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ERGODIC MEASURES AND THE DEFINABILITY OF SUBGROUPS VIA NORMAL EXTENSIONS OF SUCH MEASURES

It is shown that any subgroup H of an uncountable σ-compact locally compact topological group Γ is completely determined by a certain family of left H-invariant extensions of the left Haar measure μ on Γ. An abstract analogue of this fact is also established for a nonzero σ-finite ergodic measure given on an uncountable commutative group.

In this paper we consider an abstract space E equipped with a transformation group Γ and also endowed with some σ-finite measure μ which is invariant with respect to Γ. Various subgroups of Γ will be described in terms of corresponding invariant extensions of μ.

In the sequel, we use the following fairly standard notation.

ω = the least infinite cardinality (equivalently, the cardinality of the set $\mathbf{N}$ of all natural numbers).

$\mathbf{Z}$ = the set of all integers.

$\mathbf{R}$ = the set of all real numbers.

$X \triangle Y$ = the symmetric difference of two sets X and Y.

$\mathrm{dom}(f)$ = the domain of a given function f.

Let E be a base (ground) set. A measure μ defined on a σ-algebra of subsets of E is called diffused (or continuous) if, for each $x \in E$, we have $\{x\} \in \mathrm{dom}(\mu)$ and $\mu(\{x\}) = 0$.

$\mathcal{I}(\mu)$ = the σ-ideal generated by all μ-measure zero subsets of E.

μ^* and μ_* denote, respectively, the outer and inner measures associated with a given measure μ.

A set $X \subset E$ is called μ-thick in E if $\mu_*(E \setminus X) = 0$. Clearly, for a probability measure μ on E, the μ-thickness of $X \subset E$ is equivalent to $\mu^*(X) = 1$.

All measures considered below are assumed to be complete (without essential loss of generality).

Recall that a cardinal number $\mathbf{a}$ is two-valued measurable if there exists a two-valued diffused probability measure whose domain coincides with the family of all subsets of $\mathbf{a}$. As is well known (see, for instance, [15], [16]), two-valued measurable cardinals are very large and their existence cannot be derived from the axioms of contemporary set theory. In other words, the assumption that there are no two-valued measurable cardinals does not contradict the axioms of set theory. Detailed information on these cardinals and other type of large cardinals can be found in [6], [8], [14], [15].

Let E be a ground set endowed with some group Γ of its transformations.

A measure μ on E is called Γ-invariant if $\mathrm{dom}(\mu)$ is a Γ-invariant σ-algebra of subsets of E and $\mu(g(A)) = \mu(A)$ for all transformations $g \in \Gamma$ and all sets $A \in \mathrm{dom}(\mu)$.

A measure μ on E is called Γ-quasi-invariant if $\mathrm{dom}(\mu)$ is a Γ-invariant σ-algebra of subsets of E and, for any $g \in \Gamma$, we have

$$\mu(g(X)) = 0 \Leftrightarrow \mu(X) = 0 \ (X \in \mathrm{dom}(\mu)).$$

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A set $B \subset E$ is called almost Γ-invariant (with respect to μ) if

$$(\forall g \in \Gamma)(\mu(B \triangle g(B)) = 0).$$

In our further considerations, we will use the following auxiliary proposition on almost invariant subsets of E.

Lemma 1. *Let μ be a σ-finite Γ-invariant (respectively, Γ-quasi-invariant) measure on E and let B be a μ-thick almost Γ-invariant subset of E. Then there exists a Γ-invariant (respectively, Γ-quasi-invariant) measure μ' on E which extends μ and whose support is B, i.e., $\mu'(E \setminus B) = 0$.*

For the proof of this lemma, see e.g. [9], [12] or [18].

Let (E, Γ) be a space with a transformation group, let $\mu = \mu_\Gamma$ be a Γ-invariant (Γ-quasi-invariant) measure on E, and let G be a subgroup of Γ.

We say that μ is G-ergodic (or G-metrically transitive) if, for any set $A \in \mathrm{dom}(\mu)$ with $\mu(A) > 0$, there exists a countable family $\{g_i : i \in I\} \subset G$ such that

$$\mu(E \setminus \cup\{g_i(A) : i \in I\}) = 0.$$

Ergodic measures can frequently be met in various topics of mathematical analysis and probability theory (see e.g. [1], [4], [5], [16], [17]). The ergodicity also plays an essential role for the uniqueness of invariant measures on their domains. More precisely, suppose that $G \subset \Gamma$ is an uncountable group acting freely in E and μ is a σ-finite G-invariant G-ergodic complete measure on E. Then, for every σ-finite G-invariant measure ν with $\mathrm{dom}(\nu) = \mathrm{dom}(\mu)$, there exists a non-negative coefficient $t = t(\nu) \in \mathbf{R}$ such that $\nu = t(\nu) \cdot \mu$. The proof of this fact can be found in [9].

If we are given a space (E, Γ) with a nonzero σ-finite Γ-invariant measure $\mu = \mu_\Gamma$ and if H is any subgroup of Γ, then the same μ may be regarded as an H-invariant measure. So we can speak of H-invariant extensions of μ. Let us denote by $M(H, \mu)$ the class of all those H-invariant measures on E which extend μ. It is clear that, for any two subgroups H_1 and H_2 of Γ, the following implication holds:

$$H_1 \subset H_2 \Rightarrow M(H_2, \mu) \subset M(H_1, \mu).$$

In this paper we will be concerned with the question of whether the converse implication is also true. First, we wish to discuss the situation when a non-discrete σ-compact locally compact topological group Γ is given.

Let $(\Gamma, \cdot)$ be a non-discrete σ-compact locally compact group. Then, as is widely known, Γ can be equipped with a nonzero σ-finite left Γ-invariant Borel measure $\mu = \mu_\Gamma$, which is called the left Haar measure on Γ. This measure is Γ-ergodic and is unique with exactness to a constant non-negative coefficient (see, for instance, [1], [4], [5]).

Remark 1. It directly follows from the Baire theorem on category that the non-discreteness of a σ-compact locally compact group Γ is equivalent to its uncountability.

Remark 2. Let $(\Gamma, \cdot)$ be a σ-compact locally compact group and let G be an everywhere dense subgroup of Γ. Then the left Haar measure $\mu = \mu_\Gamma$ is G-ergodic. Conversely, if the same μ is G-ergodic for some subgroup G of Γ, then G is everywhere dense in Γ (cf. [10]).

Remark 3. Let E be an infinite-dimensional (equivalently, non-locally compact) topological vector space. In general, there exists no nonzero σ-finite Borel measure on E quasi-invariant with respect to the group of all translations of E (see e.g. [1], [2] and, especially, [19] where the case of an infinite-dimensional separable Hilbert space is considered in detail).

Dealing with the left Haar measure $\mu = \mu_\Gamma$ on an uncountable σ-compact locally compact group Γ, one may pose the question about the existence of proper left Γ-invariant extensions of μ (of course, here we mean only those extensions of μ which themselves are measures). As was demonstrated by several authors, there are many such extensions (see, for instance, [3], [5], [7], [9], [13], [18]). Moreover, if Γ is an uncountable Polish locally compact group, then there exist even nonseparable left Γ-invariant extensions of μ (see [3], [5], [7], [13]).

On the other hand, as was already mentioned, if H is an arbitrary subgroup of Γ, then μ being left Γ-invariant, is automatically left H-invariant. So it makes sense to speak of left H-invariant extensions of μ. Among all such extensions, we are especially interested in those ones which are closely connected with μ.

We shall say that a left H-invariant extension ν of μ is a *normal extension* of μ if, for any set $X \in \mathrm{dom}(\nu)$, there exists a set $Y \in \mathrm{dom}(\mu)$ such that $\nu(X \triangle Y) = 0$.

In other words, left invariant normal extensions of μ do not change the metrical structure of μ.

Let us introduce the notation:

$M_0(H, \mu)$ = the family of all left H-invariant normal extensions of μ.

Notice that if H is an everywhere dense subgroup of Γ, then all measures from $M_0(H, \mu)$ are H-ergodic (cf. Remark 2).

Our first goal is to show that, for any two subgroups H and H' of Γ, the inclusion $M_0(H', \mu) \subset M_0(H, \mu)$ implies the inclusion $H \subset H'$.

As a byproduct, we obtain that the equality $M_0(H', \mu) = M_0(H, \mu)$ implies the equality $H = H'$. Thus, we conclude that even minimal invariant extensions of μ allow to distinguish between the subgroups of Γ.

In order to demonstrate this fact, we need several auxiliary propositions.

Lemma 2. *If* $(\Gamma, \cdot)$ *is an uncountable* σ*-compact locally compact group, then* $\mathrm{card}(\Gamma) = 2^{w(\Gamma)}$*, where* $w(\Gamma)$ *denotes the topological weight of* Γ.

The above lemma is one of the most important results in the classical theory of topological groups (see [3]). It readily implies the equality

$$\mathrm{card}(\Gamma) = \mathrm{card}(\mathcal{B}(\Gamma)),$$

where $\mathcal{B}(\Gamma)$ denotes, as usual, the Borel σ-algebra of Γ. In its turn, this equality allows to carry out some Bernstein type transfinite construction for Γ.

Lemma 3. *Let* $(\Gamma, \cdot)$ *be an uncountable* σ*-compact locally compact group,* μ *be the left Haar measure on* Γ*, and let* G *be a subgroup of* Γ *represented in the form*

$$G = \cup\{G_\xi : \xi < \alpha\},$$

where α *is the least ordinal number of cardinality* $\mathrm{card}(\Gamma)$*. Suppose also that the following relations are satisfied:*

(1) $\{G_\xi : \xi < \alpha\}$ *is an increasing* α*-sequence of subgroups of* Γ*;*

(2) $\mathrm{card}(G_\xi) \le \mathrm{card}(\xi) + \omega$ *for each ordinal* $\xi < \alpha$*.*

Then there exists an α*-sequence of points* $\{x_\xi : \xi < \alpha\} \subset \Gamma$ *such that:*

(a) the set $\{x_\xi : \xi < \alpha\}$ *is* μ*-thick in* Γ*;*

(b) $(G_\xi \cdot x_\xi) \cap (G_\zeta \cdot x_\zeta) = \emptyset$ *for any two distinct ordinals* $\xi < \alpha$ *and* $\zeta < \alpha$*.*

Therefore, the set $X = \cup\{G_\xi \cdot x_\xi : \xi < \alpha\}$ *is* μ*-thick in* Γ *and*

$$(\forall g \in G)(\mathrm{card}((g \cdot X) \triangle X) < \mathrm{card}(\Gamma)).$$

As mentioned above, the proof of Lemma 3 is based on the standard argument that is usually utilized in various Bernstein type constructions. So we omit this proof here (cf. [5], [7], [9], [11], [12], [18]).

Lemma 4. *Let* $(\Gamma, \cdot)$ *be an uncountable* σ*-compact locally compact group,* G *be a subgroup of* Γ*, and let* X *be as in Lemma 3. Then there exists a left* G*-invariant normal extension* μ' *of* μ *such that* $\mu'(\Gamma \setminus X) = 0$*.*

The proof of Lemma 4 is analogous to the proof of Lemma 1.

Lemmas 2-4 enable us to establish the following result.

Theorem 1. *Let* $(\Gamma, \cdot)$ *be an uncountable* σ*-compact locally compact group and let* H *and* H' *be any two subgroups of* Γ *such that* $M_0(H', \mu) \subset M_0(H, \mu)$*. Then we have* $H \subset H'$*.*

Proof. Suppose to the contrary that there exists an element $h \in H \setminus H'$. Denote by G the group generated by $\{h\} \cup H'$. We are going to apply Lemma 3 to G. For this purpose, take

$$\{G_\xi : \xi < \alpha\},\ \{x_\xi : \xi < \alpha\},\ X = \cup\{G_\xi \cdot x_\xi : \xi < \alpha\}$$

as in Lemma 3 and denote

$$H'_\xi = G_\xi \cap H' \ (\xi < \alpha).$$

Obviously, the α-sequence of groups $\{H'_\xi : \xi < \alpha\}$ is increasing by inclusion and

$$H' = \cup\{H'_\xi : \xi < \alpha\}.$$

Further, since $h \in G$, there exists an ordinal ξ_0 such that $h \in G_\xi$ for all ordinals $\xi \in [\xi_0, \alpha[$. Consider the set

$$Y = \cup\{H'_\xi \cdot x_\xi : \xi < \alpha\}.$$

According to Lemma 3, the set Y is μ-thick in Γ and

$$(\forall g \in H')(\mathrm{card}((g \cdot Y) \triangle Y) < \mathrm{card}(\Gamma)).$$

In view of Lemma 4, there exists a left H'-invariant normal extension μ' of μ such that

$$\mu'(\Gamma \setminus Y) = 0.$$

Now, by taking into account the relations

$$h \cdot G_\xi \cdot x_\xi = G_\xi \cdot x_\xi \ (\xi_0 < \xi < \alpha),$$

$$(h \cdot H'_\xi \cdot x_\xi) \cap (H'_\xi \cdot x_\xi) = \emptyset \ (\xi < \alpha),$$

$$(G_\xi \cdot x_\xi) \cap (G_\zeta \cdot x_\zeta) = \emptyset \ (\xi < \alpha,\ \zeta < \alpha,\ \xi \neq \zeta),$$

it is not difficult to verify that

$$\mathrm{card}((h \cdot Y) \cap Y) < \mathrm{card}(\Gamma),$$

whence it follows that μ' cannot be left H-invariant (moreover, the same argument yields that μ' cannot be even left H-quasi-invariant). We thus obtain that μ' belongs to the class $M_0(H', \mu)$ but does not belong to the class $M_0(H, \mu)$, which contradicts the inclusion $M_0(H', \mu) \subset M_0(H, \mu)$. The obtained contradiction completes the proof.

The following statement is a straightforward consequence of Theorem 1.

Theorem 2. *Let* $(\Gamma, \cdot)$ *be an uncountable* σ*-compact locally compact group and let* H *and* H' *be two subgroups of* G *such that* $M_0(H', \mu) = M_0(H, \mu)$*. Then we have* $H = H'$*.*

In other words, Theorem 2 says that every subgroup H of Γ is completely determined by the corresponding class $M(H, \mu)$ of all left H-invariant normal extensions of μ.

Notice now that the considerations leading to the proof of Theorem 1 are substantially based on some topological properties of the Haar measure. Our second goal in this paper is to obtain a certain abstract analogue of Theorem 1 for the case of an uncountable commutative group $(\Gamma, +)$ equipped with a nonzero σ-finite Γ-invariant Γ-ergodic measure

$\mu = \mu_\Gamma$. We will establish this analogue under some natural set-theoretical assumptions on Γ.

In the sequel, we will assume that card(Γ) is not cofinal with ω, i.e., card(Γ) cannot be represented as a countable sum of cardinal numbers all of which are strictly less than card(Γ). So we may suppose that

$$\{A \subset \Gamma : \mathrm{card}(A) < \mathrm{card}(\Gamma)\} \subset \mathcal{I}(\mu).$$

Let H and H' be two subgroups of $(\Gamma, +)$ such that μ is H-ergodic and H'-ergodic simultaneously. Suppose that there exists at least one element $h \in H \setminus H'$. We may represent the given group Γ in the form of a transfinite sequence

$$\Gamma = \{g_\xi : \xi < \alpha\},$$

where α is the least ordinal number with $\mathrm{card}(\alpha) = \mathrm{card}(\Gamma)$. We may also assume (without loss of generality) that $g_0 = h$. Further, let us denote:

Γ^*_ξ = the group generated by $\{g_\zeta : \zeta \leq \xi\}$;

Γ_ξ = the group generated by $\{g_\zeta : \zeta < \xi\}$;

$H_\xi = \Gamma_\xi \cap H$ for any $\xi < \alpha$;

$H'_\xi = \Gamma_\xi \cap H'$ for any $\xi < \alpha$;

$T_\xi = \Gamma^*_\xi \setminus \Gamma_\xi$ for any $\xi < \alpha$.

Then we obviously have the following relations:

(a) $\Gamma = \{0\} \cup (\cup\{T_\xi : \xi < \alpha\})$;

(b) for each $\xi < \alpha$, the set T_ξ is Γ_ξ-invariant;

(c) for every set $\Xi \subset [0, \alpha[$, the set $\cup\{T_\xi : \xi \in \Xi\}$ is almost Γ-invariant with respect to μ;

(d) $H = \cup\{H_\xi : \xi < \alpha\}$;

(e) $H' = \cup\{H'_\xi : \xi < \alpha\}$.

For every $\xi < \alpha$, denote by F_ξ the group generated by $\{h\} \cup H'_\xi$.

Obviously, if $1 \leq \xi < \alpha$, then $H'_\xi \subset F_\xi \subset \Gamma_\xi$.

Also, it can easily be seen the validity of the next auxiliary proposition.

Lemma 5. *For any nonzero ordinal number $\xi < \alpha$, the factor-group F_ξ / H'_ξ is at most countable.*

Proof. Indeed, fix a nonzero ordinal $\xi < \alpha$. In view of the commutativity of Γ, we may write

$$F_\xi = \{mh + h' : m \in \mathbf{Z},\ h' \in H'_\xi\} = \mathbf{Z}h + H'_\xi,$$

whence the assertion of the lemma trivially follows.

Lemma 6. *If* card(Γ) *is not a two-valued measurable cardinal number, then there exists a subset Ξ_0 of $[0, \alpha[$ such that the set*

$$X(\Xi_0) = \cup\{T_\xi : \xi \in \Xi_0\}$$

is not μ-measurable. Moreover, $X(\Xi_0)$ turns out to be a μ-thick set in Γ and its complement $\Gamma \setminus X(\Xi_0)$ is μ-thick, too.

Proof. The argument is similar to that of [18]. Suppose to the contrary that all the sets $X(\Xi) = \cup\{T_\xi : \xi \in \Xi\}$, where $\Xi \subset [0, \alpha[$, are μ-measurable. Denote by ν a probability measure which is equivalent to μ and introduce the functional ν' as follows:

$$\nu'(\Xi) = \nu(\cup\{T_\xi : \xi \in \Xi\})\ (\Xi \subset [0, \alpha[).$$

Keeping in mind that μ is Γ-ergodic, it can readily be shown that ν' is a two-valued diffused probability measure on the σ-algebra of all subsets of $[0, \alpha[$, which is impossible in view of the equality $\mathrm{card}(\Gamma) = \mathrm{card}(\alpha)$ and the assumption on card(Γ). Consequently, there exists $\Xi_0 \subset [0, \alpha[$ for which $X(\Xi_0)$ is not μ-measurable. But the same $X(\Xi_0)$ is almost Γ-invariant with respect to μ. Utilizing once again the Γ-ergodicity of μ, we obtain that

$$\mu_*(X(\Xi_0)) = \mu_*(\Gamma \setminus X(\Xi_0)) = 0,$$

which completes the proof.

Let $\Xi_0 \subset [0, \alpha[$ be as in Lemma 6 and let $X(\Xi_0)$ be the corresponding μ-nonmeasurable set. We may assume, without loss of generality, that $0 \notin \Xi_0$. For each $\xi \in \Xi_0$, consider the set T_ξ. From the definition of T_ξ it directly follows that this set is F_ξ-invariant, so can be written as

$$T_\xi = \cup\{T_{\xi,j} : j \in J(\xi)\},$$

where all $T_{\xi,j}$ are some pairwise disjoint F_ξ-orbits. Furthermore, each F_ξ-orbit is a countable union of pairwise disjoint H'_ξ-orbits. We thus may write

$$T_{\xi,j} = \cup\{T_{\xi,j,k} : k < \omega\},$$

where all $T_{\xi,j,k}$ $(k < \omega)$ are pairwise disjoint H'_ξ-orbits. Now, since the set

$$X(\Xi_0) = \cup\{T_\xi : \xi \in \Xi_0\} = \cup\{T_{\xi,j,k} : k < \omega, j \in J(\xi), \xi \in \Xi_0\}$$

is nonmeasurable with respect to μ, there exists a natural number k_0 such that the set

$$Y(\Xi_0, k_0) = \cup\{T_{\xi,j,k_0} : j \in J(\xi),\ \xi \in \Xi_0\}$$

is also nonmeasurable with respect to μ. In addition, $Y(\Xi_0, k_0)$ is almost H'-invariant. Since μ is H'-ergodic, we conclude that $Y(\Xi_0, k_0)$ and its complement $\Gamma \setminus Y(\Xi_0, k_0)$ are μ-thick subsets of Γ.

Summarizing all the said above and keeping in mind Lemma 1, we obtain the next proposition.

Lemma 7. *There exists an H'-invariant normal extension μ' of μ such that*

$$\mu'(\Gamma \setminus Y(\Xi_0, k_0)) = 0.$$

Since μ is H'-ergodic, the measure μ' is H'-ergodic too.

We now are able to prove the following statement (preserving the notation used above).

Theorem 3. *Let H and H' be two subgroups of Γ such that μ is H-ergodic and H'-ergodic simultaneously. Then the inclusion $M_0(H', \mu) \subset M_0(H, \mu)$ implies the inclusion $H \subset H'$.*

Consequently, the equality $M_0(H', \mu) = M_0(H, \mu)$ implies the equality $H = H'$.

Proof. Indeed, suppose otherwise, i.e., the inclusion $M_0(H', \mu) \subset M_0(H, \mu)$ holds true but H is not contained in H'. Then we may choose some element $h \in H \setminus H'$. For this element h, the construction made earlier yields the H'-invariant normal extension μ' of μ concentrated on the μ'-measurable set $Y(\Xi_0, k_0)$ (see Lemma 7). So μ' belongs to the class $M_0(H', \mu)$. We know the structure of $Y(\Xi_0, k_0)$, namely, this set admits a representation

$$Y(\Xi_0, k_0) = \cup\{T_{\xi,j,k_0} : j \in J(\xi),\ \xi \in \Xi_0\},$$

where all T_{ξ,j,k_0} are some H'_ξ-orbits. Notice now that

$$T_\xi \cap T_\zeta = \emptyset\ (\xi < \alpha,\ \zeta < \alpha,\ \xi \neq \zeta).$$

$$(h + T_{\xi,j,k_0}) \cap T_{\xi,j,k_0} = \emptyset\ (j \in J(\xi)),$$

$$(h + T_{\xi,j,k_0}) \cap T_{\xi,i,k_0} = \emptyset\ (j \in J(\xi), i \in J(\xi), i \neq j).$$

We thus conclude that

$$(h + Y(\Xi_0, k_0)) \cap Y(\Xi_0, k_0) = \emptyset,$$

whence it follows that μ' is not H-invariant, so μ' does not belong to the class $M_0(H, \mu)$. The obtained contradiction finishes the proof.

Remark 4. Theorem 3 can be directly generalized to the case where an abstract set E is equipped with an uncountable commutative transformation group Γ acting freely in E, and E is also endowed with a nonzero σ-finite Γ-invariant Γ-ergodic measure μ. The proof of this generalized version of Theorem 3 substantially remains the same as above.

Remark 5. It would be interesting to get some analogue of Theorem 3 for uncountable non-commutative transformation groups acting freely on an abstract set E.

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