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POSITIVITY OF SOLUTION OF NONHOMOGENEOUS STOCHASTIC DIFFERENTIAL EQUATION WITH NON-LIPSCHITZ DIFFUSION

We give a sufficient condition on coefficients of a nonhomogeneous stochastic differential equation with non-Lipschitz diffusion for a solution starting from arbitrary nonrandom positive point to stay positive. Some examples of application of the condition mentioned above are considered.

1. Introduction

The main goal of this paper is to investigate the question about positivity of solution of nonhomogeneous stochastic differential equation with non-Lipschitz diffusion. Such stochastic differential equations arise in modelling asset prices and interest rates on financial markets.

For example, Cox-Ingersoll-Ross interest rate model has the form:

$$r_t = r_0 + \int_0^t a(b - r_s)ds + \int_0^t \sigma\sqrt{r_s}dW_s,\ t \geq 0,$$

where r_0, a, σ are real positive constants. It is easy to see that for $r_t > b$ the drift is negative and for $r_t < b$ it is positive, so the solution of this equation is mean-reverting.

Positivity of solutions of the stochastic differential equation with homogeneous coefficients of the form

$$X(t) = X_0 + \int_0^t b(X(s))ds + \int_0^t \sigma(X(s))dW(s),\ t \geq 0 \qquad (1)$$

was studied in [5], Chapter VI. This property is quite important for processes, which model interest rate dynamics on financial market, because the interest rate must be positive.

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The paper is organized as follows. Section 2 is devoted to nonhomogeneous stochastic differential equations with non-Lipschitz diffusion and contains the sufficient condition of positivity of a solution starting from a nonrandom positive value. In the paper [2] such sufficient condition was proved in a particular case, where the diffusion was of the form $\sigma(t,x)=\sigma(t)\sqrt{x}$.

Section 3 contains some examples of application of the sufficient condition mentioned above, in particular we consider a nonhomogeneous version of Cox-Ingersoll-Ross model.

2. Positivity of solution of nonhomogeneous stochastic differential equation with non-Lipschitz diffusion

Consider a stochastic differential equation

$$X(t)=X_0+\int_0^t b(s,X(s))ds+\int_0^t \sigma(s,X(s))dW(s),\ t\geq 0, \tag{2}$$

where the initial value $X_0>0$ is nonrandom, the coefficients b, $\sigma:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ are measurable, $\{W(t),t\geq 0\}$ is a Wiener process with respect to filtration $\{\mathcal{F}_t,t\geq 0\}$ on a probability space $(\Omega,\mathcal{F},P)$.

Assume that the coefficients of this equation satisfy the following Yamada conditions (Y1)–(Y4) (see e.g. [5,6]):

(Y1) the functions b and σ are jointly continuous.

(Y2) the coefficients grow at most linearly in x

$$|b(t,x)|+|\sigma(t,x)|\leq C(1+|x|),\quad t\geq 0,\ x\in\mathbb{R}.$$

(Y3) the drift is Lipschitz continuous

$$|b(t,x)-b(t,y)|\leq C\,|x-y|\,,\quad t\geq 0,\ x,y\in\mathbb{R}.$$

(Y4) there exists such an increasing function $\rho:\mathbb{R}_+\to\mathbb{R}_+$, that $\int_{0+}\rho^{-2}(u)du=+\infty$, and

$$|\sigma(t,x)-\sigma(t,y)|\leq C\rho(|x-y|),\quad t\geq 0,\ x,y\in\mathbb{R}.$$

Remark 2.1. An example of such function ρ is $\rho(x)=x^\alpha$, $\frac{1}{2}\leq\alpha\leq 1$, that is why this condition is a kind of Hölder condition for σ. Other examples are: $\rho(x)=x^{1/2}(\ln 1/x)^{1/2}$, $\rho(x)=x^{1/2}(\ln 1/x)^{1/2}(\ln\ln 1/x)^{1/2}$ etc.

Definition 2.1. We say that the pair (X,W) is a *strong* solution of the equation (2), if X is the process adapted to the filtration $\left\{\bar{\mathcal{F}}_t^W\right\}$ generated by process W.

Definition 2.2. The equation (2) has the property of uniqueness of trajectories if any two solutions (X, W) and $(\widetilde{X}, W)$, adapted to the same filtration, satisfy the equality

$$P(\forall t > 0 : \ X(t) = \widetilde{X}(t)) = 1.$$

Theorem 2.1.([6]) *Under conditions* (Y1)–(Y4) *the equation* (2) *has the unique (relatively to its trajectories) solution from the class* $\mathcal{L}_2(\Omega\times[0,T], \mathcal{F}\otimes B([0,T]), P \times \lambda)$ *for all* $T > 0$*, there* λ *is Lebesgue measure on* $\mathbb{R}_+$*.*

Assume that for all $x > 0$ and $t > 0$ $\sigma^2(t, x) > 0$, and that for all $z > 0$ the following inequality holds:

$$\inf_{x\in[z,+\infty)} \inf_{t>0} \sigma^2(t, x) = d = d(z) > 0. \tag{3}$$

Let also for some $\varepsilon > 0$ there exist a positive continuous function $A(t), t > 0$, such that for all $t > 0$ and $x \in (0, \varepsilon)$

$$\frac{2b(t, x)}{\sigma^2(t, x)} \geq \frac{A(t)}{x}. \tag{4}$$

The main result of the paper is the following one.

Theorem 2.2. *Under the assumptions* (Y1)–(Y4), (3)-(4) *and the condition*

$$\forall t > 0 \quad A(t) > 1 \quad (\textit{equivalently} \quad \forall T > 0 \quad \inf_{(0,T]\times(0,\varepsilon)} \frac{2b(t, x)x}{\sigma^2(t, x)} > 1)$$

the trajectories of the process $\{X(t), t \geq 0\}$ *will be positive with probability* 1*.*

Remark 2.2. For homogeneous Cox-Ingersoll-Ross model described by the stochastic differential equation

$$dr_t = a(b - r_t)dt + \sigma\sqrt{r_t}dW_t$$

with a initial condition $r_0 > 0$, the necessary and sufficient condition providing positivity of the solution is $ab \geq \frac{\sigma^2}{2}$ (see [5]), and in a nonhomogeneous case, if $b(t, x) = a(t)(b(t) - x)$, $\sigma(t, x) = \sigma(t)\sqrt{x}$, then the coefficient $A(t)$ is $A(t) = \frac{2a(t)b(t)}{\sigma^2(t)}$. So we obtain the condition $\frac{2a(t)b(t)}{\sigma^2(t)} > 1$ or $a(t)b(t) > \frac{\sigma^2(t)}{2}$, that is a generalization of the sufficient condition for the homogeneous case (a corresponding sufficient condition for nonhomogeneous Cox-Ingersoll-Ross model also is proved in [2])

Proof. At first we prove that the solution of the equation (2) is a semimartingale. Indeed, under condition of linear growth the process $\{Y(t), t \geq 0\}$ of the form

$$Y(t) = \int_0^t \sigma(s, X(s))dW(s)$$

is a local martingale, because

$$\int_0^t \sigma^2(s, X_s)ds \leq \int_0^t C^2(1+X_s)^2 ds \leq 2C^2 t + 2C^2 \int_0^t X_s^2 ds < +\infty \ P-\text{a.s.},$$

where the last inequality is a consequence of Theorem 2.1. So,

$$P\{\int_0^t \sigma^2(s, Y_s)ds < +\infty\} = 1$$

means that $\{Y(t), t \geq 0\}$ is a local martingale. Now let us show that $Z(t) = \int_0^t b(s, X(s))ds$, $t \geq 0$ is the process of bounded variation. Indeed, from the condition of linear growth (Y2) we can see that

$$\int_0^t |b(s, X_s)|ds \leq \int_0^t C(1 + |X_s|)ds = Ct + C \int_0^t |X_s|ds < +\infty,$$

where the last inequality follows by Theorem 2.1.

So, the process $\{X(t), t \geq 0\}$ is a sum of a local martingale and a process of bounded variation, that is why it is a semimartingale,

It is sufficient to prove that the trajectories of the process X are positive on any $[0, T]$. So, let $T > 0$ be fixed.

Let $\varepsilon > 0$ be the constant from the condition (4). For $x \in (\varepsilon, +\infty)$ by the condition $(Y2)$ there exists a constant $K > 0$ such that

$$|b(t, x)| \leq K(1 + x).$$

We denote

$$p := K\Big(1 + \frac{1}{\varepsilon}\Big),$$

and then obtain the following inequality

$$\forall x \in (\varepsilon, +\infty) \quad |b(t, x)| \leq px.$$

We define

$$V(x) = \begin{cases} \ln x, & \text{if } 0 < x < \varepsilon, \\ \frac{1}{\varepsilon} \exp\{-\frac{p\varepsilon^2}{2d}\} \int_\varepsilon^x \exp\{\frac{pu^2}{2d}\}du + \ln \varepsilon, & \text{if } \varepsilon \leq x < \infty. \end{cases}$$

It is easy to see that the function $V(x)$ is continuously differentiable on $(0, +\infty)$ and $V'(x)$ has the form

$$V'(x) = \begin{cases} \frac{1}{x}, & \text{if } 0 < x < \varepsilon, \\ \frac{1}{\varepsilon} \exp\{-\frac{p\varepsilon^2}{2d}\} \exp\{\frac{px^2}{2d}\}, & \text{if } \varepsilon \leq x < \infty. \end{cases}$$

We consider the differential operator

$$L = \frac{1}{2}\sigma^2(t,x)\frac{\partial^2}{\partial x^2} + b(t,x)\frac{\partial}{\partial x}.$$

Under condition (4) we have that for all $x \in (0,\varepsilon)$ the following holds

$$LV = \frac{1}{2}\sigma^2(t,x)(-\frac{1}{x^2}) + b(t,x)\frac{1}{x} = \frac{1}{2x}\sigma^2(t,x)(\frac{2b(t,x)}{\sigma^2(t,x)} - \frac{1}{x}) \geq$$

$$\geq \frac{1}{2x}\sigma^2(t,x)(\frac{A(t)}{x} - \frac{1}{x}) = \frac{1}{2x}\sigma^2(t,x)\frac{A(t)-1}{x} > 0.$$

For $x \in (\varepsilon, +\infty)$ the following inequality takes place

$$LV = \frac{1}{2}\sigma^2(t,x)\exp\{-\frac{p\varepsilon^2}{2d}\}\frac{px}{d\varepsilon}\exp\{\frac{px^2}{2d}\} + b(t,x)\frac{1}{\varepsilon}\exp\{-\frac{p\varepsilon^2}{2d}\}\exp\{\frac{px^2}{2d}\} \geq$$

$$\geq d\frac{px}{d\varepsilon}\exp\{-\frac{p\varepsilon^2}{2d}\}\exp\{\frac{px^2}{2d}\} - px\frac{1}{\varepsilon}\exp\{-\frac{p\varepsilon^2}{2d}\}\exp\{\frac{px^2}{2d}\} = 0.$$

Thus $LV \geq 0$ for all $x > 0, x \neq \varepsilon$.

Let $0 < m < X_0 < M$ be fixed constants. We consider the random variable

$$\tau_{m,M} = \inf\{t : X(t) = m \quad X(t) = M\}$$

We use the following generalization of Ito's formula from the article [3]. Let $X = X(t), t \geq 0$ be a continuous semimartingale and let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function of bounded variation.

We define $l_s^f(X)$ as the local time of the process X on the curve f of the form:

$$l_s^f(X) = P - \lim_{\delta \downarrow 0}\frac{1}{2\delta}\int_0^s I(f(r) - \delta < X_r < f(r) + \delta)d < X, X >_r .$$

We make a remark that in our case the curve is given by the equation $f(s) = \varepsilon$, because the second derivative of the function V has a break only at this point. The process $\{X(t), t \geq 0\}$ is a semimartingale from the proof above. That is why for any semimartingale, also for the process $\{X(t), t \geq 0\}$, the local time exists at any point, it follows from Theorem 5.52 on the page 186 in [1]. So, $l_s^f(X) = l_s^\varepsilon(X)$ exists.

We have as a consequence of Theorem 2.1 from the paper [3] that if we set $C = [0,\varepsilon]$ and $D = [\varepsilon, +\infty)$ (and taking into account that the function V does not depend on t and $V'(x)$ is continuous on interval $[0,+\infty)$ in our case), we obtain

$$V(X(\tau_{m,M} \wedge T)) = V(X_0) + \int_0^{\tau_{m,M}\wedge T} V'(X(s))dX(s) +$$

$$+\frac{1}{2}\int_0^{\tau_{m,M}\wedge T} V''(X(s))I(X(s)\neq\varepsilon)d<X,X>_s=V(X_0)+$$

$$+\int_0^{\tau_{m,M}\wedge T} V'(X(s))b(s,X(s))ds+\int_0^{\tau_{m,M}\wedge T} V'(X(s))\sigma(s,X(s))dW(s)+$$

$$+\frac{1}{2}\int_0^{\tau_{m,M}\wedge T} V''(X(s))I(X(s)\neq\varepsilon)\sigma^2(s,X(s))ds,$$

where we have used the equality

$$d<X,X>_s=\sigma^2(s,X(s))ds.$$

Functions $b(t,x)$ and $\sigma(t,x)$ satisfy Yamada conditions. The function $b(t,x)$ is bounded for $x\in[m,M]$ and separated from 0 and from ∞; $\sigma(t,\cdot)$ is uniformly at Hölder continuous. Then it follows from Theorem 3.2.1 and Corollary 3.2.2 in [4] that the transition probability function $P(s,x;t,\Gamma)$ for the process $\{X(t),t\geq 0\}$ exists and has the density

$$P(s,x;t,\Gamma)=\int_\Gamma p(s,x;t,y)dy,$$

where $p(s,x;t,y)$, $0\leq s<t$, $x,y\in\mathbb{R}$ is a positive function, which is continuous in all variables. We set $s:=0, x:=X_0$ and obtain that the distribution function of the process $\{X(t),t\geq 0\}$ has the form

$$P_1(t,\Gamma)=\int_\Gamma p_1(t,y)dy.$$

Thus, the process $\{X(t),t\geq 0\}$ has a density, so

$$P(X(t)=\varepsilon)=0,\quad \forall t>0.$$

Further,

$$0\leq |E\int_0^{\tau_{m,M}\wedge T}(V'(X(s))b(s,X(s))I(X(s)=\varepsilon)ds|\leq$$

$$\leq |V'(\varepsilon)|\max_{s\in[0,T]}|b(s,\varepsilon)|\int_0^T P(X(s)=\varepsilon)ds=0,$$

so,

$$E\int_0^{\tau_{m,M}\wedge T}(V'(X(s))b(s,X(s))I(X(s)=\varepsilon)ds=0.$$

Take an expectation of both sides

$$V(X(\tau_{m,M}\wedge T))=V(X_0)+\int_0^{\tau_{m,M}\wedge T} V'(X(s))b(s,X(s))ds+$$

$$+\int_0^{\tau_{m,M}\wedge T} V'(X(s))\sigma(s,X(s))dW(s)+$$

$$+\frac{1}{2}\int_0^{\tau_{m,M}\wedge T} V''(X(s))I(X(s)\neq\varepsilon)\sigma^2(s,X(s))ds,$$

and receive:

$$E(V(X(\tau_{m,M}\wedge T))) = V(X_0) + E\int_0^{\tau_{m,M}\wedge T} V'(X(s))b(s,X(s))ds+$$

$$+E\int_0^{\tau_{m,M}\wedge T}\frac{1}{2}V''(X(s))I(X(s)\neq\varepsilon)\sigma^2(s,X(s))ds =$$

$$= V(X_0) + E\int_0^{\tau_{m,M}\wedge T} V'(X(s))b(s,X(s))I(X(s)=\varepsilon)ds+$$

$$+E\int_0^{\tau_{m,M}\wedge T}(V'(X(s))b(s,X(s)) + \frac{1}{2}V''(X(s))\sigma^2(s,X(s))I(X(s)\neq\varepsilon)ds =$$

$$= V(X_0) + E\int_0^{\tau_{m,M}\wedge T}(V'(X(s))b(s,X(s))+$$

$$+\frac{1}{2}V''(X(s))\sigma^2(s,X(s))I(X(s)\neq\varepsilon)ds.$$

The function $V(x)$ is bounded from above on $[0,M]$, so we have

$$E(V(X(\tau_{m,M}\wedge T))) \le \max_{x\in[0,M]} V(x)P(\tau=\tau_M\wedge T) + V(m)P(\tau=\tau_m) \le$$

$$\le \max_{x\in[0,M]} V(x) + V(m)P(\tau=\tau_m),$$

where τ_M (τ_m) is the first moment that the boundary m (boundary M) is reached and $\tau=\tau_m\wedge\tau_M\wedge T$.

Thus,

$$\max_{x\in[0,M]} V(x) + V(m)P(\tau=\tau_m) \ge V(X_0)+$$

$$+E(\int_0^{\tau_{m,M}\wedge T}(V'(X(s))b(s,X(s)) + \frac{1}{2}V''(X(s))\sigma^2(s,X(s)))I(X(s)\neq\varepsilon)ds)$$

$$= V(X_0) + E(\int_0^{\tau_{m,M}\wedge T} LV(X(s))I(X(s)\neq\varepsilon)ds) \ge 0.$$

Let m tends to 0 and we obtain from above that

$$V(m)\to-\infty,\ m\to 0,$$

so the left-hand side of the inequality tends to $-\infty$ and at the same time the right-hand side of the inequality is nonnegative. Thus,

$$P(\tau=\tau_0)=0. \tag{5}$$

So, for any fixed M and T the equality(5) holds. The proof follows when M, T tend to $+\infty$. $\square$

4. Examples

Consider examples of application of Theorem 2.2 to some stochastic differential equations.

Example 3.1.
Consider a stochastic differential equation

$$X(t) = X_0 + \int_0^t (p(s) - q(s)X(s))ds + \int_0^t c(t)(X(s))^{\alpha} dW(s),\ t \geq 0, \quad (6)$$

where the initial value $X_0 > 0$ is nonrandom, the functions $p(t)$, $q(t)$ and $c(t)$ are positive, continuous and $\inf_{t>0} c(t) = d > 0$, a constant $\frac{1}{2} \leq \alpha < 1$. Then in our notations

$$b(t,x) = p(t) - q(t)x,$$

$$\sigma^2(t,x) = c^2(t)x^{2\alpha}.$$

Thus, the left-hand part of inequality (4) can be rewritten in the form

$$\frac{2b(t,x)}{\sigma^2(t,x)} = \frac{2(p(t) - q(t)x)}{c^2(t)x^{2\alpha}} = \frac{2p(t)}{c^2(t)x^{2\alpha}} - \frac{2q(t)}{c^2(t)}x^{1-2\alpha} \geq \frac{2p(t)}{c^2(t)x^{2\alpha}}$$

Then, we put in (4) $\varepsilon = 1$, $A(t) = \frac{2p(t)}{c^2(t)}$. For $\frac{1}{2} \leq \alpha < 1$ we have that

$$\frac{1}{x^{2\alpha-1}} \frac{A(t)}{x} \geq \frac{A(t)}{x},$$

because $2\alpha - 1 > 0$ and $x \in (0, \varepsilon) = (0, 1)$. So, by Theorem 2.2 we receive, under the condition $A(t) > 1$ for all $t > 0$, which in our case has the form

$$\forall t > 0 : \quad p(t) > \frac{c^2(t)}{2},$$

that the trajectories of the process $\{X(t), t \geq 0\}$ are positive with probability 1.

Example 3.2.
Consider a stochastic differential equation

$$X(t) = X_0 + \int_0^t (p(s) - q(s)X(s))ds +$$

$$+\int_0^t \sqrt{(X(s)\vee 0)((c(t)-X(s))\vee 0)}dW(s),\ t\geq 0, \tag{7}$$

where the initial value $0 < X_0 < c(0)$ is nonrandom, the functions $p(t)$, $q(t)$ and $c(t)$ are positive, continuous and $c(t)$ is also nondecreasing. Assume that

$$p(t) < q(t)c(t) \quad \forall t > 0.$$

Then we have in our notations

$$b(t,x) = p(t) - q(t)x,$$

$$\sigma(t,x) = \sqrt{(x\vee 0)((c(t)-x)\vee 0)}.$$

Let us prove that such stochastic differential equation has a solution. It is sufficient to verify the fulfillment of Yamada conditions on the coefficients. We can consider the interval $t\in[0,T]$ and then tend T to $+\infty$.

(Y1) From the initial condition on functions $p(t)$, $q(t)$ and $c(t)$ we can see that $b(t,x)$ and $\sigma(t,x)$ are jointly continuous.

(Y2)

$$|b(t,x)| + |\sigma(t,x)| \leq p(t) + q(t)|x| + |x| + c(t) + |x| = p(t) + c(t) +$$

$$+|x|(1+q(t)) \leq \max_{t\in[0,T]}(p(t)+c(t), 1+q(t))(1+|x|)$$

(Y3)

$$|b(t,x) - b(t,y)| = |q(t)(x-y)| \leq \max_{t\in[0,T]} q(t)|x-y|$$

(Y4)

$$|\sigma(t,x)-\sigma(t,y)| = |\sqrt{x(c(t)-x)\vee 0} - \sqrt{y(c(t)-y)\vee 0}|$$

Let the function $g:\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be defined as

$$g(t,x,y) = |\sqrt{x(c(t)-x)\vee 0} - \sqrt{y(c(t)-y)\vee 0}|.$$

We assume, that $t\in[0,T]$, and consider the next cases.

1) If ($x<0$ or $x>c(t)$) and ($y<0$ or $y>c(t)$) then

$$g(t,x,y) = 0 \leq \sqrt{c(t)}\sqrt{|x-y|}.$$

2) If $0\leq x\leq c(t)$ and $y<0$ then

$$g(t,x,y) = |\sqrt{x(c(t)-x)}| \leq |\sqrt{(x-y)(c(t)-x)}| \leq |\sqrt{(x-y)c(t)}| =$$

$$= \sqrt{c(t)}\sqrt{|x-y|}.$$

Note that under the symmetry the same estimation for $g(t,x,y)$ holds when $x < 0$ and $0 \le y \le c(t)$.

3) If $0 \le x \le c(t)$ and $y > c(t)$ then

$$g(t,x,y) = |\sqrt{x(c(t)-x)}| \le |\sqrt{x(y-x)}| \le |\sqrt{c(t)(y-x)}| =$$
$$= \sqrt{c(t)}\sqrt{|x-y|}.$$

Note that under the symmetry the same estimation for $g(t,x,y)$ holds when $x > c(t)$ and $0 \le y \le c(t)$.

4) If $0 \le x \le c(t)$ and $0 \le y \le c(t)$ then

$$g(t,x,y) = |\sqrt{x(c(t)-x)} - \sqrt{y(c(t)-y)}| \le \sqrt{|x(c(t)-x) - y(c(t)-y)|} =$$
$$\sqrt{|x(c(t)-x) - y(c(t)-x) + y(c(t)-x) - y(c(t)-y)|} =$$
$$= \sqrt{|(x-y)(c(t)-x) + y(c(t)-x-c(t)+y)|} =$$
$$= \sqrt{|(x-y)(c(t)-x-y)|} \le \sqrt{c(t)|x-y|} = \sqrt{c(t)}\sqrt{|x-y|}.$$

Thus, in all cases we have that

$$g(t,x,y) \le \sqrt{c(t)}\sqrt{|x-y|}.$$

That is why

$$\max_{t\in[0,T]} g(t,x,y) \le (\max_{t\in[0,T]} \sqrt{c(t)})\sqrt{|x-y|}.$$

We obtain that

$$|\sigma(t,x) - \sigma(t,y)| = |\sqrt{x(c(t)-x) \vee 0} - \sqrt{y(c(t)-y) \vee 0}| \le$$
$$\le (\max_{t\in[0,T]} \sqrt{c(t)})\sqrt{|x-y|}.$$

Then $\rho(|x-y|) = \sqrt{|x-y|}$, $\rho(x) = \sqrt{x}$ increases and

$$\int_{0+} \rho^{-2}(u)du = \int_{0+} \frac{1}{u} du = +\infty.$$

It means that Yamada's conditions are fulfilled and the equation has the unique solution.

Let us prove that $X(t) \ge 0$ a.s. for all $t \ge 0$. Define $\tau_1 = \inf\{t : X(t) = -\delta\}$, where $\delta > 0$ is some fixed constant. Assume that $P\{\tau_1 < \infty\}$. So, a constant $r < \tau_1$ exists such that

$$X(t) < 0 \quad t \in (r, \tau_1) \ \text{ a.s.}$$

However, in this case

$$dX(t) = (p(t) - q(t)X(t))dt > 0.$$

So, the function $t \to X(t)$ increases on (r, τ_1) which is impossible.

Now we prove that $X(t) \le c(t)$ a.s. for all $t \ge 0$. Define $\tau_2 = \inf\{t : c(t) - X(t) = -\delta\}$, where $\delta > 0$ is some fixed constant. Assume that $P\{\tau_2 < \infty\}$. So, a constant $R < \tau_2$ exists such that

$$-\delta < c(t) - X(t) < -\delta + \theta \quad t \in (R, \tau_2) \text{ a.s.},$$

where θ is such constant that $\frac{p(t)-q(t)c(t)}{q(t)} + \theta < 0$, $t \in (R, \tau_2)$. Then

$$c(t) + \delta - \theta < X(t) < c(t) + \delta \quad t \in (R, \tau_2).$$

But

$$dX(t) = (p(t) - q(t)X(t))dt < (p(t) - q(t)(c(t) + \delta - \theta))dt =$$

$$= ((p(t) - q(t)c(t) + q(t)\theta) - q(t)\delta)dt < 0.$$

We obtain that $dX(t) < 0, \quad t \in (R, \tau_2)$, thus, the function $t \to X(t)$ decreases on (r, τ_1), but this is impossible (because the function $c(t) - x(t)$ must decrease and the function $c(t)$ is nondecreasing). From this we obtain that the initial stochastic differential equation can be rewritten in the form

$$X(t) = X_0 + \int_0^t (p(s) - q(s)X(s))ds + \int_0^t \sqrt{X(s)(c(t) - X(s))}dW(s), \; t \ge 0.$$

We know from above that $b(t,x) = p(t) - q(t)x$, $\sigma^2(t,x) = x(c(t) - x)$ Thus, the inequality (4) can be rewritten in the form

$$\frac{2b(t,x)}{\sigma^2(t,x)} = \frac{2(p(t) - q(t)x)}{x(c(t) - x)} = \frac{2p(t)}{x(c(t) - x)} - \frac{2q(t)}{c(t) - x}.$$

As $x \in (0, \varepsilon)$, where $\varepsilon < c(0)$, then we can estimate the above expression as

$$\frac{2p(t)}{x(c(t) - x)} - \frac{2q(t)}{c(t) - x} \ge \frac{2p(t)}{xc(t)} - \frac{2q(t)}{c(t) - \varepsilon} \ge \frac{2p(t)}{xc(t)}$$

Denote $A(t) = \frac{2p(t)}{c(t)}$. Thus, by Theorem 2.2 the condition $A(t) > 1$ must be satisfied for all $t > 0$, and in our case it has the form

$$\forall t > 0 : \quad p(t) > \frac{c(t)}{2}.$$

It is sufficient for trajectories of the process $\{X(t), t \ge 0\}$ be positive a.s. So, if

$$\frac{c(t)}{2} < p(t) < q(t)c(t) \quad \forall t > 0,$$

then trajectories of the process $\{X(t), t \ge 0\}$ given by the stochastic differential equation (7) are positive with probability 1.

4. Conclusion

We declare the sufficient condition on coefficients which provides a.s. positivity of the trajectories of the solution of the stochastic differential equation with nonhomogeneous coefficients and non-Lipschitz diffusion. The result of this paper is applied to some stochastic differential equations, in particular for nonhomogeneous Cox-Ingersoll-Ross model.

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