaction of particles in a random environment depends on distances between them. Equation (3) can be considered as a stochastic analog of the Kuramoto model for interacting oscillators [6].

Theorem 1. *Let the coefficients of* (3) *satisfy the following conditions:*
1) b is global Lipschitz with some constant L,
2) $\sup_{(-\infty;0]} b < +\infty$; $\inf_{[0;\infty)} b > -\infty$,
3) $\mu_0 \in \mathfrak{M}_{1+\alpha}$ for some $\alpha > 0$.
Then

$$P\Big\{\overline{\lim_{t\to\infty}} \frac{x(0,t)}{\sqrt{2t\,\mathrm{lln}\,t}} \in D\Big\} = P\Big\{\underline{\lim_{t\to\infty}} \frac{x(0,t)}{\sqrt{2t\,\mathrm{lln}\,t}} \in -D\Big\} = 1,$$

where $D = \{|u| : u \in [\inf_{[0;+\infty)} b; \sup_{(-\infty;0]} b]\}$.

Proof. For every $u \in \mathbb{R}$, we denote

$$\zeta_u = \lim_{t\to\infty} z(u,t).$$

For every $u \in \mathbb{R}$, this limit exists almost everywhere. Now, let S be a (countable) set of all rational numbers and all atoms of measure μ_0. Then, with probability 1,

$$z(u,t) \to \zeta_u, t \to \infty \text{ for all } u \in S.$$

Let

$$\xi_u = \begin{cases} \zeta_u, \ u \in S \\ \lim_{S\ni v\to u-} \zeta_v, \ u \notin S \end{cases}$$

(the limit exists, since ζ_v is non-decreasing). ξ_u is also non-decreasing, and it is continuous except for a countable number of jumps. Hence,

$$\mu_0\{u: \ \xi_u \neq \zeta_u\} = 0,$$

and

$$\mu_0\{u: \ z(u,t) \to \xi_u, t \to \infty\} = 1.$$

With regard for the proof of Lemma 1, we have

$$\int_0^\infty \Big(\int_{\mathbb{R}} (b(z(u,s) - z(v,s)) - b(-z(v,s)))\mu_0(dv)\Big)^2 ds < \infty \text{ a.e.} \tag{4}$$

Applying Lemma 2 with $\beta = \alpha$, we have, for all $s \geq 0$,

$$|b(z(u,s) - z(v,s)) - b(-z(v,s))| \leq L|z(u,s)| \leq LC(\omega)(1 + |u|^{1+\alpha}).$$

Thus, by the Lebesgue theorem,

$$\begin{aligned} &\int_{\mathbb{R}} (b(z(u,s) - z(v,s)) - b(-z(v,s)))\mu_0(dv) \to \\ &\to \int_{\mathbb{R}} (b(\xi_u - \xi_v) - b(-\xi_v))\mu_0(dv), \ s \to \infty \text{ a.e.} \end{aligned} \tag{5}$$

From (4) and (5), we have, with probability 1,

$$\int_{\mathbb{R}} b(\xi_u - \xi_v)\mu_0(dv) = \int_{\mathbb{R}} b(-\xi_v)\mu_0(dv), \ u \in \mathbb{R}.$$

Now let $u_n \uparrow +\infty, n \to \infty$. Applying the Fatou lemma to the functions $b(\xi_{u_n} - \xi_v)1\!\!\mathrm{I}_{\{v<u_n\}}$ (which are bounded below by a constant by condition 2) of the theorem), we have

$$\int_{\mathbb{R}} \varliminf_{n\to\infty} (b(\xi_{u_n} - \xi_v)1\!\!\mathrm{I}_{\{v<u_n\}})\mu_0(dv) \le$$

$$\le \varliminf_{n\to\infty} \int_{-\infty}^{u_n} b(\xi_{u_n} - \xi_v)\mu_0(dv) = \int_{\mathbb{R}} b(-\xi_v)\mu_0(dv) -$$

$$- \varlimsup_{n\to\infty} \int_{u_n}^{+\infty} b(\xi_{u_n} - \xi_v)\mu_0(dv) = \int_{\mathbb{R}} b(-\xi_v)\mu_0(dv)$$

(we used Lemma 2 and the condition $\mu_0 \in \mathfrak{M}_{1+\alpha}$).

Now, we have

$$\varliminf_{n\to\infty} b(\xi_{u_n} - \xi_v)1\!\!\mathrm{I}_{\{v<u_n\}} \ge \inf_{z\ge 0} b(z),$$

and, thus,

$$\int_{\mathbb{R}} b(-\xi_v)\mu_0(dv) \ge \inf_{[0;+\infty)} b. \tag{6}$$

Analogously,

$$\int_{\mathbb{R}} b(-\xi_v)\mu_0(dv) \le \sup_{(-\infty;0]} b. \tag{7}$$

Now returning to the process x, we have, for some Wiener process w_0,

$$x(0,t) = \int_0^t \int_{\mathbb{R}} b(-z(v,s))\mu_0(dv)dw(s) =$$

$$= w_0\Big(\int_0^t \Big(\int_{\mathbb{R}} b(-z(v,s))\mu_0(dv)\Big)^2 ds\Big).$$

Denote

$$T(t) = \int_0^t \Big(\int_{\mathbb{R}} b(-z(v,s))\mu_0(dv)\Big)^2 ds.$$

Then

$$\lim_{t\to\infty} \frac{T(t)}{t} = \lim_{t\to\infty} \Big(\int_{\mathbb{R}} b(-z(v,t))\mu_0(dv)\Big)^2 = \Big(\int_{\mathbb{R}} b(-\xi_v)\mu_0(dv)\Big)^2.$$

If $T(t) \to \infty, t \to \infty$,

$$\varlimsup_{t\to\infty} \frac{x(0,t)}{\sqrt{2t \operatorname{lln} t}} = \varlimsup_{t\to\infty} \frac{w_0(T(t))}{\sqrt{2T(t) \operatorname{lln} T(t)}} \cdot \sqrt{\lim_{t\to\infty} \frac{T(t)}{t}} =$$

$$= \Big| \int_{\mathbb{R}} b(-\xi_v)\mu_0(dv)\Big|.$$

Else,

$$\varlimsup_{t\to\infty} \frac{x(0,t)}{\sqrt{2t \operatorname{lln} t}} = \varlimsup_{t\to\infty} \frac{w_0(T(\infty))}{\sqrt{2t \operatorname{lln} t}} = 0 = \Big| \int_{\mathbb{R}} b(-\xi_v)\mu_0(dv)\Big|.$$

Analogously,

$$\varliminf_{t\to\infty} \frac{x(0,t)}{\sqrt{2t \operatorname{lln} t}} = -\Big| \int_{\mathbb{R}} b(-\xi_v)\mu_0(dv)\Big|.$$

Together with (6) and (7), this proves the theorem. □

Corollary. *Under the conditions of the theorem, if* b *is such that*

$$u(b(u)-b(0)) \geq 0 \ (\leq 0), u \in \mathbb{R},$$

then

$$P\Big\{ \overline{\lim_{t\to\infty}} \frac{x(0,t)}{\sqrt{2t\,\mathrm{lln}\, t}} = |b(0)| \Big\} = P\Big\{ \underline{\lim_{t\to\infty}} \frac{x(0,t)}{\sqrt{2t\,\mathrm{lln}\, t}} = -|b(0)| \Big\} = 1.$$

Example. Let

$$\mu_0 = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_{\frac{\pi}{2}},$$

$$b(u) = 1 + \cos u + \sin u 1\!\!\mathrm{I}_{\{u\geq 0\}}.$$

Then

$$dx(0,t) = (\frac{1}{2}b(0) + \frac{1}{2}b(x(0,t) - x(\frac{\pi}{2},t)))dw(t)$$

$$dx(\frac{\pi}{2},t) = (\frac{1}{2}b(0) + \frac{1}{2}b(x(\frac{\pi}{2},t) - x(0,t)))dw(t).$$

Now

$$dz(\frac{\pi}{2},t) = \frac{1}{2}\sin z(\frac{\pi}{2},t)dw(t).$$

The limit

$$\xi_{\frac{\pi}{2}} = \lim_{t\to\infty} z(\frac{\pi}{2},t)$$

takes only values 0 and π. Due to the symmetry,

$$P(\xi_{\frac{\pi}{2}} = 0) = P(\xi_{\frac{\pi}{2}} = \pi) = \frac{1}{2},$$

$$P\Big\{ \int_{\mathbb{R}} b(-\xi_v)\mu_0(dv) = 2 \Big\} = P\Big\{ \int_{\mathbb{R}} b(-\xi_v)\mu_0(dv) = 1 \Big\} = \frac{1}{2}.$$

So, we have

$$P\Big\{ \overline{\lim_{t\to\infty}} \frac{x(0,t)}{\sqrt{2t\,\mathrm{lln}\, t}} = 2 \Big\} = P\Big\{ \overline{\lim_{t\to\infty}} \frac{x(0,t)}{\sqrt{2t\,\mathrm{lln}\, t}} = 1 \Big\} = \frac{1}{2}.$$

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