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PERTURBED SELF-INTERSECTION LOCAL TIME

We consider a symmetric random walk related to independent Rademacher random variables. Our aim is to study some modified versions of the so called self-intersection local time of this random walk. The modified versions of the self-intersection local time are obtained by introducing a time t and a sequence of independent with the same distribution uniform on $(0,1)$ random variables Y_i's, independent of the random walk. In this work, we study a distance between the standard self-intersection local time of the random walk and some modified versions (perturbed) of it. We also state a two-parameter strong approximation for the centered local time of the hybrids of empirical and partial sums processes by a process defined by a Wiener sheet combined with an independent Brownian motion.

1. Introduction

Let $S = \{S(n)\}_{n\geq 1}$ be the simple symmetric random walk related to a sequence $\{\epsilon_i\}_{i\geq 1}$ of independent Rademacher random variables i.e. $\epsilon_i = S(i) - S(i-1)$ with $P(\epsilon_i = 1) = P(\epsilon_i = -1) = 1/2$ and let us define the process $U(.,t) = \{U(n,t)\}_{n\geq 1}$ for $0 \leq t \leq 1$, where $U(n,t)$ is given by

$$U(n,t) = \sum_{j=1}^{n} \epsilon_j 1_{\{Y_j \leq t\}}, \tag{1}$$

where 1_A denotes the set indicator function and $\{Y_i\}_{i\geq 1}$ is a sequence of independent and identically distributed (i.i.d.) uniform on $(0,1)$ random variables, independent of the random variables ϵ_i's ($i \geq 1$).
Let us define also the process $A = \{A_n(t), 0 \leq t \leq 1, 1 \leq n < \infty\}$ given by

$$A_n(t) = \frac{U(n,t)}{\sqrt{n}}, \tag{2}$$

this last process is known in the literature as the hybrids of empirical and partial sums processes, see for instance [14] and [1].
Let $f(x), x \in \mathrm{Z}$ be a real valued function. It is well known that we have the following relation

$$\sum_{i=1}^{n} f(S(i)) = \sum_{x=-\infty}^{\infty} f(x) L_n^x(S), \ n = 1, 2, \ldots$$

where $L_n^x(S)$ denotes the local time of S given by

$$L_n^x(S) = \sum_{i=1}^{n} 1_{\{S(i)=x\}}. \tag{3}$$

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The self-intersection local time of the random walk S is given by

$$I_n(1) = \sum_{x\in \mathrm{Z}} (L_n^x(S))^2. \tag{4}$$

Let us mention that one of the best recent references on intersections local times for random walks on lattices of $\mathrm{R^d}$ and for Brownian motions is the book of Chen [4], see also [19] and [20].

We will restrict ourselves to the one-dimensional case i.e. the random walk on Z and we will denote by $\log t = \log(t \vee e)$, $\log_2 t$ the two-iterated logarithm i.e. $\log_2 t = \log\log t$ and $[x]$ denote the integer part of some real x.

The first aim of this paper is to evaluate the difference between the standard self-intersection local time of S (see (4)) and some modified version of it. The second aim is considering the local time of the process A given by $\xi_t^x(A_n)$ (see (8)), to state a strong approximation result for the centered local time process given by $\xi_t^x(A_n) - \xi_t^0(A_n)$, in terms of a two-parameter Gaussian process. This kind of approach is inspired from [7], see also [12].

In the first aim three kinds of modified version are in studying. The first one, denoted by $I_n(t)$, consider the intersection local time related to S and the process $U(.,t)$ and is defined by

$$I_n(t) = \sum\sum\nolimits_{1\le i,j\le n} 1_{\{U(i,t)=S(j)\}},\ t \in (0,1). \tag{5}$$

The second one, denoted by $J_n(t)$, consider the intersection local time of $S_n(t) = \sum_{i=1}^{[nt]} \epsilon_i$ and $U(n,t)$ for $0 < t < 1$ and is defined by

$$J_n(t) = \sum_{x\in \mathrm{Z}} L_n^x(S(.t)L_n^x\,(U(.,t)) = \sum_{x\in \mathrm{Z}}\sum\sum\nolimits_{1\le i,j\le n} 1_{\{S_i(t)=U(j,t)=x\}} \tag{6}$$

where

$$L_n^x(S(.t)) = \#\left\{i : 1 \le i \le n, S_i(t) = x\right\},\ x = 0, \pm1, \pm2, \dots,\ n = 1, 2, \dots,$$

and

$$L_n^x(U(.,t)) = \#\left\{i : 1 \le i \le n, U(i,t) = x\right\},\ x = 0, \pm1, \pm2, \dots,\ n = 1, 2, \dots.$$

Roughly speaking, $J_n(t)$ gives the number of points in common to $S_n(t)$ and $U(n,t)$.

Finally, in the third one, we consider the modification denoted by $K_n(t(n))$, where

$$K_n\,(t(n)) = \sum_{x\in \mathrm{Z}} (L_n^x(U(.,t(n))))^2 \tag{7}$$

for a sequence $\{t(n)\}_{n\ge1}$, given by $t(n) = 1 - n^{-1+\varepsilon}$, remark that $t(n) \to 1$ as $n \to \infty$. In the following section, we explain the choice of this sequence.

The second aim is, by taking in account the two-parameters to give a strong approximation result for the centered local time of the hybrids of empirical and partials sums processes by a process defined by a Wiener sheet combined with an independent Brownian motion. This is based on the approach of Csáki et al. [7] for the random walk local time approximated by a Wiener sheet combined with an independent Brownian motion (see also reference therein) and on Bass and Khoshnevisan [2] approach for the local time of the empirical process.

Let us now recall some facts related to the processes that are in studying. The process $U(.,t)$ given in (1) can be obtained from the process

$$\tilde U(.,t) = \left\{\tilde U(n,t) = \sum_{1\le i\le n} \epsilon_i 1_{\{X_i\le t\}},\right\}_{n\ge1},\ -\infty < t < \infty$$

where the sequence $\{X_i\}_{i\geq 1}$ are i.i.d. random variables with common distribution function F, independent of the sequence $\{\epsilon_i\}_{i\geq 1}$. By [14], p.5, we have without loss of generality, there is a sequence of i.i.d. random variables $\{Y_i\}_{i\geq 1}$ uniform on $(0,1)$ such that $X_i = Q(Y_i)$, with $Q(y) = \inf\{x: \ F(x) \geq y\}$ i.e. the quantile function of F, then we can consider $U(.,t)(0 \leq t \leq 1)$ in the place of $\tilde{U}(.,t)(-\infty < t < \infty)$.

Let us define the local time of $A_n(t)$ by

$$\xi_t^x(A_n) = \frac{1}{\sqrt{n}} \sum_{s\leq t} 1_{\{A_n(s)=x\}}, \ t \in [0,1], x \in \mathrm{R}. \tag{8}$$

In the light of the strong approximation given by Diebolt in [13], we define the following local time

$$\tilde{\xi}_t^x(W_n(F_n)) = \int_0^1 \delta_t^s(F_n) d_s L_s^x(W_n),$$

where $W_n(t) = W(nt)/\sqrt{n}$ with W is a standard Wiener process and $\delta_t^s(F_n)$ denote the local time at a level s up to t of the empirical distribution function $F_n(t)$ defined by

$$\delta_t^x(F_n) = \frac{1}{\sqrt{n}} \sum_{s\leq t} 1_{\{F_n(s)=x\}}. \tag{9}$$

Putting $V_n(t) = \sum_{i=1}^{[nt]} \epsilon_i/\sqrt{n}$, let us define the associated local time $\nu_t^x(V_n)$ by

$$\nu_t^x(V_n) = \sum_{j=1}^{[nt]} 1_{\left\{\sum_{i=1}^{j} \epsilon_i = \left[\sqrt{n}x\right]\right\}}, \quad x \in \mathrm{R}. \tag{10}$$

It is not difficult to see that $\nu_1^x(V_n)$, can be written as

$$\nu_1^x(V_n) = L_n^{\left[\sqrt{n}x\right]}(S) \tag{11}$$

where $L_n^{\left[\sqrt{n}x\right]}(S)$ is the local time of $S_n = \epsilon_1 + \cdots + \epsilon_n$ for $n \geq 1$, i.e. the simple symmetric random walk defined by the random variables $\epsilon_i's$ (see (3)).

Let us recall that $A_n(t) \stackrel{d}{=} V_n(F_n(t))$. By this last equality (in distribution) and from the definition of the local time $\xi_t^x(A_n)$, we have $V_n(F_n(t)) = x$ if $F_n(t)$ is the return time to x of V_n, for $x = 0, \pm 1/\sqrt{n}, \pm 2/\sqrt{n}, \ldots$. Consequently, $\xi_t^x(A_n)$ can be written as

$$\xi_t^x(A_n) = \int_0^1 \delta_t^s(F_n) d_s \nu_s^x(V_n), \tag{12}$$

for $0 \leq t \leq 1$ and $x \in \mathrm{R}$, see [1].

Bass and Khoshnevisan in [2] (section 4) stated that: on an appropriate probability space, for any $\epsilon > 0$, almost surely

$$\sup_{0\leq t\leq 1} \sup_{x\in\mathrm{R}} |L_t^x(\Xi_n) - \delta_t^x(\alpha_n)| = o\left(n^{-1/10+\epsilon}\right), \tag{13}$$

where $L_t^x(\Xi_n)$ is the local time of the Kiefer process Ξ and $\delta_t^x(\alpha_n)$ is the local time of the empirical uniform process associated to the sequence $\{Y_i\}_{i\geq 1}$ of the i.i.d. uniform on $(0,1)$ random variables. By a Kiefer process $\Xi = \{\Xi(s,t); s \geq 0, t \in [0,1]\}$, we mean a two-parameter centered Gaussian process with covariance given by

$$E\left[\Xi(s,u)\Xi(t,v)\right] = (s \wedge t)(u \wedge v - uv),$$

see [10] for further information and let $\Xi_n(.) = \Xi(n,.)$.

Our paper is organized as follows. In the following paragraphs of this section, we recall some results related to self-intersection local time. Section 2, is devoted to state our results, namely giving some properties of the local time of the process $U(.,t)$, and giving upper bound in the asymptotic behavior of $I_n(1)$ and $I_n(t)$, we will also state upper bound results for the difference between $I_n(1)$ and $I_n(t)$ for each $t \in (0,1)$ and for the difference between $I_n(1)$ and $J_n(t)$. Moreover, we study $\eta_n(t(n)) = |I_n(1) - K_n(t(n))|$ where $t(n) = 1 - n^{-1+\varepsilon}$. For this choice of $t(n)$, we obtain an upper bound for $\eta_n(t(n))$. The last result of this section, is the strong approximation for the centered local time of the hybrids of empirical and partials sums processes. Finally, Section 3 is devoted to the proof of our results.

Let us recall that the asymptotic behavior of self-intersection local times contributes in a fundamental way to the limit theorems for the random walk in random sceneries (see [5]). Namely, consider the random sequence

$$R_n = \sum_{k=1}^{n} \xi(S(k)), \ n = 1, 2, \dots$$

where $\xi(x)$ is often assumed to be an independent family of i.i.d. random variables with mean zero and strong enough integrability, independent of the random walk S. It is well known that

$$R_n = \sum_{x \in \mathrm{Z}} \xi(x) L_n^x(S), \ n = 1, 2, \dots.$$

Remark that conditionally on $L_n^x(S)$, R_n has a variance given by the quadratic form $\sum_{x\in \mathrm{Z}} \left(L_n^x(S)\right)^2$, which is essentially twice of self-intersection local time of $\{S(n)\}_{n\geq 0}$.

In 2009, Chen and Khoshnevisan studied the random quantity called a random Hamiltonian of the so-called charged polymer-model defined by

$$H_n = \sum\sum\nolimits_{1\leq i<j\leq n} q_i q_j 1_{\{S(i)=S(j)\}} \tag{14}$$

where $\{q_i\}_{i=1}^{\infty}$ is a collection of i.i.d. mean-zero random variables (random charges) with finite variance $\sigma^2 > 0$ and $E[q_1^6] < \infty$ and where $S = \{S(i)\}_{i=1}^{\infty}$ denote a random walk on Z^d with $S_0 = 0$. Each realization of S corresponds to a possible polymer path. The random variables $q_1, q_2, \dots$ are random charges that are placed on a polymer path modeled by the trajectories of S. In the last decades the model of charged polymer received an enormous amount of attention, see for instance [5], [6] and [15] (see also references therein).

Some estimates stated on Chen and Khoshnevisan (2009) are for instance (cf. Theorem 2.4. of [6]) :

$$\frac{1}{n} \sum_{x\in \mathrm{Z}^{\mathrm{d}}} (L_n^x(S))^2 \to 1 + 2\sum_{k=1}^{\infty} P(S(k) = 0) \text{ in probability } (n \to \infty). \tag{15}$$

Remark that we can also write $I_n(1)$ given in (4) as

$$I_n(1) = \sum\sum\nolimits_{1\leq i,j\leq n} 1_{\{S(i)=S(j)\}} \tag{16}$$

and (15) implies that a random walk S on Z^{d} is recurrent if and only if $I_n(1)/n \to \infty$ $(n \to \infty)$ in probability. Some other motivations for the study of intersection local times are given by Chen on [5] (see also references, therein), namely by giving results of type large deviations (section 4, d=1) and the law of the iterated logarithm (LIL) for intersection local times and related models. As mentioned below the book of Chen [4] provides un account of recent results on this topic.

2. Results

We will assume without loss of generality that all random variables and processes are defined on the same probability space.

Recall that $U(.,t) = \{U(n,t)\}_{n\geq 1}$ for $0 \leq t \leq 1$, where $U(n,t)$ is given by

$$U(n,t) = \sum_{j=1}^{n} \epsilon_j 1_{\{Y_j \leq t\}},$$

where

- (C1) the random variables $\epsilon_i = S(i) - S(i-1)$, $(i \geq 1)$ are i.i.d., with $P(\epsilon_i = 1) = P(\epsilon_i = -1) = 1/2$,
- (C2) the random variables Y_i, $(i \geq 1)$ are i.i.d. of law uniform on $(0,1)$,
- (C3) the random variables Y_i's and ϵ_i's are independent.

In the following Lemma, we establish some results related to the local time of $U(.,t)$ denoted by $L_n^x(U(.,t))$ and defined by

$$L_n^x(U(.,t)) = \sum_{i=1}^{n} 1_{\{U(i,t)=x\}}. \tag{17}$$

This Lemma is in the same vein as Lemma 2.2 of [6].

Lemma 2.1. *Assume the conditions (C1), (C2) and (C3) for the sequences* $\{\epsilon_i\}_{i\geq 1}$ *and* $\{Y_i\}_{i\geq 1}$*. Then*

$$\sum_{x\in Z} E[L_n^x(U(.,t))] = n$$

and

$$\sum_{x\in Z} E[|L_n^x(U(.,t))|^2] = n + 2\sum_{k=1}^{n-1}(n-k)P(U(k,t) = 0).$$

In the following results, all the rates of convergence are given as $n \to \infty$.
In the next Lemma, we study in an almost surely sense the behavior of $I_n(1)$, namely by giving an upper bound. Remark that this case corresponds almost surely to the self-intersection local time of the random walk S related to the sequence of the i.i.d. Rademacher random variables $\{\epsilon_i\}_{i\geq 1}$. This result is obtained by standard arguments then we will give only the main ideas of the proof.

Lemma 2.2. *Under the assumptions of Lemma 2.1, we have with probability one*

$$I_n(1) = n^{3/2+o(1)}.$$

In the following Lemma we state the behavior (giving an upper bound) of the intersection local time in the case $0 < t < 1$, i.e. $I_n(t)$ given by (5).

Lemma 2.3. *Assume the conditions of Lemma 2.1 and let* $I_n(t)$ *be as in (5). Then, we have*

$$I_n(t) = n^{3/2+o(1)} \ a.s.$$

In the case $t = 0$, $I_n(0)$ can be seen as a number of visits to zero of the symmetric random walk defined by S. We state the behavior of the difference between $I_n(1)$ and $I_n(0)$ in the following Lemma.

Lemma 2.4. *Under the same assumptions of Lemma 1, we have*

$$I_n(1) - I_n(0) = n^{3/2+o(1)}, \ a.s.$$

Our main result is given in the following theorem, where we state a result giving a rate for $I_n(1) - I_n(t)$ for $0 < t < 1$. Recall that $I_n(t)$ is giving by

$$I_n(t) = \sum_{i=1}^{n}\sum_{j=1}^{n} 1_{\{S(i)=U(j,t)\}},$$

and as explained in Section 1, we can consider

$$I_n(t) = \sum_{x\in\mathbb{Z}} L_n^x(S) L_n^x(U(.,t)) = \sum_{x\in\mathbb{Z}} \sum \sum\nolimits_{1\le i,j\le n} 1_{\{S(i)=U(j,t)=x\}} \tag{18}$$

where

$$L_n^x(S) = \#\{i : 1 \le i \le n, S(i) = x\}, \; x = 0, \pm 1, \pm 2, \ldots, \; n = 1, 2, \ldots,$$

and $L_n^x(U(.,t))$ is given by (17), see also (24).

We will write in the following S_n in the place of $S(n)$ and $S_n(t) = \sum_{i=1}^{[nt]} \epsilon_i$, where $[x]$ denote the integer part of the real x i.e. $[x] \le x < [x] + 1$. Remark that $S_n = S_n(1)$.

Theorem 2.1. *Under the assumptions of Lemma 2.1. Let $I_n(1)$ be the self-intersection local time of the random walk S, $I_n(t)$ be the intersection local time given in (18) and let $d_n = |I_n(1) - I_n(t)|$, for $0 < t < 1$. Then, we have*

$$d_n = n^{3/2+o(1)} \; a.s.$$

Now, we consider the intersection local time of $S_n(t)$ and $U(n,t)$ for $0 < t < 1$. Recall that, we have in this case (see (6))

$$J_n(t) = \sum_{x\in\mathbb{Z}} L_n^x(S(.t) L_n^x(U(.,t)) = \sum_{x\in\mathbb{Z}} \sum \sum\nolimits_{1\le i,j\le n} 1_{\{S_i(t)=U(j,t)=x\}}$$

where

$$L_n^x(S(.t)) = \#\{i : 1 \le i \le n, S_i(t) = x\}, \; x = 0, \pm 1, \pm 2, \ldots, \; n = 1, 2, \ldots.$$

Theorem 2.2. *Assume the conditions of Lemma 2.1. Let $I_n(1)$ be the self-intersection local time of the random walk S, $J_n(t)$ be the intersection local time given in (6) and let $\delta_n(t) = |I_n(1) - J_n(t)|$, for $0 < t < 1$. Then, we have with probability one*

$$\delta_n = n^{3/2+o(1)}.$$

In the following theorem, our aim is to study the behavior of

$$\eta_n(t(n)) = |I_n(1) - K_n(t(n))|$$

(see (7)) with the sequence $\{t(n)\}_{n\ge1}$ such that $t(n) = 1 - n^{-1+\varepsilon}$. The choice of this sequence is explained, because we need to use a result on increments for local times for the random walk S.

Theorem 2.3. *Assume the conditions of Lemma 2.1. Let $I_n(1)$ be the self-intersection local time of the random walk S and let $K_n(t(n))$ be the self-intersection local time of the process $U(.,t(.))$ (see (7)). Moreover, let $t(n) = 1 - n^{-1+\varepsilon}$, $0 < \varepsilon < 1$ and let $\eta_n(t(n)) = |I_n(1) - K_n(t(n))|$. Then, we have with probability one*

$$\eta_n(t(n)) = O\left(n^{\varepsilon/2+1}\left((1-\varepsilon)\log n\right)^{1/2}\right).$$

Finally, in the next theorem we state our strong approximation result. This result is in the same vein as these given by [7] (see also [12]) on the asymptotic Gaussian behavior of the centered two-time parameter local time process associated to the random walk, given by

$$\left\{L_n^k(S) - L_n^0(S), k = 0, 1, \ldots, n = 1, 2, \ldots\right\}$$

via appropriate strong approximations in terms of a Wiener sheet and an independent standard Brownian motion. By a Wiener sheet, we mean a two-parameter Gaussian process $\{W(x,y), x \ge 0, y \ge 0\}$ with mean 0 and covariance function $EW(x_1,y_1)W(x_2,y_2) = (x_1 \wedge x_2)(y_1 \wedge y_2)$.

Theorem 2.4. *For the process A (cf. (2)) related to sequences* $\{Y_i\}_{i\geq 1}$ *and* $\{\epsilon_i\}_{i\geq 1}$ *satisfying conditions (C1), (C2) and (C3) and on a suitable probability space on which one can define all random variables and process we need, for* $x \in \left(n^{-1/2}, n^{-1/3+\epsilon}\right)$, *with* $\epsilon > 0$, *we have a.s.*

$$\left|\xi_t^x(A_n) - \xi_t^0(A_n) - \int_0^1 L_t^s(K_n)\, d_s G(x, \nu_s^0(U_n))\right| = O\left(x^{5/4} n^{5/4+6\epsilon/8} (\log_2 n)^{1/2}\right),$$

where $G(x,y) = W(x,y) + W(x-1,y) - W^*(y),\ x \geq 1, y \geq 0$ *and* W^* *is a Brownian motion independent of the Wiener sheet* $\{W(x,y), x \geq 0, y \geq 0\}$.

3. Proofs.

Throughout this section we will write S_i in the place of $S(i)$.

- **Proof of Lemma 2.1.** From the definition of the local time of $U(.,t)$, we have

$$E\left[L_n^x(U(.,t))\right] = E\left[\sum_{i=1}^n 1_{\{U(i,t)=x\}}\right] = \sum_{i=1}^n P\left(U(i,t) = x\right).$$

We can sum this expression over all $x \in Z$ to find

$$\sum_{x\in Z} E\left[L_n^x(U(.,t))\right] = \sum_{i=1}^n \sum_{x\in Z} P\left(U(i,t) = x\right) = n.$$

Moreover,

$$\begin{aligned}\sum_{x\in Z} E\left[\left|L_n^x(U(.,t))\right|^2\right] &= \sum_{1\leq i\leq n} \sum_{x\in Z} P\left(U(i,t) = x\right)\\ &+2\sum_{x\in Z}\sum_{1\leq i<j\leq n} P\left(U(i,t) = x\right) P\left(U((j-i),t) = 0\right)\\ &= n + 2\sum_{k=0}^{n-1}(n-k) P\left(U(k,t) = 0\right).\end{aligned}$$

This proves the Lemma and the proof is complete.

- **Proof of Lemma 2.2.** Remark that, in this case we have to deal almost surely with the self-intersection local time of S. Then, we have

$$I_n(1) = \sum_{i=1}^n \sum_{j=1}^n 1_{\{S_i = U(j,1)\}} = \sum_{i=1}^n \sum_{j=1}^n 1_{\{S_i = S_j)\}}$$

$$= \sum_{i=1}^n 1_{\{S_i = S_i\}} + 2\sum\sum\nolimits_{1\leq i<j\leq n} 1_{\{S_i = S_j\}}$$

$$= n + 2\sum\sum\nolimits_{1\leq i<j\leq n} 1_{\{S_i = S_j\}} \tag{19}$$

by Lemma 2.3 of Hu and Khoshnevisan (2009)(case $d = 1$), we have that the second term in the right-hand side in the last equality satisfies with probability one

$$\sum\sum\nolimits_{1\leq i<j\leq n} 1_{\{S_i = S_j\}} = \frac{1}{2}\int_{-\infty}^{+\infty} \left(L_n^x(W)\right)^2 dx + n^{5/4+o(1)}. \tag{20}$$

where $L_n^x(W)$ denote the local time at a level x of the one-dimensional standard Brownian motion given by $W = \{W(t), t \geq 0\}$, starting from $t = 0$, W is a real

valued mean zero Gaussian process with covariance $E[W(t)W(s)] = t \wedge s$.
Then, we have finally that $I_n(1)$ satisfies almost surely

$$I_n(1) = n + \int_{-\infty}^{\infty} (L_n^x(W))^2 \, dx + n^{5/4+o(1)}$$
$$= n^{3/2+o(1)},$$

where we have used also Lemma 2.2 case(d=1) of Hu and Khoshnevisan (2009), see also [3] and [4], Theorem 4.4.1., p.121 for the lim sup (resp. lim inf) result (resp. exercise 4.5.1, p.132) for the L^2 norm of the local time.
This completes the proof of the Lemma.

- **Proof of Lemma 2.3.** We begin by recalling that $U(n,t) \stackrel{d}{=} S_{N_n(t)}$ where $\stackrel{d}{=}$ denote the equality in law or distribution where $N_n(t) = \sum_{i=0}^n 1_{\{Y_i \le t\}}$. Moreover, from the definition of the local time $L_n^x(U(.,t))$ (cf. (17)), we have $S_{N_n(t)} = x$ if $N_n(t)$ $(\le n)$ is the return time to x of S_n, for $x = 0, \pm 1, \pm 2, \ldots$.
By a direct argument we have in an almost surely sense

$$\sum_{x \in \mathbb{Z}} L_n^x(S) L_n^x(U(.,t)) \le \sup_{x \in \mathbb{Z}} L_n^x(S(N.(t))) \sum_{x \in \mathbb{Z}} L_n^x(S), \tag{21}$$

now, using that $\sum_{x \in \mathbb{Z}} L_n^x(S) = n$, we must evaluate $\sup_{x \in \mathbb{Z}} L_n^x(S(N.(t)))$ in order to obtain an upper bound (a.s.) in (21).
By the Kesten's law of the iterated logarithm (see [18] and [4] p.295), we have

$$\limsup_{n \to \infty} \frac{\sup_{x \in \mathbb{Z}} L_n^x(S)}{\sqrt{2n \log_2 n}}) = 1, \ a.s. \tag{22}$$

In the light of (22) jointly with the law of large numbers given $N_n(t)/n \to t$, $a.s.$, we have

$$\sup_{x \in \mathbb{Z}} L_n^x(S(N.(t))) = \frac{\sup_{x \in \mathbb{Z}} L_n^x(S(N.(t)))}{\sqrt{2N_n(t) \log_2 N_n(t)}} \times \frac{\sqrt{2N_n(t) \log_2 N_n(t)}}{\sqrt{2n \log_2 n}} \times \sqrt{2n \log_2 n}$$
$$= O\left((nt \log_2 n)^{1/2} \right) \ a.s.,$$

then the right hand side of (21) is upper bounded by $n^{3/2} (2t \log_2 n)^{1/2}$ $a.s.$
Now, let us remark that

$$\sum_{x \in \mathbb{Z}} \left(L_{[nt]}^x(S) \right)^2 L_n^{[nt]}(N.(t)) \le \sum_{x \in \mathbb{Z}} L_n^x(S) L_n^x(U(.,t)). \tag{23}$$

This last inequality is also obtained from the fact that $L_n^x\left(S_{N_n(t)}\right))$ corresponds to the local time of the process $S_{N_n(t)}$, given by

$$L_n^x(U(.,t)) = \sum_{y \in \mathbb{Z}} L_n^y(N.(t)) L_y^x(S), \tag{24}$$

Then, it is direct to obtain the lower bound given in (23). By the same argument as in the first part of the proof, we obtain that

$$\sum_{x \in \mathbb{Z}} \left(L_{[nt]}^x(S) \right)^2 L_n^{[nt]}(N.(t)) = n^{3/2+o(1)}. \tag{25}$$

By using (23) and (25), we have the announced result.

Remark 3.1. By using the law of the iterated logarithm (LIL) for S, we have that $L_n^x(S) = 0$, $a.s.$ for $x > \sqrt{2n \log_2 n}$ and in the same way, we have that $L_n^x(U(.,t)) = 0$, $a.s.$ for $x > \sqrt{2n \log_2 n}$.

- **Proof of Lemma 2.4.** The proof of this Lemma is obtained from Lemma 2, by using the fact that

$$I_n(0) = L_n^0(S),$$

and by using the Kesten's LIL i.e. $L_n^0(S) = O\left((n \log_2 n)^{1/2}\right)$, *a.s.*

- **Proof of Theorem 2.1.** Recall that $d_n(t)$ is given by

$$\begin{aligned} d_n(t) &= \left|\sum\sum\nolimits_{1\le i,j\le n} 1_{\{S_i=U(j,1)\}} - \sum\sum\nolimits_{1\le i,j\le n} 1_{\{S_i=U(j,t)\}}\right| \\ &= \left|\sum\sum\nolimits_{1\le i,j\le n} 1_{\{S(i)=S(j)\}} - \sum\sum\nolimits_{1\le i,j\le n} 1_{\{S_i=U(j,t)\}}\right| \\ &= d_{(1,n)} - d_{(2,n)} \end{aligned}$$

where $d_{(1,n)} = \sum\sum_{1\le i,j\le n} 1_{\{S(i)=S(j)\}}$ and $d_{(2,n)} = \sum\sum_{1\le i,j\le n} 1_{\{S_i=U(j,t)\}}$.

Now, the first term denoted by $d_{(1,n)}$ corresponds to the self-intersection local time of the random walk S. By Lemma 2.2, we have that

$$d_{(1,n)} = n^{3/2+o(1)},\ a.s.$$

The second term denoted by $d_{(2,n)}$, satisfies

$$d_{(2,n)} = \sum\sum\nolimits_{1\le i,j\le n} 1_{\{S_i=U(j,t)\}} = t^{1/2} n^{3/2+o(1)},\ a.s.,$$

the last equality is obtained as in the proof of Lemma 2.3.

Then

$$d_n = n^{3/2+o(1)},\ a.s.$$

Then we have obtained the announced result.

- **Proof of Theorem 2.2.** Recall that the behavior of $I_n(1)$ is given in Lemma 2. Now, let us recall that $J_n(t)$, consider the intersection local time of $S_n(t) = \sum_{i=1}^{[nt]} \epsilon_i$ and $U(n,t)$ for $0 < t < 1$ and is defined by

$$J_n(t) = \sum_{x\in\mathrm{Z}} L_n^x(S(.t)) L_n^x(U(.,t)) = \sum_{x\in\mathrm{Z}} \sum\sum\nolimits_{1\le i,j\le n} 1_{\{S_i(t)=U(j,t)=x\}}$$

where

$$L_n^x(S(.t)) = \#\{i : 1 \le i \le n, S_i(t) = x\},\ x = 0, \pm1, \pm2, \ldots,\ n = 1, 2, \ldots.$$

In the same way as in the proof of Lemma 2.3, we obtain the behavior of $J_n(t)$. This concludes the proof of the Theorem.

- **Proof of Theorem 2.3.** Our aim here is to give an upper bound for the distance $\eta_n(t(n))$ defined by

$$\eta_n(t(n)) = \left|\sum_{x\in\mathrm{Z}} \left((L_n^x(S))^2 - (L_n^x(U(.,t(n))))^2\right)\right|,$$

where $\sum_{x\in\mathrm{Z}} (L_n^x(S))^2$ is the self-intersection local time of the random walk S (see (4)), and $\sum_{x\in\mathrm{Z}} (L_n^x(U(.,t(n))))^2$ the self intersection local time of the process $U(.,t)$ (see (7)). Remark that $\eta_n(t(n))$ satisfies that

$$\eta_n(t(n)) \le \eta_{1,n} \times \eta_{2,n},$$

where

$$\eta_{1,n} = \sup_{x\in\mathrm{Z}} |L_n^x(S) - L_n^x(U(.,t(n)))|$$

and

$$\eta_{2,n} = \sum_{x\in\mathrm{Z}} (L_n^x(S) + L_n^x(U(.,t(n)))).$$

In the following, we will study separately the behavior of $\eta_{1,n}$ and $\eta_{2,n}$.
We begin by the evaluation of the first term denoted by $\eta_{1,n}$. As in the proof of

Lemma 2.3, by using that $U(n,t) \stackrel{d}{=} S_{N_n(t)}$, Remember that as in the proof of Lemma 2.3, we have $S_{N_n(t)} = x$ if $N_n(t(n))$ $(\leq n)$ is a return time to x of S, for $x = 0, \pm 1, \pm 2, \ldots$. Then, $\eta_{1,n}$ can be seen as

$$\sup_{x \in \mathrm{Z}} \left| L^x_{N_n(t(n))+h_n}(S) - L^x_{N_n(t(n))}(S) \right|$$

where, we have replaced $L^x_n(S)$ by $L^x_{N_n(t(n))+h_n}(S)$ with $h_n = n - N_n(t(n))$ and $L^x_n(S_{N_n(t)})$ by $L^x_{N_n(t(n))}(S)$.
From Theorem 1 of Csáki and Földes (1983) (see also Theorem 11.13 of [20]), we have with probability one

$$\sup_{x \in \mathrm{Z}} \left| L^x_{N_n(t(n)+h_n}(S) - L^x_{N_n(t(n))}(S) \right| = O\left(\sqrt{h_n \left(\log \frac{n}{h_n} + 2 \log_2 n \right)} \right) \tag{26}$$

where we take $h_n = [n(1 - t(n))]$ *a.s.* The choice of the above h_n is motivated by the fact that $h(n)$ as in Theorem 1 of Csáki and Földes (1983) is an integer value non-decreasing function of n, under condition that h_n/n is non-increasing and such that $\lim_{n\to\infty} h_n/\log n = \infty$. Moreover roughly speaking, we obtain $h_n = n - N_n(t(n))$ in (26), but by the Khintchine's LIL, we know that with probability one

$$|N_n(t(n)) - nt(n)| \leq \sqrt{2n \log_2 n},$$

where $nt(n) \to n$. This explains our choice of $h_n = [n(1 - t(n))]$ *a.s.*
Now, in the light of (26) and replacing $t(n)$, we have $h_n = n^{\varepsilon}(1 + o(1))$, *a.s.*, then we have

$$\eta_{1,n} = O\left(n^{\varepsilon/2} \left((1 - \varepsilon) \log n) + 2 \log_2 n \right)^{1/2} \right), \ a.s. \tag{27}$$

Now, we study the behavior of the second term denoted by $\eta_{2,n}$, namely giving an upper bound for this. Recall that in this case, we must evaluate

$$\eta_{2,n} = \sum_{x \in \mathrm{Z}} \left(L^x_n(S) + L^x_n(S_{N_n(t)}) \right).$$

As in the proof of Theorem 2.1, we have that $L^x_n(S(N_n(t(n))) \leq L^x_n(S)$ then

$$\eta_{2,n} \leq 2 \sum_{x \in \mathrm{Z}} L^x_n(S),$$

we have finally

$$\eta_{2,n} = O\left(n\right), \ a.s. \tag{28}$$

In the light of (27) and (28), we obtain finally that with probability one

$$\eta_n(t(n)) = O\left(n^{\varepsilon/2+1} \left((1 - \varepsilon) \log n) \right)^{1/2} \right).$$

This last relation gives us the announced result.

Remark 3.2. . By the approximations for the hybrid process given by Horváth (2000) and by Theorem 1.2.1 of Csörgő-Révész (1981), for $k_n = n(1-t(n)) = n^{\varepsilon}$, we have with probability one

$$\sup_{0 \leq t \leq n-k_n} \sup_{0 \leq s \leq n-k_n} |W(t+s) - W(t)| = O\left(\sqrt{2k_n \left(\log \frac{n}{k_n} + \log_2 n \right)} \right).$$

Proof of Theorem 2.4.
The proof of the theorem is a direct consequence of the three following Lemma's. Remark that the centered two-time parameter local time process $\xi_t^x(A_n) - \xi_t^0(A_n)$ can be written as

$$\xi_t^x(A_n) - \xi_t^0(A_n) = \int_0^1 \delta_t^s(F_n) d_s \left(\nu_s^x(U_n) - \nu_s^0(U_n)\right). \tag{29}$$

Now, we will establish an almost sure representation of the local time $\delta_t^1(F_n)$.

Lemma 3.1. *Under condition $(C2)$, let $\delta_t^1(F_n)$ given by (9) and let Ξ be a Kiefer process and $L_t^x(\Xi_n)$ be the related local time (see (13)), all defined on an appropriate probabilty space. Then*

$$\delta_t^1(F_n) = L_t^{\sqrt{n}\mu}(\Xi_n) + o\left(\frac{1}{n^{1/10-\epsilon}}\right), \; a.s.$$

for all $\mu \in (0,1)$.

Proof of lemma 3.1. To establish this result, we make use the following arguments. Our definition of the local time in (8) follows from the definition of the local time of the compensated compound Poisson process $\{Z(t)\}$ given in (1.1) of [17], see also [1]. More precisely, by (3.1) of [16], we have that

$$\{\alpha_n(t);\; t \geq 0\} \equiv \{Z_n(t);\; t \geq 0 \;|Z_n(1) = 0\}, \tag{30}$$

where $\alpha_n(t)$ is the uniform empirical process and $Z_n(t) = Z(nt)/\sqrt{n}$ with $\{Z_n(t)\}_{n\geq 1}$ is a sequence of compensated Poisson process with expected arrival rate of $1/n$. Remark that $\{Z_n(1) = 0\} \equiv \{N(n) = n\}$ for $N(n)$ a Poisson random variable with mean n.
To study (9), in the light of crossing comparison (see p. 339 of [16]), we can use

$$1_{\{\alpha_n(t)\equiv Z_n(t)|Z_n(1)=0\}} \Leftrightarrow 1_{\left\{F_n(t)\equiv \frac{1}{N_n(n)} N_n(t)|N_n(n)=n\right\}}. \tag{31}$$

Then by using the precedent arguments and (13), we obtains finally that $\delta_t^1(F_n)$ can be written almost surely as

$$L_t^{\sqrt{n}\mu}(\Xi_n) + o\left(\frac{1}{n^{1/10-\epsilon}}\right), \; \mu \in (0,1).$$

The parameter $\mu \in (0,1)$ is obtained when we replace $\delta_t^1(F_n)$ by the local time of the uniform empirical process. Let us indicate, that there are some asymptotic results of the local time of the uniform empirical process in [11].

Let us recall that

$$\limsup_{n\to\infty} \frac{L_1^0(\Xi_n)}{\sqrt{2\log_2 n}} = \limsup_{n\to\infty} \frac{L_1^*(\Xi_n)}{\sqrt{2\log_2 n}} = 1, \; a.s.$$

and

$$\liminf_{n\to\infty} \sqrt{\log_2 n} L_1^*(\Xi_n) = \sqrt{2}\pi, \; a.s.,$$

where $L_1^*(\Xi_n) = \sup_x L_1^x(\Xi_n)$. In the two last relations n need not be integer-valued (cf. [2]).

Lemma 3.2. *Under the same conditions as in Theorem 2.4, we have with probability one*

$$g_n(t) = \left|\int_0^1 \left\{\delta_t^{s\sqrt{n}}(\alpha_n) - L_t^{s\sqrt{n}}(\Xi_n)\right\} d_s \left(\nu_s^x(V_n) - \nu_s^0(V_n)\right)\right|$$
$$= O\left(\left(2x\sqrt{n}-1\right)^{1/2} n^{2/5+\epsilon} \log_2 n\right).$$

Proof of Lemma 3.2. Let us recall that $\nu_s^x(V_n)-\nu_s^0(V_n)=L_{sn}^{[\sqrt{n}x]}(S)-L_{sn}^0(S),\quad s\in[0,1]$, a.s. Then, we have with probability one, as $n\to\infty$

$$g_n(t)=O\left(\frac{\left(2\left[\sqrt{n}x\right]-1\right)^{1/2}\left(\nu_1^0(V_n)\log_2 n\right)^{1/2}}{n^{1/10-\epsilon}}\right),$$

is a direct consequence of (13) and Theorem C2 of [7] (see also (4.1a) of [8]) and by the law of the iterated logarithm for $\nu_n^0(S_n)$, we get finally the announced result.

Lemma 3.3. *Under the same conditions as in Theorem 2.4. We have a.s.*

$$\left|\int_0^1 L_t^{s\sqrt{n}}(\Xi_n)d_s\left(\nu_s^x(V_n)-\nu_s^0(V_n)\right)-\int_0^1 L_t^{s\sqrt{n}}(\Xi_n)d_sG\left(\left[\sqrt{n}x\right],\nu_s^0(U_n)\right)\right|$$
$$=O\left(x^{5/4}n^{5/4+6\epsilon/8}\right)$$

Proof of Lemma 3.3. Remark that

$$d_n(t)=\left|\int_0^1 L_t^{s\sqrt{n}}(\Xi_n)d_s\left(\nu_s^x(V_n)-\nu_s^0(V_n)\right)-\int_0^1 L_t^{s\sqrt{n}}(\Xi_n)d_sG\left(\left[\sqrt{n}x\right],\nu_s^0(V_n)\right)\right|$$
$$=\left|\int_0^1 L_t^{s\sqrt{n}}(\Xi_n)d_s\left(L_{sn}^{[\sqrt{n}x]}(S)-L_{sn}^0(S)-G\left(\left[\sqrt{n}x\right],L_{sn}^0(S)\right)\right)\right|$$

and by given an upper estimation obtained from Stieljes integral, we have

$$d_n(t)\le d_{1,n}(t)+d_{2,n}(t)$$

where

$$d_{1,n}(t)=\left|L_t^{\sqrt{n}}(\Xi_n)\left(L_n^{[\sqrt{n}x]}(S)-L_n^0(S)-G\left(\left[\sqrt{n}x\right],L_n^0(S)\right)\right)\right|$$

and

$$d_{2,n}(t)=\sup_{0\le s\le 1}\left|L_{sn}^{[\sqrt{n}x]}(S)-L_{sn}^0(S)-G\left(\left[\sqrt{n}x\right],L_{sn}^0(S)\right)\right|\int_0^1 d_sL_t^{s\sqrt{n}}(\Xi_n).$$

Now, by Theorem 1.1 of [7], we have for $n^{-1/2}\le x\le n^{-1/3+\epsilon}$ and in an almost surely sense that

$$\left|L_n^{[\sqrt{n}x]}(S)-L_n^0(S)-G\left(\left[\sqrt{n}x\right],L_n^0(S)\right)\right|=O\left(x^{5/4}n^{3/4+5\epsilon/8}\right),$$

moreover, from (13), we have a.s. as $n\to\infty$

$$\sup_{0\le t\le 1}\left|L_t^{s\sqrt{n}}(\Xi_n)\right|=\sup_{0\le t\le 1}\left|L_t^{s\sqrt{n}}(\alpha_n)\right|+O(n^{-1/10+\epsilon})$$
$$=O\left(\sqrt{\log_2 n}\right),$$

the last equality is obtained from Theorem 1.1 of [2]. Then

$$d_{1,n}(t)=O\left(x^{5/4}n^{3/4+5\epsilon/8}\left(\log_2 n\right)^{1/2}\right),\ a.s.$$

For the term denoted by $d_{2,n}(t)$ and tacking in mind again Theorem 1.1 of [7] (uniform case), then

$$d_{2,n}(t)=O\left(x^{5/4}n^{3/4+5\epsilon/8}\right),\ a.s.$$

where we have used also $\int_0^1 d_sL_t^{s\sqrt{n}}(\Xi_n)=O(1)$.

This concludes the proof of the Lemma.

The proof of Theorem 2.4 is obtained by Lemma's 3.1, 3.2 and 3.3.

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