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ON THE GENERALIZATION OF THE MCKEAN–VLASOV EQUATION TO THE CASE WHERE THE TOTAL MASS OF PARTICLES IS INFINITE

The McKean–Vlasov equation describing the motion of a system of particles with infinite total mass is considered. The theorem of existence and uniqueness of a solution is proved. The solution is constructed by passing to the limit from that for the systems of particles having a finite total mass.

1. Introduction

Stochastic flows with interaction is an important mathematical object that can have applications to statistical physics, chemistry, biology. Equations of motion with interaction were studied from different points of view (see [1]-[13] and references therein). The aim of this paper is to generalize the McKean–Vlasov equation to the case where the total mass of particles is infinite. Recall that the McKean–Vlasov equation can be introduced in the following way.

Assume that there are N particles and the mass of each particle is $1/N$. By $X_t^{i,N}$, we denote a coordinate of the i-th particle at the time moment t. Let $\{w_t^i\}$ be Wiener processes, and let $\{x_0^i\}$ be identically distributed random variables. Suppose that $\{\{w_t^i, t \geq 0\},\ x_0^i,\ i = 1, ..., N\}$ are jointly independent. Denote, by μ, the distribution of x_0^i. Suppose that the motion of particles is described by the equation

$$(1)\qquad \begin{cases} dX_t^{i,N} = dw_t^i + \frac{1}{N}\sum_{j=1}^N b(X_t^{i,N}, X_t^{j,N})dt,\ i = 1, ..., N,\ t \in [0,T], \\ \hfill X_0^{i,N} = x_0^i, \end{cases}$$

Here, the function b is related to the interaction between the particles. It was proved (see [8], [12], [13]) that if b is a bounded Lipshitz function, then there exists a limit X_t^i of the sequence $\{X_t^{i,N},\ N \geq 1\}$, and X_t^i is a unique solution of the equation

$$(2)\qquad \begin{cases} dX_t^i = dw_t^i + \int\limits_{\mathbf{R}} b(X_t^i, y)\nu_t^i(dy)dt,\ t \in [0,T], \\ \hfill X|_{t=0} = x_0^i, \end{cases}$$

where $\nu_t^i(dy)$ is a distribution of the random variable X_t^i. Notice that $\nu_0^i = \mu$ for all $i \geq 1$. It can be seen that ν_t^i is independent of i. Denote $\nu_t(dy) = \nu_t^i(dy)$. So, the equation

$$(3)\qquad \begin{cases} dX_t = dw_t + \int\limits_{\mathbf{R}} b(X_t, y)\nu_t(dy)dt,\ t \in [0,T], \\ \hfill X|_{t=0} = x_0, \\ \hfill \nu_t(dy)\ \textit{is the distribution of}\ X_t, \end{cases}$$

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can be interpreted as an equation that determines a motion of the system of interacting particles with total mass 1. Moreover, it can be proved that

$$\nu_t^{(N)} := \sum_{i=1}^{N} N^{-1}\delta_{X_t^{i,N}} \Rightarrow \nu_t,\ N \to \infty.$$

So, the measure $\nu_t(dv)$ can be considered as the mass distribution of the system of particles at the time t, and the function $X_\cdot(\omega)$ can be interpreted as the trajectory of a single particle.

If the total mass of particles is finite, then we can proceed similarly. For example, let μ be an initial distribution of mass, $\mu(\mathbb{R}) = M$. Introduce a probability measure $\nu_0(du) = \mu(du)/M$. Let x_i be jointly independent random variables with distribution ν_0. Assume that there are N particles in $\mathbb{R}$, the mass of each particle is equal to M/N, and the motion is described by the equation

$$\text{(4)}\qquad \begin{cases} dX_t^{i,N} = dw_t^i + \frac{M}{N}\sum_{j=1}^{N} b(X_t^{i,N}, X_t^{j,N})dt,\ i = 1, ..., N,\ t \in [0,T], \\ \hfill X_0^{i,N} = x_0^i. \end{cases}$$

Or we may assume that there are $[MN]$ particles with masses $1/N$, and the motion of particles is given by the equation

$$\text{(5)}\qquad \begin{cases} dX_t^{i,N} = dw_t^i + \frac{1}{N}\sum_{j=1}^{[MN]} b(X_t^{i,N}, X_t^{j,N})dt,\ i = 1, ..., N,\ t \in [0,T], \\ \hfill X_0^{i,N} = x_0^i. \end{cases}$$

Denote $b_1(\cdot,\cdot) = Mb(\cdot,\cdot)$. Similarly to the case $M = 1$, there exists a limit X_t^i of the sequence $\{X_t^{i,N},\ N \geq 1\}$, and X_t^i is a unique solution of the equation

$$\begin{cases} dX_t = dw_t + \int\limits_{\mathbf{R}} b_1(X_t, y)\nu_t(dy)dt,\ t \in [0,T], \\ \hfill X_0 = x_0, \\ \hfill \nu_t(dy) \textit{ is the distribution of } X_t. \end{cases}$$

Denote $\mu_t = M\nu_t(dy) = |\mu|\nu_t(dy)$. Then the last equation may be rewritten in the form

$$\begin{cases} dX_t = dw_t + \int\limits_{\mathbf{R}} b(X_t, y)\mu_t(dy)dt,\ t \in [0,T], \\ \hfill X_0 = x_0, \\ \hfill \mu_t = |\mu|\nu_t,\ \nu_t \textit{ is the distribution of } X_t. \end{cases}$$

However, if the total mass of particles is infinite, then we cannot proceed similarly, because we cannot normalize the measure μ_t to get a probability measure ν_t. Denote, by P, a Wiener measure on $C = C[0,T]$. Let us make the change of a variable under the sign of integral in (3). Then we obtain the equation

$$\text{(6)}\qquad \begin{cases} dX_t(u,\omega) = dw_t(\omega) + \int\limits_{\mathbf{R}}\int\limits_{C} b(X_t(u,\omega), X_t(v,\widetilde{\omega}))P(d\widetilde{\omega})\mu(dv)dt, \\ \hfill X_0(u) = u. \end{cases}$$

Note that this representation fits for infinite measure μ too. It can be considered as an analog of the McKean–Vlasov equation with infinite initial mass. The aim of this paper is to prove that there exists a unique solution of Eq. (6) with locally finite measure μ.

In this paper, it is also proved that the unique solution of Eq. (6) can be obtained by passing to the limit as $n \to \infty$ in the equation with finite initial mass distribution $\mu_n(dx) = I_{[-n,n]}(x)\mu(dx)$.

2. Equation of motion of the system of interacting particles with infinite total mass

Suppose $(\Omega, \Im, (\Im_t, t \geq 0), P)$ is a filtered probability space, w_t is a Wiener process, and μ is a measure on $\mathbb{R}$. Consider the equation that determines the motion of a continual system of interacting particles

$$(7) \qquad \begin{cases} dX_t(u,\omega) = dw_t(\omega) + \int_{\mathbb{R}} \int_{\Omega} b(X_t(u,\omega), X_t(v,\widetilde{\omega}))P(d\widetilde{\omega})\mu(dv)dt, \\ X_0(u) = u, \end{cases}$$

Further, we assume that the function b depends only on the difference of its arguments. Then Eq. (7) can be rewritten in the following way:

$$(8) \qquad \begin{cases} dX_t(u,\omega) = dw_t(\omega) + \int_{\mathbb{R}} \int_{\Omega} b(X_t(v,\widetilde{\omega}) - X_t(u,\omega))P(d\widetilde{\omega})\mu(dv)dt, \\ X_0(u) = u, \end{cases}$$

Here, $X_s(v)$ can be interpreted as a coordinate at the time moment s of a particle that started from a point v. The measure μ can be treated as the initial mass distribution. The function b determines the interaction between particles.

Definition 2.1. Suppose that $X_t = X_t(u,\omega)$ is a stochastic process continuous in t, measurable in (u,t,ω), and $\Im_t$-adapted. If, for almost all ω, the stochastic process $X_t(u,\omega)$ satisfies (8), then $X_t(u,\omega)$ is called a solution of Eq. (8).

Assume that the measure μ and the function b satisfy the following assumptions:

(A1) There exists a constant $c_\mu > 0$ such that, for every interval $[a,b]$,

$$\mu([a,b]) \leq C_\mu(b-a+1).$$

(A2) $b \in C^1(\mathbb{R})$.

(A3) There exists a function $R : \mathbb{R} \to [0,+\infty)$ non-decreasing on $(-\infty,0]$ and non-increasing on $[0,+\infty)$ such that $|b(z)| \leq R(z)$ for all $z \in \mathbb{R}$ and

$$C_R := 2 \sup_{a \in \mathbb{R}} \int_{\mathbb{R}} R(z+a)\mu(dz) < +\infty.$$

(A4) There exists a function $Q : \mathbb{R} \to [0,+\infty)$ non-decreasing on $(-\infty,0]$ and non-increasing on $[0,+\infty)$ such that $|b'(z)| \leq Q(z)$ for all $z \in \mathbb{R}$, and

$$C_Q := 2 \sup_{a \in \mathbb{R}} \int_{\mathbb{R}} Q(z+a)\mu(dz) < +\infty.$$

Remark 2.1. If R and Q are integrable with respect to the Lebesgue measure on $\mathbb{R}$, and if μ satisfies (A1), then $C_R < +\infty$, and $C_Q < +\infty$.

Denote

$$M(t,\omega) = \sup_{u \in \mathbb{R}, s \in [0,t]} |X_s(u,\omega) - u|.$$

The main result of this paper is the following theorem.

Theorem 2.1. *There exists a solution of Eq.* (8) *such that*

$$\forall \alpha > 0 \ E \exp(\alpha M_T) < +\infty.$$

The solution is unique in the following sense: if $X_t(u) = X_t(u,\omega)$ and $Y_t(u) = Y_t(u,\omega)$ are solutions of Eq. (8), *and if*

$$E \sup_{u \in \mathbb{R}, s \in [0,T]} |X_s(u) - u| < +\infty, \ E \sup_{u \in \mathbb{R}, s \in [0,T]} |Y_s(u) - u| < +\infty,$$

then

$$P(\forall t \in [0,T] \ \forall u \in \mathbb{R} : \ X_t(u) = Y_t(u)) = 1.$$

We will prove the existence for a solution of Eq. (8) in the following way. We approximate the measure μ in (8) by the measures $\mu_n(dx) = I_{[-n,n]}(u)\mu(dx)$ and prove that the obtained sequence of solutions is relatively compact. Then we verify that a limit of any convergent subsequence satisfies (8).

We need the following estimate for a deviation of particles from the initial position.

Lemma 2.1. *There exists a function $C(t) = C(C_\mu, C_R, R(0), t)$ such that, for every solution $X_t(u,\omega)$ of Eq. (8) that satisfies the inequality $EM(T) < +\infty$, we have*

$$\forall t \in [0,T]: \ EM(t) \le C(t). \tag{9}$$

Proof. By Eq. (8), we have

$$|X_t(u,\omega) - u| \le |w_t(\omega)| + \int_0^t \int_\Omega \int_{\mathbb{R}} R(X_s(v,\widetilde{\omega}) - X_s(u,\omega))\mu(dv)P(d\widetilde{\omega})ds \tag{10}$$

Using the definition of $M(t,\omega)$, we get

$$X_s(v,\widetilde{\omega}) - X_s(u,\omega) = (X_s(v,\widetilde{\omega}) - v) - (X_s(u,\omega) - u) + v - u \ge$$

$$\ge v - u - M(s,\omega) - M(s,\widetilde{\omega}).$$

We will divide the integral on the right-hand side of inequality (10) into three parts: by intervals $[u - M(t,\omega) - M(t,\widetilde{\omega}), u + M(t,\omega) + M(t,\widetilde{\omega})]$, $(-\infty, u - M(t,\omega) - M(t,\widetilde{\omega}))$, and $(u + M(t,\omega) + M(t,\widetilde{\omega}), +\infty)$. From the last inequality and assumption (A3), we get $R(X_s(v,\widetilde{\omega}) - X_s(u,\omega)) \le R(v - u - M(s,\omega) - M(s,\widetilde{\omega}))$, when $v \ge u + M(s,\omega) + M(s,\widetilde{\omega})$.

Analogously, $R(X_s(v,\widetilde{\omega}) - X_s(u,\omega)) \le R(v - u + M(s,\omega) + M(s,\widetilde{\omega}))$, when $v \le u - M(s,\omega) - M(s,\widetilde{\omega})$. From these inequalities, assumptions (A1)-(A3), and inequality (10), we obtain

$$|X_t(u,\omega) - u| \le |w_t(\omega)| + \int_0^t (2R(0)C_\mu(EM(s) + M(s,\omega) + 1) + C_Q)ds. \tag{11}$$

After taking supremum in $u \in \mathbb{R}$, $s \in [0,t]$ and expectation on both sides of inequality (11), we get

$$EM(t) \le E|w_t| + \int_0^t (2R(0)C_\mu(2EM(s) + 1) + C_R)ds.$$

We now use Gronwall's lemma in the following formulation.

Lemma 2.2. *Let f, g, a be nonnegative measurable functions. Suppose that g is nondecreasing, a and f are locally bounded, and*

$$\forall t \ge 0 \ f(t) \le g(t) + \int_0^t a(s)f(s)ds.$$

Then

$$\forall t \ge 0 \ f(t) \le g(t)e^{\int_0^t a(s)ds}.$$

Since $E|w_t| = \sqrt{\frac{2t}{\pi}} \le \sqrt{t}$, we have, by using Lemma 2.2,

$$EM(t) \le (\sqrt{t} + (2C_\mu R(0) + C_R)t)e^{4C_\mu R(0)t} =: C(t).$$

This finishes the proof of Lemma 2.1.

Lemma 2.3. *There exists a constant $C_4 = C_4(C_Q, C_\mu, Q(0), T)$ such that, for every solution $X_t(u,\omega)$ of Eq. (8), the inequality $EM(T) < +\infty$ implies*

$$\forall t \in [0,T] \ \forall u_1, u_2 \in \mathbb{R}: \ |X_t(u_1) - X_t(u_2)| \le C_4|u_1 - u_2|.$$

Proof. Denote $\widetilde{b}(s,u) = \int_\Omega \int_{\mathbb{R}} b(X_s(v,\widetilde{\omega}) - u)\mu(dv)P(d\widetilde{\omega})$. Then Eq. (8) can be rewritten in the form

$$X_t(u,\omega) = u + w_t + \int_0^t \widetilde{b}(s, X_s(u,\omega))ds \tag{12}$$

Assume that we have proved that

$$\exists C_5 \ \forall t \in [0,T] \ \forall u_1, u_2 : \ |\widetilde{b}(t,u_1) - \widetilde{b}(t,u_2)| \le C_5 |u_1 - u_2|. \tag{13}$$

Subtract Eqs. (12) for $X_t(u_1,\omega)$ and $X_t(u_2,\omega)$. Then Lemma 2.3 can be easily deduced from Lemma 2.2.

Let us verify (13):

$$|\widetilde{b}(s,u_1) - \widetilde{b}(s,u_2)| \le \int_\Omega \int_{\mathbb{R}} |b(X_s(v,\widetilde{\omega}) - u_1) - b(X_s(v,\widetilde{\omega}) - u_2)|\mu(dv)P(d\widetilde{\omega}). \tag{14}$$

So, the Lagrange theorem yields

$$|b(X_s(v,\widetilde{\omega}) - u_1) - b(X_s(v,\widetilde{\omega}) - u_2)| =$$
$$= |b'(\theta(u_1,u_2,v,s,\widetilde{\omega}))| \cdot |(X_s(v,\widetilde{\omega}) - u_1) - (X_s(v,\widetilde{\omega}) - u_2)| \le$$
$$\le Q(\theta(u_1,u_2,v,s,\widetilde{\omega}))|u_1 - u_2|.$$

Here, $\theta(u_1,u_2,v,s,\widetilde{\omega})$ is a point between $X_s(v,\widetilde{\omega}) - u_1$ and $X_s(v,\widetilde{\omega}) - u_2$. Estimate now the right-hand side of inequality (14) analogously to the proof of Lemma 1.

Without loss of generality, we may assume that $u_1 \le u_2$. Then

$$v - u_2 - M(s,\widetilde{\omega}) \le \theta(u_1,u_2,v,s,\widetilde{\omega}) \le v - u_1 + M(s,\widetilde{\omega})$$

Divide the integral on the right-hand side of Eq. (14) into three parts: by intervals $[u_1 - M(s,\widetilde{\omega}), u_2 + M(s,\widetilde{\omega})]$, $(-\infty, u_1 - M(s,\widetilde{\omega}))$, and $(u_2 + M(s,\widetilde{\omega}), +\infty)$. If $v \le u_1 - M(s,\widetilde{\omega})$, then $|b(X_s(v,\widetilde{\omega}) - u_1) - b(X_s(v,\widetilde{\omega}) - u_2)| \le Q(0)|u_1 - u_2|$. Analogously to the proof of Lemma 2.1, we will use assumption (A1) to estimate the integrand. So, from (14), we get the inequality

$$|\widetilde{b}(s,u_1) - \widetilde{b}(s,u_2)| \le C_\mu(1 + u_2 - u_1 + 2EM(s))Q(0)(u_2 - u_1) + C_Q|u_2 - u_1|. \tag{15}$$

If $|u_1 - u_2| \le 1$, then it follows from (15) that

$$|b(s,u_1) - b(s,u_2)| \le C_5|u_1 - u_2|, \tag{16}$$

where $C_5 = C_Q + (2 + 2EM(T))Q(0)C_\mu$. Thus, inequality (16) holds for all u_1, u_2 such that $|u_1 - u_2| \le 1$. Hence, (16) holds for all u_1, u_2. Therefore, $|X_t(u_1) - X_t(u_2)| \le |u_1 - u_2| + \int_0^t |\widetilde{b}(s, X_s(u_1)) - \widetilde{b}(s, X_s(u_2))|ds \le |u_1 - u_2| + C_5 \int_0^t |X_s(u_1) - X_s(u_2)|ds$. Now, by Lemma 2.2, we obtain $|X_t(u_1) - X_t(u_2)| \le |u_1 - u_2| \exp(C_5 T) = C_4|u_1 - u_2|$, where $C_4 = \exp(C_5 T)$. The lemma is proved.

Lemma 2.4. *Let $\{X_t^n(u,\omega),\ n \ge 1\}$ be solutions of the equations*

$$X_t^n(u,w) = u + w_t(\omega) +$$

$$\int_0^t \int_\Omega \int_{\mathbb{R}} b(X_s^n(v,\widetilde{\omega}) - X_s^n(u,\omega))\mu_n(dv)P(d\widetilde{\omega})ds,\ t \in [0,T], \tag{17}$$

where $\mu_n(dx) = I_{[-n,n]}(x)\mu(dx)$. By λ_n^m, we denote the distribution of the restriction of $X_\cdot^n(\cdot,\omega)$ to $[0,T] \times [-m,m]$. Then, for all $m \in \mathbb{N}$, the set of probability measures $\{\lambda_n^m | n \ge 1\}$ is tight in $C([0,T] \times [-m,m])$.

Remark 2.2. For every n, the measure μ_n is finite. Hence, there exists a unique solution of Eq. (17) (see [12]).

Remark 2.3. For every n, the measure μ_n satisfies assumptions (A1)-(A4) with the same constants as μ. Hence, without loss of generality, we may assume that the constants from Lemmas 2.1 and 2.3 for the measure μ_n are the same as those for the measure μ.

Proof. It is enough to check two conditions (see [14], Theorem 8.2):
(B1) $\forall \varepsilon > 0\ \exists \delta > 0\ \forall n \geq 1\ P(|X_0^n(0)| > \delta) < \varepsilon$,
(B2) $\forall \varepsilon > 0\ \exists \delta \in (0,1)\ \exists n_0\ \forall n \geq n_0\ P(\omega_X^n(\delta) \geq \varepsilon) \leq \varepsilon$, where

$$\omega_X^n(\delta) = \sup_{|t_1 - t_2| \leq \delta, |u_1 - u_2| \leq \delta} |X_{t_1}^n(u_1) - X_{t_2}^n(u_2)|$$

is the module of continuity.

From (17), we have $X_0^n(0) = 0$. Hence, condition (B1) is fulfilled. Let us check that condition (B2) is satisfied. Using Lemma 2.3, we get the inequalities

$$|X_{t_1}^n(u_1, \omega) - X_{t_2}^n(u_2, \omega)| \leq |X_{t_1}^n(u_1, \omega) - X_{t_1}^n(u_2, \omega)| + |X_{t_1}^n(u_2, \omega) - X_{t_2}^n(u_2, \omega)| \leq$$

$$C_4|u_1 - u_2| + |w_{t_1} - w_{t_2}| + \int_{t_1}^{t_2} \int_\Omega \int_\mathbb{R} R(X_s^n(v, \widetilde{\omega}) - X_s^n(u_2, \omega))\mu_n(dv)P(d\widetilde{\omega})ds.$$

From the proof of Lemma 2.1, it follows that

$$\int_\Omega \int_\mathbb{R} R(X_s^n(v, \widetilde{\omega}) - X_s^n(u_2, \omega))\mu_n(dv)P(d\widetilde{\omega}) \leq$$

$$2R(0)C_\mu(EM_n(s) + M_n(s, \omega) + 1) + C_R.$$

Hence,

$$|X_{t_1}^n(u_1, \omega) - X_{t_2}^n(u_2, \omega)| \leq C_4|u_1 - u_2| + |w_{t_1} - w_{t_2}| +$$

$$(C_R + 2R(0)C_\mu(EM(T) + M(t, w)))|t_1 - t_2|.$$

If $|u_1 - u_2| \leq \delta_1 := \varepsilon/(3C_4)$, then $C_4|u_1 - u_2| < \varepsilon/3$. The Wiener process is a continuous stochastic process, hence (see [14], the proof of Theorem 8.2),

$$\exists \delta_2 \in (0,1) : P(\omega_w(\delta_2) \geq \varepsilon/3) \leq \varepsilon/2.$$

From Lemma 2.1, we have

$$\forall n \geq 1 :\ EM^n(t) \leq C(t).$$

So, the Chebyshev inequality yields

$$P(\sup_{|t_1 - t_2| \leq \delta_3} (C_R + 2R(0)C_\mu(EM_n(T) + M_n(T, \omega)))|t_1 - t_2| \geq \varepsilon/3) \leq$$

$$\leq (C_R + 4R(0)C_\mu C_T \delta_3)/(\varepsilon/3) \leq \varepsilon/2$$

for $\delta_3 = \varepsilon^2/(6(C_R + 4R(0)C_\mu C_T))$. Therefore, for $\delta = min(\delta_1, \delta_2, \delta_3)$, we have

$$P(\sup_{|t_1 - t_2| \leq \delta, |u_1 - u_2| \leq \delta} |X_{t_1}^n(u_1, \omega) - X_{t_2}^n(u_2, \omega)| \geq \varepsilon) \leq P(\sup_{|u_1 - u_2| \leq \delta_1} C_4|u_1 - u_2| \geq \varepsilon/3)$$

$$+P(\sup_{|t_1 - t_2| \leq \delta_2} |w_{t_1} - w_{t_2}| \geq \varepsilon/3) + P(\sup_{|t_1 - t_2| \leq \delta_3} (C_R + 2R(0)C_1(EM_n(T) +$$

$$M_n(T, \omega)))|t_1 - t_2| \geq \varepsilon/3) \leq 0 + \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

The lemma is proved.

Lemma 2.5. *Let X_t be a solution of (8), $EM_T < \infty$. Then*

$$\forall \alpha > 0\ Ee^{\alpha M_T} < +\infty.$$

Proof. Analogously to the proof of Lemma 2.1, we get

$$|X_t(u,\omega)-u| \le |w_t| + \int_0^t (C_\mu R(0)(EM_s + M_s(\omega)+1) + C_R)ds \le$$

$$\le \sup_{t\in[0,T]} |w_t| + TC_\mu R(0)(C_T+1) + TC_R + \int_0^T C_\mu R(0) M_s(\omega)ds. \tag{18}$$

After taking supremum by u and t on both sides of the inequality, we get

$$M_t(\omega) \le (\sup_{t\in[0,T]} |w_t| + TC_1R(0)(C(T)+1) + TC_R) + C_\mu R(0)\int_0^t M_s(\omega)ds.$$

Denote $C_6 = TC_\mu R(0)(C(T)+1) + TC_R$. Then, by Lemma 2.2,
$M_t(\omega) \le (\sup_{t\in[0,T]} |w_t| + C_6)e^{C_\mu R(0)t}$. Hence,

$$\forall \alpha > 0 \ Ee^{\alpha M_T} < +\infty.$$

The lemma is proved.

Lemma 2.6. *Let X, Y be two solutions of Eq. (8). Suppose that*

$$\forall \alpha > 0 \ Ee^{\alpha \sup_{t\in[0,T],u\in\mathbb{R}} |X_t(u,\omega)-u|} < +\infty, Ee^{\alpha \sup_{t\in[0,T],u\in\mathbb{R}} |Y_t(u,\omega)-u|} < +\infty.$$

Then $P(\forall t\in[0,T]\ \forall u\in\mathbb{R}:\ X_t(u) = Y_t(u)) = 1$.

Proof. Denote

$$A_t(\omega) = \sup_{s\le t,u\in\mathbb{R}} |X_s(u)-u| + \sup_{s\le t,u\in\mathbb{R}} |Y_s(u)-u|,\ g_t(\omega) = \sup_{s\le t,u\in\mathbb{R}} |X_s(u)-Y_s(u)|.$$

Subtracting Eq. (8) for X from Eq. (8) for Y, analogously to the proof of Lemma 2.1, we obtain

$$|X_t(u,\omega) - Y_t(u,\omega)| \le$$

$$\le \int_0^t \int_\Omega (C_\mu(A_s(\omega) + A_s(\widetilde{\omega}) + 1)Q(0) + C_Q)(g_s(\omega) + g_s(\widetilde{\omega}))P(d\widetilde{\omega})ds. \tag{19}$$

Denote $C_7 = max(C_\mu, Q(0), C_\mu Q(0), C_Q)$. After taking supremum by $u\in\mathbb{R},\ s\in[0,t]$ on both sides of inequality (19), we get

$$g_t(\omega) \le C_7 \int_0^t \int_\Omega (A_s(\omega)g_s(\omega) + A_s(\omega)g_s(\widetilde{\omega}) + A_s(\widetilde{\omega})g_s(\omega) + A_s(\widetilde{\omega})g_s(\omega) +$$
$$g_s(\widetilde{\omega}) + g_s(\omega))P(d\widetilde{\omega})ds.$$

Denote $B_s(\omega) = A_s(\omega)+1$. Then the last inequality yields

$$g_t \le C_7 \int_0^t (B_sg_s + B_sEg_s + g_sEB_s + E(g_sB_s))ds.$$

Multiplying both sides by B_t and additionally multiplying some terms on the right-hand side by B_s(notice that $B_s \ge 1$), we obtain

$$B_tg_t \le C_7B_t \int_0^t (B_sg_s + B_sE(g_sB_s) + B_sg_sEB_s + EB_sg_s)ds$$

Denote $\xi_t = B_tg_t$. Then

$$\xi_t \le C_8B_t \int_0^t (B_sE\xi_s + \xi_sEB_s)ds.$$

Applying Lemma 2.2 to the function ξ_t, we get

$$\xi_t \le C_8B_t \int_0^t B_sE\xi_s ds \exp{(C_8B_t \int_0^t EB_sds)} \le$$

$$\leq C_8 B_t \int_0^t B_s E\xi_s ds \exp\left(C_8 B_t T(C_T + 1)\right).$$

After taking expectation on both sides of the inequality and applying Gronwall's lemma to the function $f(t) = E\xi_t$, we get $E\xi_t \leq 0$. The lemma is proved.

Lemma 2.7. *There exists a solution of Eq. (8) such that $EM_T < +\infty$.*

Proof. First, let us prove the weak existence. It follows from Lemma 2.4 that there exists a subsequence $X_t^{n_1(k)}(u,\omega)$ that is weakly convergent in $C([-1,1]\times[0,T])$. Similarly, there exists a subsubsequence $X_t^{n_2(k)}(u,\omega)$ that is weakly convergent in $C([-2,2]\times[0,T])$, and so on. Using Cantor's diagonal method, we choose a subsequence $X_t^{n(k)}(u,\omega)$ that is weakly convergent on $[-m,m]\times[0,T]$ for every $m \geq 1$, i.e., $X_t^{n(k)}(u,\omega)$ is convergent in the topology of uniform convergence on compact sets. By Skorokhod's representation theorem (see [15], Section 1.6), there exists a probability space and random elements $(w_t^{n(k)}, X_t^{n(k)})$, such that the pair $(w_\cdot,\ X_t^{n(k)}(u))$ is convergent almost surely to $(w_\cdot, X_t(u))$. To simplify the notation, we denote the obtained sequence by $X_t^n(u)$. We verify now that $X_t(u)$ is a solution of (8).

Fatou's lemma implies that, for any $m > 0$,

$$E \sup_{u\in[-m,m]} |X_t(u,\omega) - u| \leq \liminf_{n\to\infty} E \sup_{u\in[-m,m]} |X_t^n(u,\omega) - u| \leq C(t).$$

By B. Levi's theorem, we have

$$E \sup_{u\in\mathbb{R}} |X_t(u,\omega) - u| = \lim_{n\to\infty} E \sup_{u\in[-m,m]} |X_t(u,\omega) - u| \leq C(t).$$

Denote

$$M_t^n(\omega) = \sup_{u\in\mathbb{R}, s\in[0,t]} |X_s^n(u,\omega) - u|,$$

$$A^n = A_T^n(\omega,\widetilde{\omega}) = M_T(\omega) + M_T(\widetilde{\omega}) + M_T^n(\omega) + M_T^n(\widetilde{\omega}). \tag{20}$$

We now prove that

$$\int_0^t \int_\Omega \int_\mathbb{R} b(X_s(v,\widetilde{\omega}) - X_s(u,\omega))\mu(dv)P(d\widetilde{\omega})ds-$$

$$\int_0^t \int_\Omega \int_\mathbb{R} b(X_s^n(v,\widetilde{\omega}) - X_s^n(u,\omega))\mu_n(dv)P(d\widetilde{\omega})ds \to 0,\ n\to\infty. \tag{21}$$

Using the definition of μ_n, we obtain that the left-hand side of (21) is equal to

$$\int_0^t \int_\Omega \int_\mathbb{R} b(X_s(v,\widetilde{\omega}) - X_s(u,\omega))I_{|v>n|}\mu(dv)P(d\widetilde{\omega})ds+$$

$$\int_0^t \int_\Omega \int_{[-n,n]} (b(X_s(v,\widetilde{\omega}) - X_s(u,\omega)) - b(X_s^n(v,\widetilde{\omega}) - X_s^n(u,\omega)))\mu(dv)P(d\widetilde{\omega})ds. \tag{22}$$

It follows from the proof of Lemma 2.1 that

$$\int_0^t \int_\Omega \int_\mathbb{R} |b(X_s(v,\widetilde{\omega}) - X_s(u,\omega))|\mu(dv)P(d\widetilde{\omega})ds < +\infty.$$

Hence, the first integral in (22) tends to 0 as $n\to\infty$.

Divide the integral in Eq. (22) into three parts. Let I_1^n be the integral by $[(-n)\wedge(u-a), u-a)$, let I_2^n be the integral by $[u-a, u+a]$, and let I_3^n be the integral by $(u+a, (u+a)\vee n]$. The constant $2R(0)$ is an integrable majorant for I_2^n; hence, $I_2^n \to 0,\ n\to\infty$. In I_3^n, the argument may be estimated as $|b(X_s(v,\widetilde{\omega}) - X_s(u,\omega)) - b(X_s^n(v,\widetilde{\omega}) - X_s^n(u,\omega))| \leq$

$2R(0)$ for $v < u + A^n$, and $|b(X_s(v,\widetilde{\omega}) - X_s(u,\omega)) - b(X_s^n(v,\widetilde{\omega}) - X_s^n(u,\omega))| \le 2R(v - u - A^n)$ for $v \ge u + A^n$, where A^n is defined in (20). Suppose that $0 \le \alpha \le a$. Then

$$\int_\Omega \int_{[u+a,(u+a)\vee n]} |b(X_s(v,\widetilde{\omega}) - X_s(u,\omega)) - b(X_s^n(v,\widetilde{\omega}) - X_s^n(u,\omega))|\mu(dv)P(d\widetilde{\omega}) \le$$

$$\int_\Omega (I_{A^n\le\alpha} \int_{(u+a,(u+a)\vee n]} 2R(v-u-A^n)\mu(dv) + I_{A^n>\alpha}(C_\mu R(0)(A^n-a+1)+C_R))P(d\widetilde{\omega})$$

$$\le \int_\Omega \int_{(u+a,(u+a)\vee n))} 2R(v-u-\alpha)\mu(dv)P(d\widetilde{\omega}) +$$

$$\int_\Omega I_{(M_T^n(\widetilde{\omega})+M_T(\widetilde{\omega})\ge\alpha-M_T^n(\omega)-M_T(\omega))} C_\mu(M_T^n(\widetilde{\omega}) + M_T(\widetilde{\omega}) + M_T^n(\omega) - M_T(\omega)))P(d\widetilde{\omega}).$$

By Lebesgue's dominated convergence theorem, the second term tends to 0 as $\alpha \to \infty$ for every ω. We now choose $\alpha = \alpha(\omega)$ so that the second term is less than $\varepsilon/4$. Then we select $a = a(\omega)$ so that the first term is less than $\varepsilon/4$. Estimating I_1, we obtain that, for every ω $\limsup_{n\to\infty} I_1^n + I_3^n \le \varepsilon$, $\lim_{n\to\infty} I_2^n = 0$. Hence, $\limsup_{n\to\infty} I_1^n + I_2^n + I_3^n \le \varepsilon$. Turning ε to 0, we obtain $\lim_{n\to\infty} I_1^n + I_2^n + I_3^n = 0$. So, $X_t(u,\omega)$ is a solution of Eq. (8). Lemma 5 and the Yamada–Watanabe theorem imply that there exists a strong solution of Eq. (8). The lemma is proved.

Theorem 2.1 follows from the proved lemmas.

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