Theory of Stochastic Processes
Vol. 18 (34), no. 2, 2012, pp. 8–14

E. V. GLINYANAYA

ASYMPTOTICS OF DISORDERING IN THE DISCRETE APPROXIMATION OF AN ARRATIA FLOW

We propose an approach to study the geometrical properties of stochastic flows with coalescence. We consider the discrete time approximation of a stochastic flow. In contrast to flows with continuous time, the ordering of particles in the discrete-time flows does not hold. The disordering in the approximation scheme can be considered as geometrical properties of a stochastic flow. We establish the rate of decrease to zero of the time which two particles spend in the opposite order.

1. Introduction

The main object of the article is a discrete flow of random mappings which approximates the Arratia flow [2, 3]. The Arratia flow consists of the Brownian motions that start from every point of the real line and move independently up to the meeting, when they coalesce and move together. One of the main features of the Arraita flow $\{x(u,t),\ u\in\mathbb{R},\ t\geq 0\}$ is the ordering of particles: $x(u_1,t)\leq x(u_2,t)$ if $u_1\leq u_2$. It is known that the $x(\cdot,t)$-image of any bounded subset of $\mathbb{R}$ is finite for any positive t due to the coalescence [2, 4]. So, the graph of the mapping $x(\cdot,t)$ is a step function. The graph of the mapping $x(\cdot,t)$ does not give an idea of the characteristic features of a behavior of particles that coalesce. To detect a characteristic property of the joint behavior of particles up to the time moment of their meeting and coalescence, we consider the discrete-time approximation of the Arratia flow. On the one hand, the motion of particles in the approximation scheme is more complicated: in this case, the change of the initial order of particles is possible. On the other hand, in the flow with discrete time, it is easier to analyze a joint behavior of particles: at each time instant from a discrete set, we can estimate the probability of disordering in the system of particles. In other words, the approximation scheme of the Arratia flow is driven by a finite number of random fields. The idea of considering a finite number of points instead of a continuous flow was advanced by H. Helmholtz. He proposed an approach to approximate a two-dimensional incompressible flow of a liquid with a finite set of vortices. In other words, in order to study a flow which consists of a continuum set of particles, one can consider a finite set of points with infinitesimal geometrical characteristics. In such approach, the Euler equation is replaced by combined equations which describe the evolution of vortices [1].

In the construction of an approximation, we follow the procedure proposed in [5, 6]. Namely, we start from a sequence of independent stationary Gaussian processes $\{\xi_n; n\geq 1\}$ on $\mathbb{R}$, which have zero mean and a covariance function Γ. Define the iterations:

$$(1)\qquad x_0(u)=u,\ x_{n+1}(u)=x_n(u)+\frac{1}{\sqrt{n}}\xi_n\left(x_n(u)\right),\ n\geq 0,\ u\in\mathbb{R}.$$

Using these iterations, one can built the random process $\tilde{x}_n$ on $[0,1]$ as follows. Let $\tilde{x}_n$ be a polygonal line with edges $\left(\frac{k}{n},x_k(u)\right)$, $k=0,\ldots,n$. It was proved in [5] that if the covariance Γ approximates the function $\mathbb{I}_{\{0\}}$ in some sense (a precise statement will be given later), then the $m-$point motions of $\tilde{x}_n$, i.e., the sets of processes $\{\tilde{x}_n(u_1),\ldots,\tilde{x}_n(u_m)\}$

2000 *Mathematics Subject Classification.* Primary 60K35; 60H35.
Key words and phrases. Stochastic flow, numerical approximation.

weakly converge to the $m-$point motions of the Arratia flow. Hence, one can expect that the time of the disordering between the particles in $\tilde{x}_n$ will go to zero, when it approximates the Arratia flow. Here, we will try to find the rate of this convergence.

2. Approximation of Arratia's flow and the time of disordering

Let us start from the definition of a flow of Brownian motions on the real line and the precise statement about flow's approximation.

Definition 1. The Arratia flow is a family $\{x(u,\cdot),\ u\in\mathbb{R}\}$ of Brownian martingales with respect to the join filtration such that:
1)for every $u\in\mathbb{R}$,

$$x(u,0)=u;$$

2) for every $u_1\le u_2$ and $t\ge 0$,

$$x(u_1,t)\le x(u_2,t);$$

3) the joint characteristics are

$$d\langle x(u_1,\cdot),x(u_2,\cdot)\rangle(t)=\mathbb{I}_{\{x(u_1,t)=x(u_2,t)\}}dt.$$

The Arratia flow describes the joint behavior of Brownian particles on the real line, which move independently till the time instant of their meeting and then coalesce and move together.

To define a discrete approximation of the Arratia flow, we consider a sequence of series of random processes $\{\xi^n_k,\ k=\overline{0,n}\}_{n\ge1}$, where, for every $n\ge 1$, $\{\xi^n_k,\ k=\overline{0,n}\}$ are independent identically distributed stationary Gaussian processes with zero mean and a covariance function Γ_n. Suppose that, for every $n\ge1$, Γ_n is a continuous function. Define the sequence of random mappings $\{x^n_k,\ k=\overline{0,n}\}_{n\ge1}$ via the recurrence equation

$$x_0(u)=u,\ x^n_{k+1}(u)=x^n_k(u)+\frac{1}{\sqrt{n}}\xi^n_{k+1}(x^n_k(u)),\quad k=\overline{0,n-1},\ u\in\mathbb{R}. \tag{2}$$

The continuity of Γ_n implies that the processes $\{\xi^n_k\}$ have measurable modifications. This allows us to substitute x^n_k into ξ^n_{k+1}. Consider the sequence of random functions $\{x_n(u,t),\ t\in[0,1],\ n\ge1\}$, where we define $x_n(u,\cdot)$ to be the piecewise linear version of $x^n_k(u)$ with the interpolation interval $\frac{1}{n}$:

$$x_n(u,t)=n\left(\frac{k+1}{n}-t\right)x^n_k(u)+n\left(t-\frac{k}{n}\right)x^n_{k+1}(u),$$
$$u\in\mathbb{R},\ t\in\left[\frac{k}{n},\frac{k+1}{n}\right],\ k=\overline{0,n-1}$$

The following proposition is known for such approximation scheme [5].

Theorem 1. *Suppose that, for every $n\ge1$, x_n is built upon a sequence $\{\xi^n_k,\ n\ge1\}$, where the independent identically distributed processes ξ^n_k have the covariance function Γ_n, which satisfies the Lipschitz condition, and $\Gamma_n(0)=1$. For every $n\ge1$, we define*

$$C_n=\sup_{\mathbb{R}}\frac{2-2\Gamma_n(x)}{x^2}.$$

If
1) $\lim_{n\to\infty}\frac{C_ne^{C_n}}{n}=0$,
2) for every $\delta>0$, $sup_{\mathbb{R}\setminus[-\delta,\delta]}|\Gamma_n(x)|\to0,\ n\to0,$
then the random processes $\{x_n(u_1,\cdot),\dots,x_n(u_l,\cdot),\ n\ge1\}$ weakly converge to the l-point motions of Arratia's flow starting from $u_1,\dots,u_l$.

In the next example, we give some covariance functions Γ_n, which satisfy the conditions of Theorem 1.

Example 1. Consider a function $\Gamma_n(u) = e^{-\frac{a_n u^2}{2}}$. One can calculate that Γ_n satisfies the Lipschitz condition, and $C_n = a_n$. To satisfy the conditions of the previous theorem, we can take $a_n = \alpha \ln n,\ \alpha < 1$.

Since our aim is to describe the time of disordering between the particles in the discrete-time approximation scheme, we consider the two-point motions

$$\{x_n(u_1, \cdot), x_n(u_2, \cdot)\}$$

and define a sequence of random processes $\{y_n(t),\ t \in [0,1]\}_{n \geq 0}$:

$$\begin{cases} y_n(t) = x_n(u_2, t) - x_n(u_1, t) \\ y_0(t) = u_2 - u_1 > 0. \end{cases} \tag{3}$$

Denote $B_n(u) = \sqrt{2 - 2\Gamma_n(u)}$. The sequence of random variables $\left\{y_n\left(\frac{k}{n}\right),\ k = \overline{0,n}\right\}_{n\geq 1}$, is equidistributed with a sequence of random variables $\{y^n_k,\ k = \overline{0,n}\}_{n\geq 1}$, which is defined via the recurrence equation:

$$y^n_{k+1} = y^n_k + B_n(y^n_k)\Delta w(\frac{k}{n}),\ y^n_0 = u_2 - u_1, \tag{4}$$

where $\Delta w(\frac{k}{n}) = w(\frac{k+1}{n}) - w(\frac{k}{n})$ are the increments of a standard Brownian motion w. From the recurrence equation (4), one can see that the random process $\{y_n(t),\ t \in [0,1]\}$ is equidistributed with a difference approximation of a solution to the Cauchy problem:

$$\begin{cases} d\tilde{y}_n(t) = B_n(\tilde{y}_n(t))dw(t), \\ \tilde{y}_n(0) = u_2 - u_1. \end{cases} \tag{5}$$

It is known [7] that

$$\mathbb{E} \max_{[0,1]} |y_n(t) - \tilde{y}_n(t)|^2 \leq c\frac{C_n e^{C_n}}{n}. \tag{6}$$

We will analyze the joint behavior of two particles in the discrete scheme, using the functional

$$\Phi_n = \int_0^1 \mathbb{I}_{\{y_n(s)<0\}} ds.$$

We obtain the rate of decreasing to zero for Φ_n.

To prove the main theorem, we need a preliminary lemma, in which we compare the solution of SDE (5) with the Brownian motion. Without loss of generality, we make an assumption: $u_2 - u_1 = 1$.

Lemma 1. *Let $\tilde{y}_n$ be a solution to (5). Suppose that $B_n \in C^1((0, +\infty))$ and*

1) $B_n(u)$ increases for $u > 0$;

2) $B_n\left(\frac{1}{\sqrt{C_n}}\right) \geq \frac{1}{K},\ n \geq 1$.

Let $\tau = \inf\{t :\ \tilde{y}_n(t) = \frac{1}{\sqrt{C_n}}\}$. Then, for $t \in [0, \tau)$,

$$\tilde{y}_n(t) \leq \sqrt{2}(K + w(t)) + \frac{1}{\sqrt{C_n}},$$

where w is the very same Wiener process as in (5).

Proof. For the function

$$f(x) = \int_{\frac{1}{\sqrt{C_n}}}^x \frac{1}{\sqrt{B_n(u)}} du,$$

the Itô formula yields

$$f(\tilde{y}_n(t)) - f(1) = \int_0^t dw(s) - \frac{1}{2}\int_0^t B_n'(\tilde{y}_n(s))ds.$$

Since $B_n(0) = 0$ and $\tilde{y}_n(0) = 1 > 0$, the process $\tilde{y}_n(t)$ is positive for all $t \geq 0$ as a solution of SDE (5). So, using condition 1) of this lemma, $B_n'(\tilde{y}_n(s)) > 0$ for $s \geq 0$, and we have

$$f(\tilde{y}_n(t)) \leq f(1) + w(t).$$

Using the estimation $B_n(u) \leq \sqrt{2}$ and condition 2), we obtain

$$f(x) \geq \frac{1}{\sqrt{2}}\left(x - \frac{1}{\sqrt{C_n}}\right), \text{ if } x \geq \frac{1}{\sqrt{C_n}},$$

$$f(1) \leq \frac{1}{B_n\left(\frac{1}{\sqrt{C_n}}\right)} \leq K.$$

Combining these inequalities, one can get

$$\frac{1}{\sqrt{2}}\left(\tilde{y}_n(t) - \frac{1}{\sqrt{C_n}}\right) \leq f(\tilde{y}_n(t)) \leq K + w(t),$$

for all t such that $\tilde{y}_n(t) \geq \frac{1}{\sqrt{C_n}}$. □

The main result of this paper is as follows.

Theorem 2. *Suppose that, for every $n \geq 1$, the covariance function Γ_n satisfies the conditions of Theorem 1. Then*

$$\mathbb{P}\{\Phi_n > 0\} \leq nF\left(\sqrt{\frac{n}{C_n}}\right).$$

If Γ_n is such that

1)$\lim_{n\to\infty} \frac{C_n^2 e^{C_n}}{n} = 0$,

2)$\frac{u}{B_n(u)}$ *increases for* $u > 0$,

3)$B_n\left(\frac{1}{\sqrt{C_n}}\right) \geq \frac{1}{K},\ n \geq 1$,

then, for any $\varepsilon > 0$,

$$\varliminf_{n\to\infty} \frac{\mathbb{P}\{\Phi_n > \varepsilon\}}{F\left(K\sqrt{\frac{n}{C_n}}\right)} \geq const > 0,$$

where $F(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du$

Proof. Using (4) for any $k \geq 1$ we have:

$$\mathbb{I}_{\{y_{k-1}^n > 0\}}\mathbb{P}\{y_k^n > 0 | y_{k-1}^n\} = \mathbb{I}_{\{y_{k-1}^n > 0\}}\mathbb{P}\Big\{y_{k-1}^n + B_n(y_{k-1}^n)\Delta w\left(\frac{k}{n}\right) > 0 | y_{k-1}^n\Big\} =$$

$$= \mathbb{I}_{\{y_{k-1}^n > 0\}}\mathbb{P}\Big\{\Delta w\left(\frac{k}{n}\right) > -\frac{y_{k-1}^n}{B_n(y_{k-1}^n)}\Big| y_{k-1}^n\Big\} \geq \mathbb{I}_{\{y_{k-1}^n > 0\}}\int_{-\sqrt{\frac{n}{C_n}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx =$$

$$= \mathbb{I}_{\{y_{k-1}^n > 0\}}\left(1 - F\left(\sqrt{\frac{n}{C_n}}\right)\right),$$

where we used the inequality $\frac{|u|}{B_n(u)} \geq \frac{1}{\sqrt{C_n}}$, which follows from the definition of C_n. Using this inequality, we obtain

$$\mathbb{P}\{\Phi_n = 0\} = \mathbb{P}\{y_1^n > 0, y_2^n > 0, \ldots, y_n^n > 0\} =$$
$$= \mathbb{E}\left(\mathbb{I}_{\{y_1^n>0\}} \ldots \mathbb{I}_{\{y_{n-1}^n>0\}} \mathbb{E}\left[\mathbb{I}_{\{y_n^n>0\}} \Big| \Delta w\left(0\right), \Delta w\left(\frac{1}{n}\right), \ldots, \Delta w\left(\frac{n-1}{n}\right)\right]\right) =$$
$$= \mathbb{E}\left(\mathbb{I}_{\{y_1^n>0\}} \ldots \mathbb{I}_{\{y_{n-1}^n>0\}} \cdot \mathbb{P}\{y_n^n > 0 | y_{n-1}^n\}\right) \geq$$
$$\geq \left(1 - F\left(\frac{\sqrt{n}}{\sqrt{C_n}}\right)\right) \mathbb{E}(\mathbb{I}_{\{y_1^n>0\}} \ldots \mathbb{I}_{\{y_{n-1}^n>0\}}) \geq$$
$$\geq \ldots \geq \left(1 - F\left(\sqrt{\frac{n}{C_n}}\right)\right)^n \geq 1 - nF\left(\sqrt{\frac{n}{C_n}}\right).$$

Thus, we have

$$\mathbb{P}\{\Phi_n > 0\} = 1 - \mathbb{P}\{\Phi_n = 0\} \leq nF\left(\sqrt{\frac{n}{C_n}}\right).$$

To prove the second part of this theorem, let us introduce the following stopping times:

$$\kappa_n = \min\left\{\frac{k}{n} : y_k^n < \sqrt{\frac{2}{C_n}}\right\}.$$

Then, for any $\varepsilon > 0$,

$$\mathbb{P}\{\Phi_n > \varepsilon\} \geq \sum_{k=1}^{[n(1-\varepsilon)]} \mathbb{P}\Big\{\kappa_n = \frac{k}{n},\ y_{k+j}^n < 0,\ j = 1, \ldots, n-k\Big\}.$$

We will estimate separately each term of the previous sum:

$$\mathbb{P}\Big\{\kappa_n = \frac{k}{n},\ y_{k+j}^n < 0,\ j = 1, \ldots, n-k\Big\} =$$
$$\mathbb{E}\left(\mathbb{I}_{\{\kappa_n = \frac{k}{n}\}} \mathbb{I}_{\{y_{k+1}^n<0\}} \ldots \mathbb{I}_{\{y_{n-1}^n<0\}} \mathbb{E}[\mathbb{I}_{\{y_n^n<0\}} | y_{n-1}^n]\right) \geq$$
$$\geq \left(1 - F\left(\frac{\sqrt{n}}{\sqrt{C_n}}\right)\right) \mathbb{P}\Big\{\kappa_n = \frac{k}{n},\ y_{k+j}^n < 0,\ j = 1, \ldots, n-k-1\Big\} \geq \ldots \geq$$
$$\geq \left(1 - F\left(\frac{\sqrt{n}}{\sqrt{C_n}}\right)\right)^{n-k-1} \mathbb{P}\Big\{\kappa_n = \frac{k}{n}, y_{k+1}^n < 0\Big\}.$$

For $\mathbb{P}\{\kappa_n = \frac{k}{n}, y_{k+1}^n < 0\}$, we obtain the estimation:

$$\mathbb{P}\Big\{\kappa_n = \frac{k}{n}, y_{k+1}^n < 0\Big\} = \mathbb{P}\Big\{y_1^n > \sqrt{\frac{2}{C_n}}, \ldots, y_{k-1}^n > \sqrt{\frac{2}{C_n}}, y_k^n < \sqrt{\frac{2}{C_n}}, y_{k+1}^n < 0\Big\} =$$
$$= \mathbb{E}\left(\mathbb{I}_{\{\kappa_n = \frac{k}{n}\}} \mathbb{P}\Big\{y_{k+1}^n < 0 \Big| y_k^n < \sqrt{\frac{2}{C_n}}\Big\}\right) =$$
$$= \mathbb{E}\left(\mathbb{I}_{\{\kappa_n = \frac{k}{n}\}} \mathbb{P}\Big\{\Delta w\left(\frac{k}{n}\right) < -\frac{y_k^n}{B_n(y_k^n)} \Big| y_k^n < \sqrt{\frac{2}{C_n}}\Big\}\right) \geq$$
$$\geq \mathbb{P}\Big\{\Delta w\left(\frac{k}{n}\right) < -K\sqrt{\frac{2}{C_n}}\Big\} \mathbb{P}\Big\{\kappa_n = \frac{k}{n}\Big\}.$$

So we get the estimation

$$\mathbb{P}\{\Phi_n > \varepsilon\} \geq F\left(K\sqrt{\frac{n}{C_n}}\right) \sum_{k=1}^{[n(1-\varepsilon)]} \left(1 - F\left(\sqrt{\frac{n}{C_n}}\right)\right)^{n-k-1} \mathbb{P}\Big\{\kappa_n = \frac{k}{n}\Big\} \geq$$

$$\geq \left(1 - F\left(\sqrt{\frac{n}{C_n}}\right)\right)^{n-2} F\left(K\sqrt{\frac{n}{C_n}}\right) \sum_{k=1}^{[n(1-\varepsilon)]} \mathbb{P}\Big\{\kappa_n = \frac{k}{n}\Big\}.$$

To get the estimation for $\sum_{k=1}^{[n(1-\varepsilon)]} \mathbb{P}\{\kappa_n = \frac{k}{n}\}$, we use estimation (6) and Lemma 1.

Denote

$$\tau_n = \inf\Big\{t : \tilde{y}_n(t) = \sqrt{\frac{1}{C_n}}\Big\},$$

where $\tilde{y}_n(\cdot)$ is a solution to SDE (5). We have

$$\sum_{k=1}^{[n(1-\varepsilon)]} \mathbb{P}\Big\{\kappa_n = \frac{k}{n}\Big\} \geq \mathbb{P}\Big\{\kappa_n \leq \frac{[n(1-\varepsilon)]}{n}, \max_{[0,1]}|\tilde{y}_n(t) - y_n(t)| \leq \sqrt{\frac{1}{C_n}}\Big\} \geq$$

$$\geq 1 - \mathbb{P}\Big\{\tau_n > \frac{[n(1-\varepsilon)]}{n}\Big\} - \mathbb{P}\Big\{\max_{[0,1]}|\tilde{y}_n(t) - y_n(t)|^2 > \frac{1}{C_n}\Big\} \geq$$

$$\geq \mathbb{P}\Big\{\tau_n \leq \frac{[n(1-\varepsilon)]}{n}\Big\} - c\frac{C_n^2 e^{C_n}}{n}.$$

We put $\Theta_n = \inf\Big\{t : \sqrt{2}(w(t) + K) + \sqrt{\frac{1}{C_n}} = \sqrt{\frac{1}{C_n}}\Big\} = \inf\{t : w(t) + K = 0\}$. Then, by Lemma 1,

$$\mathbb{P}\Big\{\tau_n \leq \frac{[n(1-\varepsilon)]}{n}\Big\} \geq \mathbb{P}\Big\{\Theta_n \leq \frac{[n(1-\varepsilon)]}{n}\Big\} = 2\mathbb{P}\Big\{w\left(\frac{[n(1-\varepsilon)]}{n}\right) > K\Big\}.$$

Finally, we get

$$\mathbb{P}\{\Phi_n > \varepsilon\} \geq F\left(K\sqrt{\frac{n}{C_n}}\right) A_n,$$

where

$$A_n = \left(1 - (n-2)F\left(\sqrt{\frac{n}{C_n}}\right)\right)\left(2\mathbb{P}\Big\{w\left(\frac{[n(1-\varepsilon)]}{n}\right) > K\Big\} - c\frac{C_n^2 e^{C_n}}{n}\right) \to$$
$$\to 2\mathbb{P}\{w(1-\varepsilon) > K.\}$$

□

In the corollary, we obtain the asymptotics of the logarithm of $\mathbb{P}\{\Phi_n > \varepsilon\}$.

Corollary 1. *Under conditions of the previous theorem, we have*

$$\overline{\lim_{n\to\infty}} \frac{2C_n}{n} \ln \mathbb{P}\{\Phi_n > 0\} \leq -1$$
$$\underline{\lim_{n\to\infty}} \frac{2C_n}{n} \ln \mathbb{P}\{\Phi_n > \varepsilon\} \geq -K^2.$$

Proof. From the previous theorem,

$$\ln \mathbb{P}\{\Phi_n > 0\} \leq \ln n + \ln F\left(\sqrt{\frac{n}{C_n}}\right).$$

Using

$$F(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du \sim \frac{1}{x\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \text{as } x \to \infty,$$

we obtain the first part of the corollary. To get the second one, we use the estimation from the proof of the theorem:

$$\mathbb{P}\{\Phi_n > \varepsilon\} \geq F\left(K\sqrt{\frac{n}{C_n}}\right) A_n.$$

□

References

1. V. I. Arnold and B. F. Khesin, *Topological Methods in Hydrodynamics*, Springer, New York, 1998.
2. R. Arratia, *Coalescing Brownian motions on the line*, Thesis (Ph.D.), Univ. of Wisconsin, Madison, 1979.
3. Y. Le Jan and O. Raimond,*Flows, coalescence and noise*, Ann. Probab. **32** (2004), no. 2, 1247–1315.
4. A. A. Dorogovtsev, *Measure-Valued Processes and Stochastic Flows*, Inst. of Math., Kyiv, 2007 (in Russian).
5. I. I. Nishchenko, *Discrete time approximation of coalescing stochastic flows on the real line*, Theory Stoch. Process. **17** (2011), no. 1.
6. E. V. Glinyanaya, *Discrete analogue of the Krylov-Veretennikov expansion*, Theory Stoch. Process. **17** (2011), no. 1.
7. D. F. Kuznetsov, *Stochastic Differential Equations: Theory and Practice of Numerical Solution*, St.-Petersburg– Polytechn.- Univ., St.-Petersburg, 2010 (in Russian).

Institute of Mathematics of Ukrainian Academy of Sciences
E-mail address: glinkate@gmail.com