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**THE UNIFIED FORM OF POLLACZEK–KHINCHINE FORMULA
 FOR LÉVY PROCESSES WITH MATRIX-EXPONENTIAL NEGATIVE
 JUMPS**

For Lévy processes with matrix-exponential negative jumps, the unified form of the Pollaczek-Khinchine formula is established.

1. INTRODUCTION

The celebrated Pollaczek-Khinchine (PK) formula is known in queuing theory and risk theory in different forms. Following [1], under the safety loading condition, PK formula represents the cumulative distribution function of the waiting time W of an ordinary $M/G/1$ queue in terms of the arrival rate λ , the cumulative distribution function $F(x)$ of the service time Y , and its mean as

$$(1) \quad \mathbf{P}\{W < x\} = (1 - \rho) \sum_{n=0}^{\infty} \rho^n F_0^{*n}(x),$$

where $\rho = \lambda \mathbf{E}Y$, $F_0(x) = (\mathbf{E}Y)^{-1} \int_0^x (1 - F(y)) dy$, F_0^{*n} denotes the convolution of the distribution F_0 with itself. Following [2], formula (1) in risk theory is called the Beekman’s convolution formula and represents the survival probability for the classical risk process (with $\rho = \lambda \mathbf{E}Y/c$, where c is the premium rate).

Formula (1) is a special case of a more general result on the distribution of the maximum of a spectrally positive Lévy process. If $X_t, t \geq 0$ is a Lévy process with no negative jumps and the cumulant function $k(r) = \ln \mathbf{E}e^{rX_1}$, then the moment generating function for the absolute maximum $X^+ = \sup_{0 \leq t < \infty} X_t$ has the following representation (see, e.g., [3]):

$$(2) \quad \mathbf{E}e^{rX^+} = -\frac{rk'(0+)}{k(r)}.$$

Inverting formula (2) with respect to r , we can get the generalization of formula (1) (see [4]). For the interpretation of the generalized PK for general perturbed risk processes under the Cramer–Lundberg conditions, see [5] and references therein.

In [6], the analog of formula (2) was obtained without specific restrictions on the distribution of negative jumps. Using this analog, the generalization of (1) for risk processes with exponentially distributed premiums (with parameter $b > 0$) can be represented in terms of the convolutions of $\bar{F}_0(x) = \bar{F}(x) + b\bar{\bar{F}}(x)$ (see [12] Table IV in appendix). For some alternative representations of the supremum distribution for a Lévy process, when the Lévy measure is nonzero on $(-\infty, 0)$, see [7, 8, 9] and references therein.

In this article, we combine the results of [10, 11, 6] to get the Lévy triplet for the supremum of a Lévy process, when the negative jumps have a matrix-exponential distribution. We consider two special cases of matrix-exponential distributions, namely the Erlang and hyperexponential ones, in detail.

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2. THE UNIFIED FORM OF THE POLLACZEK–KHINCHINE FORMULA

Consider a Lévy process $X_t, t \geq 0$ with the cumulant function

$$k(r) = a'r + \frac{\sigma^2}{2}r^2 + \int_{-\infty}^{\infty} (e^{rx} - 1 - rxI_{\{|x| \leq 1\}}) \Pi(dx),$$

where a' and σ are real constants, and Π is a non-negative measure defined on $\mathbb{R} \setminus \{0\}$ such that $\int_{\mathbb{R}} \max\{x^2, 1\} \Pi(dx) < \infty$. The components $\{a', \sigma, \Pi\}$ are called a Lévy triplet. Hereinafter, we assume that $\gamma = \int_{-1}^1 |x| \Pi(dx) < \infty$. Then the cumulant function has the form

$$(3) \quad k(r) = ar + \frac{\sigma^2}{2}r^2 + \int_{-\infty}^{\infty} (e^{rx} - 1) \Pi(dx),$$

where $a = a' - \gamma$.

We write X_t^+ and X_t^- for the supremum and infimum processes and X^+ and X^- for the absolute extrema:

$$X_t^\pm = \sup(\inf)_{0 \leq t' \leq t} X_{t'}, X^\pm = \sup(\inf)_{0 \leq t < \infty} X_t.$$

Denote, by θ_s , an exponentially distributed random variable with parameter $s > 0$: $\mathbb{P}\{\theta_s > t\} = e^{-st}, t > 0$, independent of X_t , and $\theta_0 = \infty$. Then X_{θ_s} is called a Lévy process killed at the rate s (see [13]), and its moment generating function is as follows:

$$\mathbb{E}e^{rX_{\theta_s}} = \frac{s}{s - k(r)}, \operatorname{Re}[r] = 0.$$

The moment generating functions of killed extrema $\mathbb{E}e^{rX_{\theta_s}^\pm}$ represent the Wiener–Hopf factors of $\mathbb{E}e^{rX_{\theta_s}}$, i.e.,

$$\mathbb{E}e^{rX_{\theta_s}} = \mathbb{E}e^{rX_{\theta_s}^+} \mathbb{E}e^{rX_{\theta_s}^-}.$$

Having a representation of one of the factors, we can get that for another one.

In the general case, we can represent $\mathbb{E}e^{rX_{\theta_s}^+}$ in terms of the convolution of the cumulative distribution function of the killed infimum $P_-(s, y) = \mathbb{P}\{X_{\theta_s}^- < y\}$ with the integral transform of the measure Π : $\tilde{\Pi}(x, u) = \int_x^\infty e^{u(x-z)} \Pi(dz)$.

In [12] (see corollary 2.2), it was established that if $\operatorname{DX}_1 < \infty, \mathbb{E}X_{\theta_s}^+ < \infty$, then

$$(4) \quad \mathbb{E}e^{rX_{\theta_s}^+} = (1 - r(A_*(s) + s^{-1}U(s, r)))^{-1},$$

where

$$A_*(s) = \begin{cases} (2s)^{-1} \sigma^2 \frac{\partial}{\partial y} P_-(s, y) \Big|_{y=0} & \sigma > 0, \\ s^{-1} \mathbb{P}\{X_{\theta_s}^- = 0\} \max\{0, a\} & \sigma = 0, \end{cases}$$

$$U(s, r) = \int_0^\infty e^{rx} \int_{-\infty}^0 \tilde{\Pi}(x - y, 0) dP_-(s, y) dx.$$

Integrating by parts, we get $U(s, r) = \int_0^\infty (e^{rx} - 1) \int_{-\infty}^0 \Pi(dx - y) dP_-(s, y)$. As was shown in [12], if $\mu = \mathbb{E}X_1 = k'(0) < 0$, then there exist $\lim_{s \rightarrow 0} A_*(s) = a_*$ and

$$\begin{aligned} \lim_{s \rightarrow 0} s^{-1}U(s, r) &= - \int_0^\infty (e^{rx} - 1) \int_{-\infty}^0 \Pi(dx - y) d \left(\int_0^\infty \mathbb{P}\{X_t^- > y\} dt \right) = \\ &= - \int_0^\infty \int_{-\infty}^0 (e^{rx} - 1) \Pi(dx - y) d\mathbb{E}\tau^-(y), \end{aligned}$$

where $\tau^-(y) = \inf\{t > 0 : X_t < y\}, y < 0$. Denote $p_-(0, y) = -\mathbb{E}\tau^-(y)$. Then we have the following statement.

Proposition 2.1. For a Lévy process X_t , if $DX_1 < \infty$, $EX_{\theta_s^+} < \infty$, then

$$(5) \quad \mathbb{E}e^{rX_{\theta_s^+}} = \left(1 - \left(rA_*(s) + \int_0^\infty (e^{rx} - 1) \pi_s(dx) \right) \right)^{-1},$$

where $\pi_s(dx) = \int_{-\infty}^0 \Pi(dx - y) s^{-1} dP_-(s, y)$, $x > 0$.

If $\mu = EX_1 < 0$ the moment generating function of absolute supremum X^+ can be represented as the moment generating function of the subordinator X^* (a Lévy process with bounded variation, drift $a_* \geq 0$ and measure $\Pi_*(dx) = \int_{-\infty}^0 \Pi(dx - y) dp_-(0, y)$ is concentrated on $(0, \infty)$) killed at rate 1:

$$(6) \quad \mathbb{E}e^{rX^+} = \frac{1}{1 - k_*(r)} = \mathbb{E}e^{rX_{\theta_1^*}}.$$

This proposition offers the following scheme for finding the m.g.f. for the supremum of a Lévy process. First, we should find the distribution of the infimum killed at a rate s and its convolution with $\tilde{\Pi}(x, 0) = \int_x^\infty \Pi(dx)$. Then, under the condition $\mu < 0$ after the limit transition as s tends to 0, we can get the Lévy triplet of X^+ .

We call formula (6) the unified Pollaczek-Khinchine formula. To justify the name, we follow the approach from [4].

Assume $\Pi_*(0, +\infty) < \infty$ and $a_* > 0$. Then, with the use of the notation $\rho = (1 + \Pi_*(0, +\infty))^{-1}$, $c_* = (a_*(1 - \rho))^{-1}$, and $\hat{\Pi}_*(r) = \int_{(0, \infty)} e^{rx} \Pi_*(dx) / \Pi_*(0, +\infty)$, relation (6) can be rewritten as

$$(7) \quad \mathbb{E}e^{rX^+} = \frac{1}{1 - \frac{r}{c_*} - \rho \hat{\Pi}_*(r)} = \frac{c_*}{c_* - r} \frac{1}{1 - \frac{c_*}{c_* - r} \rho \hat{\Pi}_*(r)} = \\ = (1 - \rho) \sum_{n=0}^{\infty} \left(\rho \hat{\Pi}_*(r) \right)^n \left(\frac{c_*}{c_* - r} \right)^{n+1}.$$

If X_t is a compound Poisson process with drift -1 , positive jumps, and $EX_1 < 0$, then formula (7) can be reduced to $\mathbb{E}e^{rX^+} = (1 - \rho) \sum_{n=0}^{\infty} \left(\rho \int_0^\infty e^{rx} dF_0(x) \right)^n$ with $dF_0(x) = (\Pi_*(0, +\infty))^{-1} \Pi_*(dx)$. After the inversion with respect to r , we get the classical PK formula (1).

3. MATRIX-EXPONENTIAL NEGATIVE JUMPS

To be more precise, we use some additional conditions on $\Pi(dx)$ if $x < 0$. Assume that $\int_{(-\infty, 0)} \Pi(dx) = \lambda_- < \infty$, and the negative jumps have a distribution with rational characteristic function (or matrix-exponential distribution). Then, following [11], the Lévy measure for negative x has the representation

$$\Pi(-\infty, x) = \sum_{i=1}^m \sum_{j=0}^{k_i} \int_{-\infty}^x a_j^{(i)} (-y)^j e^{b_i y} dy; \quad x < 0,$$

where $\text{Re}[b_m] \geq \dots \geq \text{Re}[b_2] \geq b_1 > 0$.

We study separately two cases:

(NS): If $\sigma > 0$ or $\sigma = 0, a < 0$.

(S): If $\sigma = 0, a \geq 0$.

In [11], it was established that the cumulant equation $k(r) = s$ have N roots $-r_i(s)$ in the half-plane $\text{Re}[r] < 0$, and $-r_1(s)$ is the unique root in $[-b_1, 0]$, where

$$N = \begin{cases} \sum_{i=1}^m k_i + m + 1, & \text{in case (NS)}, \\ \sum_{i=1}^m k_i + m, & \text{in case (S)}. \end{cases}$$

We assume that the roots are ordered in the ascending order of their real parts

$$\operatorname{Re} [r_N (s)] \geq \dots \geq \operatorname{Re} [r_2 (s)] > r_1 (s).$$

By [11] (see also formula (2.25) [10]),

$$(8) \quad \mathbf{E} e^{r X_{\theta_s}^-} = \frac{\prod_{i=1}^N r_i (s) \prod_{i=1}^m (r + b_i)^{k_i+1}}{\prod_{i=1}^m b_i^{k_i+1} \prod_{i=1}^N (r + r_i (s))}, \quad \operatorname{Re} [r] = 0.$$

Let the cumulant equation have l distinct roots $r_k (s)$ with multiplicity n_k , $k = \overline{1, l}$. Then the density of $X_{\theta_s}^-$ has the representation

$$(9) \quad P'_-(s, y) = \frac{\partial}{\partial y} P_-(s, y) = D_-(s, y) + \frac{\prod_{i=1}^N r_i (s)}{\prod_{i=1}^m b_i^{k_i+1}} \sum_{k=1}^l e^{r_k (s) y} \sum_{j=1}^{n_k} \frac{A_{k,j} (s)}{(j-1)!} y^{j-1}, y < 0,$$

where

$$A_{k,j} (s) = \frac{1}{(n_k - j)!} \lim_{r \rightarrow -r_k (s)} \left[\frac{d^{(n_k - j)}}{dr^{(n_k - j)}} \left(\frac{(r + r_k (s))^{n_k} \prod_{i=1}^m (r + b_i)^{k_i+1}}{\prod_{i=1}^N (r + r_i (s))} \right) \right],$$

$$D_-(s, x) = \begin{cases} 0, & \text{in case (NS),} \\ \frac{\prod_{i=1}^N r_i (s)}{\prod_{i=1}^m b_i^{k_i+1}} \delta (x), & \text{in case (S),} \end{cases}$$

and $\delta (x)$ is the Dirac delta function.

It was shown in [11] that if $\mu = \mathbf{E} X_1 < 0$, then, as $s \rightarrow 0$, $s^{-1} r_1 (s) \rightarrow |\mu|^{-1}$, $r_i (s) \rightarrow r_i$, $\operatorname{Re} [r_i] > 0$, $2 \leq i \leq N$. Hence,

$$\frac{\prod_{i=1}^N r_i (s)}{\prod_{i=1}^m b_i^{k_i+1}} \rightarrow \frac{\prod_{i=2}^N r_i}{|\mu| \prod_{i=1}^m b_i^{k_i+1}}$$

and

$$(10) \quad p'_-(0, y) = \frac{\partial}{\partial y} p_-(0, y) = \\ = D_-(y) + \frac{\prod_{i=2}^N r_i}{|\mu| \prod_{i=1}^m b_i^{k_i+1}} \sum_{k=2}^l e^{r_k y} \sum_{j=1}^{n_k} \frac{A_{k,j} (0)}{(j-1)!} y^{j-1}, y < 0,$$

$$D_-(y) = \begin{cases} 0, & \text{in case (NS),} \\ \frac{|\mu|^{-1} \prod_{i=2}^N r_i}{\prod_{i=1}^m b_i^{k_i+1}} \delta (x), & \text{in case (S).} \end{cases}$$

If the multiplicity of a real root $r_k (s)$ is 1, then the corresponding addends in $P'_-(s, y)$ are simplified to the exponential function $A_k (s) e^{r_k (s) y}$. If $r_k (s) = v_k (s) + i w_k (s)$ is a complex root of multiplicity 1, then the corresponding addend in $P'_-(s, y)$ reduces to the product of exponential and cosine functions $A_k (s) e^{v_k (s) y} \cos (w_k (s) y + \varphi_k (s))$, where $A_k (s) = 2 |\operatorname{res} (r_k (s))|$, $\varphi_k (s) = \arg (\operatorname{res} (r_k (s)))$, $\operatorname{res} (r_k (s)) = \left[(r + r_k (s)) \mathbf{E} e^{r X_{\theta_s}^-} \right]_{r \rightarrow -r_k (s)}$.

4. SPECIAL CASES

Further, we consider several examples, when the negative jumps have the Erlang and hyperexponential distributions. In both cases in the half-plane $\operatorname{Re} [r] < 0$, the cumulant equation $k (r) = s$ has $N = d + 1$ roots in case (NS) and $N = d$ roots in case (S) with negative real parts.

4.1. Hyperexponentially distributed negative jumps. If a Lévy process X_t have hyperexponential negative jumps, then all negative roots of the cumulant equation are distinct and real (see, e.g., [15] or [14],) and the distribution of the killed infimum $X_{\theta_*}^-$ is mixed exponential.

Corollary 4.1. *Let a process X_t have hyperexponentially distributed negative jumps. If $\mu = \mathbb{E}X_1 < 0$, then the absolute maximum X^+ is distributed as the subordinator with Lévy triplet $(a_*, 0, \Pi_*(dx))$ killed rate 1, where, in case (NS),*

$$(11) \quad a_* = \frac{\sigma^2}{2|\mu|} \left(1 - \sum_{k=2}^{d+1} \frac{\prod_{i=1}^d (1 - r_k/b_i)}{\prod_{i=2, i \neq k}^N (1 - r_k/r_i)} \right),$$

$$\Pi_*(dx) = |\mu|^{-1} \left(\tilde{\Pi}(x, 0) - \sum_{k=2}^{d+1} \frac{\prod_{i=1}^d (1 - r_k/b_i)}{\prod_{i=2, i \neq k}^{d+1} (1 - r_k/r_i)} \tilde{\Pi}(x, r_k) \right) dx,$$

and, in case (S),

$$(12) \quad a_* = a|b_1\mu|^{-1} \prod_{i=2}^d (r_i/b_i),$$

$$\Pi_*(dx) = \frac{1}{|\mu|} \left(\frac{1}{b_1} \prod_{i=2}^d \frac{r_i}{b_i} \Pi(dx) + \left(\tilde{\Pi}(x, 0) - \sum_{k=2}^d \frac{\prod_{i=1}^d (1 - r_k/b_i)}{\prod_{i=2, i \neq k}^d (1 - r_k/r_i)} \tilde{\Pi}(x, r_k) \right) dx \right).$$

Proof. As all $r_k(s)$ are distinct and real, formula (9) yields

$$(13) \quad P'_-(s, y) = D_-(s, y) + \frac{\prod_{i=1}^N r_i(s)}{\prod_{i=1}^d b_i} \sum_{k=1}^N \frac{\prod_{i=1}^d (b_i - r_k(s))}{\prod_{i=1, i \neq k}^N (r_i(s) - r_k(s))} e^{r_k(s)y}, y < 0,$$

where

$$D_-(s, x) = \begin{cases} 0, & \text{in case (NS),} \\ \prod_{i=1}^d (r_i(s)/b_i) \delta(x), & \text{in case (S).} \end{cases}$$

If $\mu < 0$, then

$$(14) \quad p'_-(0, y) = D_-(y) + |\mu|^{-1} \left(1 - \sum_{k=2}^N \frac{\prod_{i=1}^d (1 - r_k/b_i)}{\prod_{i=2, i \neq k}^N (1 - r_k/r_i)} e^{r_k y} \right), y < 0,$$

where

$$D_-(x) = \begin{cases} 0, & \text{in case (NS),} \\ |b_1\mu|^{-1} \prod_{i=2}^d (r_i/b_i) \delta(x), & \text{in case (S).} \end{cases}$$

Combining formula (14) with Proposition 2.1, we get (11) and (12). Note that $\tilde{\Pi}(x, 0)$ is just $\int_x^\infty \Pi(dy)$, $x > 0$. \square

Example 4.1. If $d = 2$ in Corollary 4.1, then the parameters of X^* in case (NS) are

$$a_* = \frac{\sigma^2}{2|\mu|} \frac{r_2 r_3}{b_1 b_2},$$

$$\Pi_*(dx) = \frac{1}{|\mu|} \left[\tilde{\Pi}(x, 0) + \frac{r_3 (r_2 - b_1) (b_2 - r_2)}{b_1 b_2 (r_3 - r_2)} \tilde{\Pi}(x, r_2) + \frac{r_2 (r_3 - b_1) (r_3 - b_2)}{b_1 b_2 (r_3 - r_2)} \tilde{\Pi}(x, r_3) \right] dx.$$

In case (S), we have

$$a_* = \frac{ar_2}{|\mu| b_1 b_2},$$

$$\Pi_*(dx) = \frac{r_2}{|\mu|b_1b_2} \left[\Pi(dx) + \left(\frac{b_1b_2}{r_2} \tilde{\Pi}(x, 0) + \frac{(r_2 - b_1)(b_2 - r_2)}{r_2} \tilde{\Pi}(x, r_2) \right) dx \right].$$

4.2. Erlang-distributed negative jumps. If the negative jumps have the Erlang distribution, then we will use a matrix representation alternative to (9) for the inversion of m.g.f. of $X_{\theta_s}^-$ (for details, see [2]): if

$$\int_{-\infty}^0 e^{rx} f(x) dx = \frac{\sum_{k=1}^n \alpha_k r^{k-1}}{\sum_{k=1}^n t_k r^{n-k} + r^n},$$

then

$$(15) \quad f(x) = \boldsymbol{\alpha} e^{\mathbf{T}x} \mathbf{t}, \quad x < 0,$$

$$\text{where } \boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n), \mathbf{t} = (0, \dots, 0, 1)^\top, \mathbf{T} = \begin{pmatrix} 0 & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -1 \\ t_n & t_{n-1} & \dots & t_1 \end{pmatrix}.$$

Corollary 4.2. *Let the process X_t have Erlang-distributed negative jumps. If $\mu = \mathbf{E}X_1 < 0$, then the absolute maximum X^+ is distributed as the subordinator with Lévy triplet $(a_*, 0, \Pi_*(dx))$ killed rate 1, where, in case (NS), we have*

$$a_* = \frac{\sigma^2}{2|\mu|} \prod_{i=2}^{d+1} \left(\frac{r_i}{b} \right),$$

$$(16) \quad \Pi_*(dx) = |\mu|^{-1} \prod_{i=2}^{d+1} \left(\frac{r_i}{b} \right) \boldsymbol{\alpha} \int_x^\infty e^{\mathbf{T}(x-y)} \Pi(dy) \mathbf{t} dx,$$

$\alpha_k = \binom{d}{k-1} b^{d+1-k}$, $t_k = \sum_{1 \leq i_1 < \dots < i_k \leq d+1} r_{i_1} \dots r_{i_k}$, $\binom{d}{k-1}$ are binomial coefficients, $k = \overline{1, d+1}$.

In case (S),

$$a_* = a|b\mu|^{-1} \prod_{i=2}^d (r_i/b),$$

$$(17) \quad \Pi_*(dx) = |b\mu|^{-1} \prod_{i=2}^d \left(\frac{r_i}{b} \right) \left(\Pi(dx) + \boldsymbol{\alpha} \int_x^\infty e^{\mathbf{T}(x-y)} \Pi(dy) \mathbf{t} dx \right),$$

$\alpha_k = \binom{d}{k-1} b^{d+1-k} - t_{d+1-k}$, $t_k = \sum_{1 \leq i_1 < \dots < i_k \leq d} r_{i_1} \dots r_{i_k}$, $k = \overline{1, d}$.

Proof. Using the inversion formula (15) from (8), we deduce in case (NS) that

$$(18) \quad P'_-(s, y) = r_1(s) \prod_{i=2}^{d+1} \frac{r_i(s)}{b} \boldsymbol{\alpha} e^{\mathbf{T}(s)y} \mathbf{t}, y < 0.$$

In case (S),

$$(19) \quad P'_-(s, y) = r_1(s) \frac{\prod_{i=2}^d r_i(s)}{b^d} \left(\delta(y) + \boldsymbol{\alpha} e^{\mathbf{T}(s)y} \mathbf{t} \right), y < 0.$$

As s approaches 0, we can get the representation for $p'_-(0, y)$ from (18) and (19). From Proposition 2.1, we deduce (16) and (17). \square

To expand the matrix exponents in (16) and (17), we should know the values of their multiplicities and whether the roots r_i are real.

Example 4.2. Assume that $d = 2$ in Corollary 4.2. In case (S) as s approaches 0: $r_1(s) \rightarrow 0, r_2(s) \rightarrow r_2 > 0$. Then $\alpha = (b^2, 2b - r_2)$, $\mathbf{T} = \begin{pmatrix} 0 & -1 \\ 0 & r_2 \end{pmatrix}$. Hence,

$$p'_-(0, y) = \frac{r_2}{|\mu|b^2} \left(\delta(x) + \frac{b^2}{r_2} - \frac{(b-r_2)^2}{r_2} e^{r_2 y} \right)$$

and

$$a_* = \frac{ar_2}{|\mu|b^2},$$

$$\begin{aligned} \Pi_*(dx) &= \frac{r_2}{|\mu|b^2} \left(\Pi(dx) + \alpha \int_x^\infty e^{\mathbf{T}(x-y)} \Pi(dy) \mathbf{t} dx \right) = \\ &= \frac{r_2}{|\mu|b^2} \left(\Pi(dx) + \left(\frac{b^2}{r_2} \tilde{\Pi}(x, 0) - \frac{(b-r_2)^2}{r_2} \tilde{\Pi}(x, r_2) \right) dx \right). \end{aligned}$$

In case (NS) as s approaches 0: $r_1(s) \rightarrow 0, r_2(s) \rightarrow r_2, r_3(s) \rightarrow r_3$. Then

$$a_* = \frac{\sigma^2}{2|\mu|} \frac{r_2 r_3}{b^2},$$

$\alpha = (b^2, 2b, 1)$, $\mathbf{T} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & r_2 r_3 & (r_2 + r_3) \end{pmatrix}$. In this case, $\Pi_*(dx)$ can contain, along with the integral transform $\tilde{\Pi}(x, u)$, also the integral transform

$$B(x, u) = \int_x^\infty (x-z) e^{u(x-z)} \Pi(dz)$$

and integral transforms $C_1(x, v, w) = \int_x^\infty e^{v(x-z)} \cos(w(x-z)) \Pi(dz)$, $C_2(x, v, w) = \int_x^\infty e^{v(x-z)} \sin(w(x-z)) \Pi(dz)$. More precisely, we have 3 possibilities:

1) $r_{2,3}$ are real and distinct, then

$$\alpha e^{\mathbf{T}x \mathbf{t}} = \frac{b^2}{r_2 r_3} + \frac{e^{r_3 x} r_2 (b-r_3)^2 - e^{r_2 x} r_3 (b-r_2)^2}{r_2 r_3 (r_3 - r_2)},$$

so $\Pi_*(dx) = |\mu|^{-1} \left(\tilde{\Pi}(x, 0) - \frac{r_3(b-r_2)^2}{b^2(r_3-r_2)} \tilde{\Pi}(x, r_2) + \frac{r_2(b-r_3)^2}{b^2(r_3-r_2)} \tilde{\Pi}(x, r_3) \right) dx$. Here, we get the same formula as for the hyperexponential distribution with $b_1 = b_2 = b$.

2) $r_{2,3}$ are real and equal, then

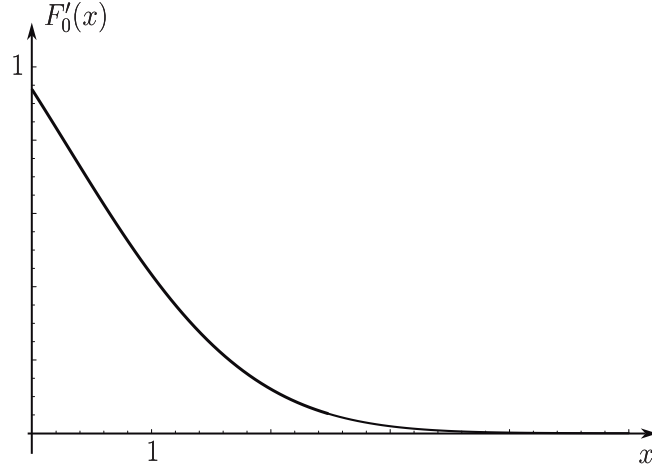
$$\alpha e^{\mathbf{T}x \mathbf{t}} = \frac{b^2}{r_2^2} \left(1 + e^{r_2 x} \left(\frac{r_2^2 - b^2}{b^2} + \frac{(b-r_2)^2 r_2}{b^2} x \right) \right),$$

$\Pi_*(dx) = |\mu|^{-1} \left(\tilde{\Pi}(x, 0) - \frac{r_2^2 - b^2}{b^2} \tilde{\Pi}(x, r_2) + \frac{(b-r_2)^2}{b^2} B(x, r_2) \right) dx$.

3) $r_{2,3}$ are complex-valued: $r_{2,3} = v \pm iw$, then

$$\begin{aligned} \alpha e^{\mathbf{T}x \mathbf{t}} &= \frac{b^2}{v^2 + w^2} + \\ &+ e^{vx} \left(\frac{(v^2 + w^2 - b^2)}{v^2 + w^2} \cos(wx) + \frac{\left(v + b \left(\frac{bv}{v^2 + w^2} - 2 \right) \right)}{w} \sin(wx) \right), \end{aligned}$$

$\Pi_*(dx) = |\mu|^{-1} \left(\tilde{\Pi}(x, 0) dx + \frac{|r_2|^2 - b^2}{b^2} C_1(x, v, w) + \frac{b^2 v + |r_2|^2 (v-2b)}{b^2 w} C_2(x, v, w) \right) dx$.

FIGURE 1. Density of η_i

Example 4.3. In conclusion, we consider a numerical example, when a process X_t have Lévy triplet $(a, \sigma, \Pi(dx))$, where

$$\Pi(dx) = \begin{cases} \lambda_+ f_+(x) & x > 0, \\ \lambda_- f_-(x) & x < 0, \end{cases} = \begin{cases} \lambda_+ \frac{2\beta}{\pi} e^{-\frac{x^2\beta^2}{\pi}} & x > 0, \\ \lambda_- \left(1 + \frac{1}{4\pi^2}\right) (1 - \cos(2\pi x)) e^x & x < 0, \end{cases}$$

i.e., the positive jumps are half-normal(β) distributed, and the negative jumps are matrix-exponential distributed with $b_1 = 1$, $b_{2,3} = 1 \pm 2\pi i$ (a special case of the matrix-exponential distribution, which is not a phase-type distribution [2]). The cumulant of X_t is

$$k(r) = ar + \frac{\sigma^2}{2}r^2 + \lambda_+ \left(e^{\frac{\pi r^2}{4\beta^2}} \left(1 + \operatorname{erf}\left(\frac{\sqrt{\pi}r}{2\beta}\right) \right) - 1 \right) + \lambda_- \left(\frac{1 + 4\pi^2}{(1+r)(4\pi^2 + (1+r)^2)} - 1 \right),$$

where $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$.

For $a = 0.2$, $\sigma = 2$, $\lambda_+ = 2$, $\beta = 1$, $\lambda_- = 4$, $m = \mathbf{E}X_1 = \frac{49+36\pi^2}{-5-20\pi^2} < 0$, the cumulant equation $k(r) = 0$ has the roots $-r_1 = 0$, $-r_{2,3} \approx 1.023 \pm 6.290i$, $-r_4 \approx 2.159$ in the half-plane $\operatorname{Re}[r] \leq 0$ and, by formula (10),

$$p'_-(0, x) \approx 0.501 + 0.582e^{2.159x} + e^{1.023x} (0.002 \cos(6.290x) + 0.008 \sin(6.290x)).$$

Hence, $a_* = \frac{\sigma^2}{2} p'_-(0, 0) \approx 2.169$ and $\Pi_*(dx) = \int_x^\infty p'_-(0, x-y) \lambda_+ f_+(y) dy$ can be expressed in terms of $\operatorname{erf}(x)$. Using formula (7), we can represent X^+ as the sum of a geometrically distributed number $\nu \sim G(1-\rho)$, exponentially distributed random variables $\xi_0, \dots, \xi_n, \dots \sim \operatorname{Exp}(c_*)$, and random variables $\eta_1, \dots, \eta_n, \dots \sim F_0$, where $1-\rho \approx 0.586$, $c_* \approx 0.792$, $F'_0(x)dx = (\Pi_*(0, +\infty))^{-1} \Pi_*(dx)$ (see Fig. 1). That is,

$$X^+ = \xi_0 + \sum_{n=1}^{\nu} (\xi_n + \eta_n).$$

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