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THE DISTRIBUTION OF RANDOM MOTION IN SEMI-MARKOV MEDIA

This paper deals with the random motion with finite speed along uniformly distributed directions, where the direction alternations occur according to renewal epochs of a general distribution. We derive a renewal equation for the characteristic function of a transition density of multidimensional motion. By using the renewal equation, we study the behavior of the transition density near the sphere of its singularity in two- and three-dimensional cases. For $(n-1)$-Erlang distributed steps of the motion in an n-dimensional space $(n \geq 2)$, we have obtained the characteristic function as a solution of the renewal equation. As an example, we have derived the distribution for the three-dimensional random motion.

1. Introduction

Most of the papers on the random motion with uniformly distributed directions in a multidimensional space are devoted to the analysis of models, in which motions are driven by a homogeneous Poisson process, so their processes are Markovian [1], [2], and so on. Papers [3]-[6] considered a non-Markovian generalization of one-dimensional random evolutions of the telegrapher's random process, where the motion is driven by an alternating semi-Markov process with Erlang distributed interrenewal times. Random flights in R^n with K-Erlang distributed displacements and uniformly distributed directions have been studied in [7]. A planar random motion performed by a particle, which changes its direction at even-valued Poisson events is studied in [8]. Papers [9] and [10] analyzed a random walk with steps of uniform orientation and Dirichlet-distributed lengths. The transition densities which have simple analytical forms for two- and four-dimensional Markovian random motions were derived in [1] and [2].

In the present work, we consider multidimensional random motions with uniformly distributed directions with general distributed steps, by extending some results of [1], [2], and [7].

Let us consider the renewal process $\nu(t) = \max\{m \geq 0 : \ \tau_m \leq t\}$, $t \geq 0$, where $\tau_m = \sum_{k=0}^{m} \theta_k$, $\tau_0 = 0$, and $\theta_k \geq 0$, $k = 1, 2, \ldots$, are i.i.d. with a distribution function $G(t)$ and the probability density function (pdf) $g(t) = \frac{d}{dt} G(t)$.

We assume that a particle starting from the coordinate origin $(0, 0, \ldots, 0)$ of the space R^n at time $t = 0$ continues its motion with a constant velocity $v > 0$ along the direction of $\overrightarrow{\eta}_0^{(n)}$, where $n \geq 2$, $\overrightarrow{\eta}_0^{(n)} = (x_1, x_2, \cdots, x_n)$ is a random n-dimensional vector uniformly distributed on the unit sphere $\Omega_1^{n-1} = \left\{(x_1, x_2, \cdots, x_n) : \ x_1^2 + x_2^2 + \cdots + x_n^2 = 1\right\}$.

At the instant τ_1, the particle changes its direction to $\overrightarrow{\eta}_1^{(n)}$, where $\overrightarrow{\eta}_1^{(n)}$ and $\overrightarrow{\eta}_0^{(n)}$ are independent and identically distributed on Ω_1^{n-1}, and continues its motion with a velocity v along the direction of $\overrightarrow{\eta}_1^{(n)}$. Then at the instant τ_2, the particle changes its direction to $\overrightarrow{\eta}_2^{(n)}$, where $\overrightarrow{\eta}_2^{(n)}$ is also uniformly distributed on Ω_1^{n-1} and independent of

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$\overrightarrow{\eta}_0^{(n)}$, $\overrightarrow{\eta}_1^{(n)}$, and continues its motion with a velocity v along the direction of $\overrightarrow{\eta}_2^{(n)}$, and so on.

By $\overrightarrow{x}^{(n)}(t)$, $t \geq 0$, we denote the particle position at the time t. We have

$$\overrightarrow{x}^{(n)}(t) = v \sum_{j=1}^{\nu(t)} \overrightarrow{\eta}_{j-1}^{(n)} (\tau_j - \tau_{j-1}) + v \overrightarrow{\eta}_{\nu(t)}^{(n)} \left(t - \tau_{\nu(t)}\right). \tag{1}$$

Here and in the sequel, we assume that $\sum_{j=1}^{0} = 0$.

Basically, this equation determines the random evolution in a semi-Markov medium $\nu(t)$. It is easily seen that $\nu(t)$ is the number of velocity alternations occurred in the interval $(0, t)$.

The probabilistic properties of a random vector $\overrightarrow{x}^{(n)}(t)$ are completely determined by those of its projection $x^{(n)}(t) = v \sum_{j=1}^{\nu(t)} \eta_{j-1}^{(n)} (\tau_j - \tau_{j-1}) + v\eta_{\nu(t)}^{(n)} \left(t - \tau_{\xi(t)}\right)$ on a fixed line, where $\eta_j^{(n)}$ is the projection of $\overrightarrow{\eta}_j^{(n)}$ on the line.

Indeed, let us consider the distribution function $F_x(y) = P\left(x^{(n)}(t) \leq y\right)$. Then the characteristic function $H(t)$ of $\overrightarrow{x}^{(n)}(t)$ is given by

$$\begin{aligned} H(t) &= E\exp\left\{i\left(\overrightarrow{\alpha}, \overrightarrow{x}^{(n)}(t)\right)\right\} = E\exp\left\{i\,\|\overrightarrow{\alpha}\|\left(\overrightarrow{e}, \overrightarrow{x}^{(n)}(t)\right)\right\} \\ &= E\exp\left\{i\,\|\overrightarrow{\alpha}\|\, x^{(n)}(t)\right\} = \int_0^\infty \exp\{i\,\|\overrightarrow{\alpha}\|\, y\}\, dF_x(y), \end{aligned}$$

where $\|\overrightarrow{\alpha}\| = \sqrt{\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2}$, $\overrightarrow{e} = \frac{\overrightarrow{\alpha}}{\|\overrightarrow{\alpha}\|}$.

By $f_{\eta^{(n)}}(x)$, we denote the pdf of the projection $\eta_j^{(n)}$ of the vector $\overrightarrow{\eta}_j^{(n)}$ onto a fixed line. In [5], we proved that

$$f_{\eta^{(n)}}(x) = \begin{cases} \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)}\left(1 - x^2\right)^{(n-3)/2}, & x \in [-1, 1]; \\ 0, & x \notin [-1, 1]. \end{cases} \tag{2}$$

By $\varphi_{\eta^{(n)}}(t) = Ee^{-it\eta^{(n)}} = \int_{-\infty}^{\infty} e^{-itx}\, f_{\eta^{(n)}}(x)dx$, we denote the characteristic function of $\eta_j^{(n)}$. We note that the function $\varphi(t) = \varphi_{\eta^{(n)}}(\alpha t v)$, where $\alpha = \|\overrightarrow{\alpha}\|$, is also used in [2], where it was obtained by different methods. It is well known [2], [5] that

$$\varphi(t) = 2^{\frac{n-2}{2}} \Gamma\left(\frac{n}{2}\right) \frac{J_{\frac{n-2}{2}}(\alpha t v)}{(\alpha t v)^{\frac{n-2}{2}}}.$$

It is easily seen that $\varphi(t) = \varphi_{\eta^{(n)}}(\alpha t v) = Ee^{-itv\left(\overrightarrow{\alpha}, \overrightarrow{\eta}_j^{(n)}\right)} = \int_{-\infty}^{\infty} e^{-i\alpha t v x}\, f_{\eta^{(n)}}(x)dx$.

2. Renewal Equation for the Characteristic Function

The characteristic function of a random motion $\overrightarrow{x}^{(n)}(t)$ is given by

$$H(t) = \exp\left\{i\left(\overrightarrow{\alpha}, \overrightarrow{x}^{(n)}(t)\right)\right\}.$$

Theorem 2.1. *The characteristic function $H(t)$, $t \geq 0$, is a solution of the Volterra integral equation*

$$H(t) = (1 - G(t))\,\varphi(t) + \int_0^t g(u)\,\varphi(u)\,H(t-u)\,du. \tag{3}$$

Proof. It follows from Eq. (1) that

$$
\begin{aligned}
H(t) &= E\exp\left\{ i\left(\overrightarrow{\alpha}, \overrightarrow{x}^{(n)}(t)\right)\right\} \\
&= E\exp\left\{ i\left(\overrightarrow{\alpha}, v\sum_{j=1}^{\xi(t)} \overrightarrow{\eta}_{j-1}^{(n)}\theta_j + v\overrightarrow{\eta}_{\xi(t)}^{(n)}\left(t-\tau_{\xi(t)}\right)\right)\right\} \\
&= E\exp\left[I_{[\tau_1>t]}e^{itv\left(\overrightarrow{\alpha}, \overrightarrow{\eta}_0^{(n)}\right)}\right] + \int_0^t E\left(I_{[\tau_1\in du]}e^{iuv\left(\overrightarrow{\alpha}, \overrightarrow{\eta}_0^{(n)}\right)}\right)H(t-u) \\
&= (1-G(t))\,Ee^{itv\left(\overrightarrow{\alpha}, \overrightarrow{\eta}_0^{(n)}\right)} + \int_0^t g(u)\,Ee^{iuv\left(\overrightarrow{\alpha}, \overrightarrow{\eta}_0^{(n)}\right)}H(t-u)\,du.
\end{aligned}
$$

To complete the proof, we observe that $\varphi(t) = Ee^{iv\left(\overrightarrow{\alpha}, \overrightarrow{\eta}_0^{(n)}\right)}$.
It is worth noting that this theorem was proved in [7] for the Erlang case.

Passing to the Laplace transform $\hat{H}(s) = \mathcal{L}(H(t)) = \int_0^\infty H(t)\,e^{-st}dt$ in Eq.(3), we get

$$
\hat{H}(s) = \frac{\int_0^\infty (1-G(t))\,\varphi(t)\,e^{-st}dt}{1-\int_0^\infty g(t)\,\varphi(t)\,e^{-st}dt}. \tag{4}
$$

By $f_n(t, \overrightarrow{x})$, we denote the pdf of particles position at the time t. It is easily seen that $f_n(t, \overrightarrow{x}) = \mathcal{F}^{-1}(H(t))$.
Our purpose is to study $f_n(t, \overrightarrow{x})$.
We now introduce the function

$$
H_{n-2}(t) = e^{-\lambda t}\sum_{j=0}^{\infty} \frac{(\lambda t)^{n-2+(n-1)j}}{(n-2+(n-1)j)!} \times \frac{2^{\frac{n-2+(n-1)j}{2}}\Gamma\left(\frac{n-2+(n-1)j}{2}+1\right)}{(vt\alpha)^{\frac{n-2+(n-1)j}{2}}} J_{\frac{n-2+(n-1)j}{2}}(vt\alpha).
$$

The following theorem generalizes the result of [7] (see Section 3) for any $n \geq 2$.

Theorem 2.2. *Suppose* $g(t) = e^{-\lambda t}\frac{\lambda^{n-1}t^{n-2}}{(n-2)!}I_{\{t\geq 0\}}$, $n \geq 2$, *i.e.* θ_k *is* $(n-1)$*-Erlang distributed. Then*

$$
\begin{aligned}
H_{n-2}(t) &= e^{-\lambda t}\frac{(\lambda t)^{n-2}}{(n-2)!}\frac{2^{n-2/2}\Gamma((n-2)/2+1)}{(vt\alpha)^{(n-2)/2}} J_{\frac{n-2}{2}}(vt\alpha) \\
&+ \int_0^t g(u)\,\varphi(u)\,H_{n-2}(t-u)\,du.
\end{aligned} \tag{5}
$$

Proof. In what follows, we use the equation (see [13], Formula 6.581(3))

$$
\int_0^t u^{\mu}\, J_\mu(u)\,(t-u)^{\nu}J_\nu(t-u)\,du = \frac{\Gamma\left(\mu+\frac{1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{2\pi}\,\Gamma(\mu+\nu+1)}t^{\mu+\nu+\frac{1}{2}}J_{\mu+\nu+\frac{1}{2}}(t), \quad \mu > -\frac{1}{2},\ \nu > -\frac{1}{2}. \tag{6}
$$

It is easily verified that

$$
\frac{2^{\nu+\mu}}{\sqrt{\pi}}\Gamma\left(\frac{\nu+\mu+1}{2}\right)\Gamma\left(\frac{\nu+\mu}{2}\right) = \Gamma(\nu+\mu). \tag{7}
$$

Let us fix an integer $r \geq 1$. Combining Eqs. (6) and (7), for $j = 1, 2, \ldots,$ we obtain

$$
\begin{aligned}
&\int_0^t g(u)\varphi(u)\frac{e^{-\lambda(t-u)}\lambda^r}{r!}\frac{(2(t-u))^{\frac{r}{2}}\Gamma\left(\frac{r}{2}+1\right)}{(v\alpha)^{\frac{r}{2}}}J_{\frac{r}{2}}(v(t-u)\alpha)\,du\\
=\;&\frac{e^{-\lambda t}\left(\sqrt{2}\lambda\right)^{n+r-1}\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{r}{2}+1\right)}{\sqrt{2}(\alpha v)^{\frac{n+r-2}{2}}(n-2)!r!}\int_0^t u^{\frac{n-2}{2}}\;J_{\frac{n-2}{2}}(vu\alpha)(t-u)^{\frac{r}{2}}J_{\frac{r}{2}}(v(t-u)\alpha)\,du\\
=\;&\frac{e^{-\lambda t}\left(\sqrt{2}\lambda\right)^{n+r-1}\Gamma\left(\frac{r+1}{2}\right)\Gamma\left(\frac{r}{2}+1\right)}{(\alpha v)^{\frac{n+r-2}{2}}(n-2)!r!}\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}{2\sqrt{\pi}\Gamma\left(\frac{n+r}{2}\right)}t^{\frac{n+r-1}{2}}J_{\frac{n+r-1}{2}}(t)\\
=\;&\frac{e^{-\lambda t}\left(\sqrt{2}\lambda\right)^{n+r-1}\sqrt{\pi}}{(\alpha v)^{\frac{n+r-2}{2}}2^r}\frac{t^{\frac{n+r-1}{2}}\Gamma\left(\frac{n+r-1}{2}+1\right)}{2^n\Gamma\left(\frac{n+r}{2}\right)\Gamma\left(\frac{n+r-1}{2}+1\right)}J_{\frac{n+r-1}{2}}(t)\\
=\;&\frac{e^{-\lambda t}\left(\sqrt{2}\lambda\right)^{n+r-1}}{(\alpha v)^{\frac{n+r-2}{2}}}\frac{t^{\frac{n+r-1}{2}}\Gamma\left(\frac{n+r-1}{2}+1\right)}{\Gamma(n+r)}J_{\frac{n+r-1}{2}}(t).
\end{aligned}
$$

By putting $r=n-2+(n-1)j$, we conclude the proof.

Taking Eq. (3) into account, we now solve the equation

$$
\begin{aligned}
H(t)=&\sum_{i=0}^{n-2}e^{-\lambda t}\frac{(\lambda t)^i}{i!}\frac{2^{n-2/2}\Gamma((n-2)/2+1)}{(vt\alpha)^{(n-2)/2}}J_{\frac{n-2}{2}}(vt\alpha)\\
&+\int_0^t g(u)\varphi(u)H(t-u)\,du. \qquad (8)
\end{aligned}
$$

By $H^{(k)}(t)$, $k=0,1,\dots,n-2$, we denote solutions of the equation

$$
\begin{aligned}
H^{(k)}(t)=&e^{-\lambda t}\frac{\lambda^k t^k}{k!}\frac{2^{n-2/2}\Gamma((n-2)/2+1)}{(vt\alpha)^{(n-2)/2}}J_{\frac{n-2}{2}}(vt\alpha)\\
&+\int_0^t g(u)\varphi(u)H^{(k)}(t-u)\,du. \qquad (9)
\end{aligned}
$$

It is easily seen that $H(t)=\sum_{k=0}^{n-2}H^{(k)}(t)$ is the solution of Eq. (8).

Lemma 2.1. *For each $k=0,1,\dots,n-2$, the following equations hold:*

$$H^{(k)}(t)=e^{-\lambda t}\frac{\lambda^k t^k}{k!}\varphi(t)+\lambda\int_0^t e^{-\lambda u}\frac{\lambda^k u^k}{k!}\varphi(u)H_{n-2}(t-u)\,du.$$

Proof. Denote $g_k(t)=e^{-\lambda t}\frac{\lambda^k t^k}{k!}$. Performing the Laplace transformation $\hat{H}_{n-2}(s)=\int_0^\infty H_{n-2}(t)e^{-st}dt$ in Eq.(5) and $\hat{H}^{(k)}(s)=\int_0^\infty H^{(k)}(t)e^{-st}dt$ in Eq.(9), we get, respectively,

$$\hat{H}_{n-2}(s)=\frac{1/\lambda\int_0^\infty g_k(t)\varphi(t)e^{-st}dt}{1-\int_0^\infty g_k(t)\varphi(t)e^{-st}dt}=1/\lambda\sum_{j=1}^{\infty}\left(\int_0^\infty g_k(t)\varphi(t)e^{-st}dt\right)^j \qquad (10)$$

and

$$
\begin{aligned}
\hat{H}^{(k)}(s)\;=\;&\int_0^\infty g_k(t)\varphi(t)e^{-st}dt\\
+\;&\int_0^\infty g_k(t)\varphi(t)e^{-st}dt\sum_{j=1}^{\infty}\left(\int_0^\infty g_k(t)\varphi(t)e^{-st}dt\right)^j. \qquad (11)
\end{aligned}
$$

The inverse Laplace transformation in Eqs. (10) and (11) concludes the proof.

Let us calculate $\mathcal{F}^{-1}\left(H_{n-2}\left(t\right)\right)$, where $\mathcal{F}^{-1}$ is the n-dimensional inverse Fourier transform $\mathcal{F}^{-1}$ w.r.t. $\overrightarrow{\alpha}$. Now, we need the following integral (see [12], page 69):

$$\int_0^{\infty} J_{\nu+1}\left(az\right) J_{\mu}\left(bz\right) z^{\mu-\nu} dz = \frac{\left(a^2-b^2\right)^{\nu-\mu} b^{\mu}}{2^{\nu-\mu} a^{\nu+1} \Gamma\left(\nu-\mu+1\right)}, \quad \nu+1>\mu>0.$$

By reducing the n-dimensional inverse Fourier transformation to the Hankel one, we obtain, for $|x|<vt$, $j\geq 1$,

$$\begin{aligned}
\mathcal{F}^{-1}\left(\frac{J_{\frac{n-2+(n-1)j}{2}}\left(vt\alpha\right)}{\left(vt\alpha\right)^{\frac{n-2+(n-1)j}{2}}}\right) &= \frac{1}{\left(2\pi\right)^{\frac{n}{2}}}\int_0^{\infty}\frac{J_{\frac{n-2+(n-1)j}{2}}\left(vt\alpha\right)}{\left(vt\alpha\right)^{\frac{n-2+(n-1)j}{2}}}\alpha^{n-1}\frac{J_{\frac{n-2}{2}}\left(\|\overrightarrow{x}\|\,\alpha\right)}{\left(\|\overrightarrow{x}\|\,\alpha\right)^{\frac{n-2}{2}}}d\alpha \\
&= \frac{\left(\frac{v^2t^2-\|\overrightarrow{x}\|^2}{2}\right)^{\frac{(n-1)j}{2}-1}}{\left(2\pi\right)^{\frac{n}{2}}\left(vt\right)^{n-2+(n-1)j}\Gamma\left(\frac{(n-1)j}{2}\right)}.
\end{aligned}$$

It was obtained in [2] that $\mathcal{F}^{-1}\left(\frac{J_{\frac{n-2}{2}}(vt\alpha)}{(vt\alpha)^{\frac{n-2}{2}}}\right)=\frac{\delta\left(v^2t^2-\|\overrightarrow{x}\|^2\right)}{(2\pi)^{n/2}(vt)^{n-1}}$.

Then, by using Eq. (6), we have

$$\begin{aligned}
\mathcal{F}^{-1}\ \left(H_{n-2}\left(t\right)\right) &= e^{-\lambda t}\sum_{j=0}^{\infty}\frac{\left(\lambda t\sqrt{2}\right)^{n-2+(n-1)j}\Gamma\left(\frac{n-2+(n-1)j}{2}+1\right)}{\left(n-2+\left(n-1\right)j\right)!} \\
&\quad\times\mathcal{F}^{-1}\left(\frac{J_{\frac{n-2+(n-1)j}{2}}\left(vt\alpha\right)}{\left(vt\alpha\right)^{\frac{n-2+(n-1)j}{2}}}\right) \\
&= \frac{\left(\lambda t\right)^{n-2}e^{-\lambda t}\Gamma\left(\frac{n}{2}\right)}{2\left(n-2\right)\pi^{\frac{n}{2}}\left(vt\right)^{n-1}}\delta\left(v^2t^2-\|\overrightarrow{x}\|^2\right) \\
&+ e^{-\lambda t}\sum_{j=1}^{\infty}\frac{\left(\lambda t\sqrt{2}\right)^{n-2+(n-1)j}\Gamma\left(\frac{n-2+(n-1)j}{2}+1\right)}{\left(n-2+\left(n-1\right)j\right)!} \\
&\quad\times\mathcal{F}^{-1}\left(\frac{J_{\frac{n-2+(n-1)j}{2}}\left(vt\alpha\right)}{\left(vt\alpha\right)^{\frac{n-2+(n-1)j}{2}}}\right) \\
&= e^{-\lambda t}\frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{\frac{n}{2}}\left(vt\right)^{n-1}}\delta\left(v^2t^2-\|\overrightarrow{x}\|^2\right) \\
&+ e^{-\lambda t}\sum_{j=1}^{\infty}\frac{\left(\lambda t\sqrt{2}\right)^{n-2+(n-1)j}\Gamma\left(\frac{n-2+(n-1)j}{2}+1\right)}{\left(n-2+\left(n-1\right)j\right)!\Gamma\left(\frac{(n-1)j}{2}\right)} \\
&\quad\times\frac{\left(v^2t^2-x^2\right)^{\frac{(n-1)j}{2}-1}}{\left(2\pi\right)^{\frac{n}{2}}\left(vt\right)^{n-2+(n-1)j}} \\
&= e^{-\lambda t}\frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{\frac{n}{2}}\left(vt\right)^{n-1}}\delta\left(v^2t^2-\|\overrightarrow{x}\|^2\right) \\
&+ e^{-\lambda t}\sum_{j=1}^{\infty}\frac{\left(\lambda t\right)^{n-2+(n-1)j}}{2\Gamma\left(\frac{(n-1)j}{2}\right)\Gamma\left(\frac{n-1+(n-1)j}{2}\right)}\frac{\left(v^2t^2-x^2\right)^{\frac{(n-1)j}{2}-1}}{\left(2\pi\right)^{\frac{n-1}{2}}\left(vt\right)^{n-2+(n-1)j}}.
\end{aligned}$$

By using Lemma 2.1, we can calculate $H^{(k)}\left(t\right)$, $k=0,1,\ldots,n-2$.

Then, passing to the inverse Fourier transformation, we obtain

$$f\left(t, \overrightarrow{x}\right) = \sum_{k=0}^{n-2} \mathcal{F}^{-1}\left(H^{(k)}\left(t\right)\right).$$

Example 2.1. Let us consider the three-dimensional case and 2-Erlang distributed $g\left(t\right)$, i.e. $n = 3$ and $g\left(t\right) = \lambda^2 t e^{-\lambda t}$, $\lambda > 0$. In this case, we obtain

$$\mathcal{F}^{-1}\left(H^{(1)}\left(t\right)\right) = \frac{e^{-\lambda t}}{4\pi v^2 t}\delta\left(v^2t^2 - \|\overrightarrow{x}\|^2\right) + e^{-\lambda t}\sum_{j=1}^{\infty}\frac{(\lambda t)^{1+2j}}{\Gamma\left(2+2j\right)}\frac{\sqrt{8}\Gamma\left(\frac{1+2j}{2}+1\right)}{\Gamma\left(j\right)(vt)^{2j+1}}\left(v^2t^2 - \|\overrightarrow{x}\|^2\right)^{j-1}.$$

For the second term, we get

$$\begin{aligned}
&\sum_{j=1}^{\infty}\frac{(\lambda t)^{2j+1}}{\Gamma\left(2+2j\right)}\frac{\sqrt{8}\Gamma\left(\frac{1+2j}{2}+1\right)}{\Gamma\left(j\right)(vt)^{2j+1}}\left(v^2t^2 - \|\overrightarrow{x}\|^2\right)^{j-1} \\
= \;& \sum_{j=1}^{\infty}\frac{\lambda^{2j+1}\sqrt{2\pi}}{2^j\Gamma\left(j\right)\Gamma\left(j+1\right)v^{2j+1}}\left(v^2t^2 - \|\overrightarrow{x}\|^2\right)^{j-1} \\
= \;& \frac{\sqrt{\pi}(\lambda/v)^2}{\sqrt{\left(v^2t^2 - \|\overrightarrow{x}\|^2\right)}}\sum_{m=0}^{\infty}\frac{1}{\Gamma\left(m+1\right)\Gamma\left(m+2\right)}\left(\frac{\frac{\lambda}{v}\sqrt{2\left(v^2t^2 - \|\overrightarrow{x}\|^2\right)}}{2}\right)^{2m+1} \\
= \;& \frac{\sqrt{\pi}(\lambda/v)^2}{\sqrt{\left(v^2t^2 - \|\overrightarrow{x}\|^2\right)}}I_1\left(\frac{\lambda}{v}\sqrt{2\left(v^2t^2 - \|\overrightarrow{x}\|^2\right)}\right),
\end{aligned}$$

where I_1 is the modified Bessel function of the first kind.

Therefore,

$$\mathcal{F}^{-1}\left(H^{(1)}\left(t\right)\right) = \frac{e^{-\lambda t}}{4\pi v^2 t}\delta\left(v^2t^2 - \|\overrightarrow{x}\|^2\right) + e^{-\lambda t}\frac{\sqrt{\pi}(\lambda/v)^2}{\sqrt{\left(v^2t^2 - \|\overrightarrow{x}\|^2\right)}}I_1\left(\frac{\lambda}{v}\sqrt{2\left(v^2t^2 - \|\overrightarrow{x}\|^2\right)}\right).$$

It follows from Lemma 2.1 that

$$H^{(0)}\left(t\right) = e^{-\lambda t}\frac{\sin\left(vt\alpha\right)}{vt\alpha} + \lambda\left(e^{-\lambda t}\frac{\sin\left(vt\alpha\right)}{vt\alpha}\right) * H_1\left(t\right).$$

Passing to the inverse Fourier transformation, we get

$$\begin{aligned}
\mathcal{F}^{-1}\left(H_0\left(t\right)\right) = \;& \frac{e^{-\lambda t}}{4\pi v^2 t^2}\delta\left(v^2t^2 - \|\overrightarrow{x}\|^2\right) \\
&+ \frac{e^{-\lambda t}}{16\pi^2 v^4}\int_{\|\overrightarrow{u}\|\le tv}\int_0^t \frac{\delta\left(v^2s^2 - \|\overrightarrow{u}\|^2\right)}{s^2} \\
&\times \frac{\delta\left(v^2(t-s)^2 - \|\overrightarrow{x} - \overrightarrow{u}\|^2\right)}{(t-s)}dsd\overrightarrow{u} \\
&+ \frac{e^{-\lambda t}\lambda^3}{4\sqrt{\pi}v^2}\int_{\|\overrightarrow{u}\|\le tv}\frac{I_1\left(\frac{\lambda}{v}\sqrt{2\left(\left(tv - \|\overrightarrow{u}\|\right)^2 - \|\overrightarrow{x} - \overrightarrow{u}\|^2\right)}\right)}{\|\overrightarrow{u}\|^2\sqrt{\left(\left(tv - \|\overrightarrow{u}\|\right)^2 - \|\overrightarrow{x} - \overrightarrow{u}\|^2\right)}}d\overrightarrow{u}.
\end{aligned}$$

Therefore,

$$\begin{aligned} f_3(t,\overrightarrow{x}) &= \mathcal{F}^{-1}\left(H^{(0)}(t)\right)+\mathcal{F}^{-1}\left(H^{(1)}(t)\right)=\frac{e^{-\lambda t}+te^{-\lambda t}}{4\pi(vt)^2}\delta\left(v^2t^2-\|\overrightarrow{x}\|^2\right)\\ &+ \frac{e^{-\lambda t}}{16\pi^2v^4}\int_{\|\overrightarrow{u}\|\le tv}\int_0^t\frac{\delta\left(v^2s^2-\|\overrightarrow{u}\|^2\right)}{s^2}\frac{\delta\left(v^2(t-s)^2-\|\overrightarrow{x}-\overrightarrow{u}\|^2\right)}{(t-s)}dsd\overrightarrow{u}\\ &+ \frac{e^{-\lambda t}\lambda^3}{4\sqrt{\pi}v^2}\int_{\|\overrightarrow{u}\|\le tv}\frac{I_1\left(\frac{\lambda}{v}\sqrt{2\left((tv-\|\overrightarrow{u}\|)^2-\|\overrightarrow{x}-\overrightarrow{u}\|^2\right)}\right)}{\|\overrightarrow{u}\|^2\sqrt{\left((tv-\|\overrightarrow{u}\|)^2-\|\overrightarrow{x}-\overrightarrow{u}\|^2\right)}}d\overrightarrow{u}\\ &+ e^{-\lambda t}\frac{\sqrt{\pi}(\lambda/v)^2}{\sqrt{\left(v^2t^2-\|\overrightarrow{x}\|^2\right)}}I_1\left(\frac{\lambda}{v}\sqrt{2\left(v^2t^2-\|\overrightarrow{x}\|^2\right)}\right). \end{aligned}$$

As we showed in [7], $f_3(t,\overrightarrow{x})\uparrow\infty$ *as* $\|\overrightarrow{x}\|\uparrow vt$.

Lemma 2.2. *Suppose that* $g(t)>0$ *for any* $t\ge 0$. *Then, for* $n=2,3$,

$$f_n(t,\overrightarrow{x})\uparrow\infty \text{ as } \|\overrightarrow{x}\|\uparrow vt.$$

Proof. Since $f_n(t,\overrightarrow{x})=\mathcal{F}^{-1}(H(t))$, where $\mathcal{F}^{-1}$ is the inverse n-dimensional Fourier transform of $H(t)$ w.r.t. $\overrightarrow{\alpha}$. It follows from Eqs. (3) that

$$\begin{aligned} f_n(t,\overrightarrow{x})=\mathcal{F}^{-1}(H(t)) &= (1-G(t))\frac{\Gamma\left(\frac{n}{2}\right)\delta\left(v^2t^2-\|\overrightarrow{x}\|^2\right)}{2\pi^{\frac{n}{2}}(vt)^{n-1}}\\ &+\frac{\left(\Gamma\left(\frac{n}{2}\right)\right)^2}{4\pi^n}\int_0^t\int_{\|\overrightarrow{u}\|\le vt}\frac{(1-G(t-s))\,\delta\left(v^2(t-s)^2-\|\overrightarrow{x}-\overrightarrow{u}\|^2\right)}{(v(t-s))^{n-1}}\\ &\times\frac{g(s)\,\delta\left(v^2s^2-\|\overrightarrow{u}\|^2\right)}{(vs)^{n-1}}dsd\overrightarrow{u}+\dots \end{aligned}$$

For $n=3$, we have $\varphi(t)=\frac{\sin(vt\|\overrightarrow{\alpha}\|)}{vt\|\overrightarrow{\alpha}\|}$. It is well known that

$$\mathcal{L}\left(\frac{\sin(vt\|\overrightarrow{\alpha}\|)}{vt\|\overrightarrow{\alpha}\|}\right)=\frac{1}{v\|\overrightarrow{\alpha}\|}\operatorname{arctg}\left(\frac{v\|\overrightarrow{\alpha}\|}{s}\right).$$

By using the result in [2], we obtain

$$\begin{aligned} &\frac{\left(\Gamma\left(\frac{3}{2}\right)\right)^2}{4\pi^3}\int_0^t\int_{\|\overrightarrow{u}\|\le vt}\frac{\delta\left(v^2(t-s)^2-\|\overrightarrow{x}-\overrightarrow{u}\|^2\right)}{(v(t-s))^2}\frac{\delta\left(v^2s^2-\|\overrightarrow{u}\|^2\right)}{(vs)^2}dsd\overrightarrow{u}\\ &= \mathcal{F}^{-1}\left(\frac{1}{v^2\|\overrightarrow{\alpha}\|^2}\mathcal{L}^{-1}\left[\left(\operatorname{arctg}\left(\frac{v\|\overrightarrow{\alpha}\|}{s}\right)\right)^2\right]\right)\\ &= \frac{1}{4\pi v^2t\|\overrightarrow{x}\|}\ln\left(\frac{vt+\|\overrightarrow{x}\|}{vt-\|\overrightarrow{x}\|}\right). \end{aligned}$$

Since, for every $t\ge 0$, $g(t)>0$, it is easy to verify that

$$C_t=\inf_{0\le s\le t}(1-G(s))\,g(s)>0\ ,$$

and we have

$$\frac{C_t}{4\pi v^2t^2\|\overrightarrow{x}\|}\ln\left(\frac{vt+\|\overrightarrow{x}\|}{vt-\|\overrightarrow{x}\|}\right)\ \le f_3(t,\overrightarrow{x}).$$

Therefore, $f_3(t,\overrightarrow{x})\uparrow\infty$ *as* $\|\overrightarrow{x}\|\uparrow vt$.

For $n = 2$ (i.e., $\overrightarrow{u}$, $\overrightarrow{x} \in \mathrm{R}^2$), we have [2], [7]:

$$\frac{2}{4\pi^2}\int_0^t \int_{\|\overrightarrow{u}\| \le vt} \frac{\delta\left(v^2(t-s)^2 - \|\overrightarrow{x} - \overrightarrow{u}\|^2\right)}{\left(v\,(t-s)\right)^2} \times \frac{\delta\left(v^2s^2 - \|\overrightarrow{u}\|^2\right)}{(vs)^2} ds d\overrightarrow{u} = \frac{\left(v^2s^2 - \|\overrightarrow{u}\|^2\right)^{-\frac{1}{2}}}{4\pi vt}.$$

In the same way as for $n = 3$, we can show that $f_2(t, \overrightarrow{x}) \uparrow \infty$ *as* $\|\overrightarrow{x}\| \uparrow vt$.

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