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APPROXIMATION OF RANDOM PROCESSES BY CUBIC SPLINES

Approximation of some classes of random processes by cubic splines with given accuracy and reliability is considered. Estimations of deviation of approximating spline from original process are obtained. A few examples of approximation are considered. Application of splines for simulation of processes is also studied.

1. Introduction

Let $\{X(t), t \in T = [a,b]\}$ be a random $L_2(\Omega)$-process.

Denote by $\Delta := \{a = t_0 < ... < t_N = b\}$ – the partition of the segment $[a,b]$ into N parts. Assume that the values $\{y_i, i = \overline{0,N}\}$ of the process $X(t)$ in corresponding points $\{t_i, i = \overline{0,N}\}$ are known.

The problem of approximation of such process $X(t)$ with given accuracy and reliability in norms of different spaces $(C([a,b]), L_p(T)$ etc.) by cubic splines, constructed on known values of the process in partition points, with given boundary conditions, is considered.

We recall some basic definitions and facts used in the article.

Let (Ω, B, P) be a standard probability space.

Definition 1. The process $\widetilde{X}(t)$ approximates the process $X(t)$ with given accuracy $\varepsilon > 0$ and reliability $1-\delta, 0 < \delta < 1$ in space A if the next inequality is satisfied:

$$P\left\{\left\|X(t) - \widetilde{X}(t)\right\|_A > \varepsilon\right\} \le \delta.$$

Definition 2. [1] Function $S_{\Delta,y}(t)$ (or $S_\Delta(t)$), continuous on $[a,b]$ with its first and second derivatives, which is a cubic polynomial on every segment $[t_{i-1}, t_i], i = \overline{1,N}$, and which satisfies the conditions $S_\Delta(t_i) = y_i, i = \overline{0,N}$, is called a cubic spline on Δ, which interpolates values y_i in the knots of Δ.

Denote by $Y_N(t) := X(t) - S_\Delta(t), t \in T$, the deviation random process.

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Definition 3. [2] A continuous even convex function $u = (u(x), x \in R)$ is called an Orlicz N-function, if it is monotonically increasing for $x > 0$, $u(0) = 0$ and $\frac{u(x)}{x} \to 0$ as $x \to 0$ and $\frac{u(x)}{x} \to \infty$ as $x \to \infty$.

Assumption Q: [3] The assumption Q holds for an Orlicz N-function φ, if

$$\underline{\lim}_{x\to 0} \frac{\varphi(x)}{x^2} = c > 0.$$

Remark 1. The constant c can be equal to ∞.

Definition 4. [3] Let φ be an Orlicz N-function satisfying the assumption Q. A zero mean random variable ξ belongs to the space $Sub_\varphi(\Omega)$ (the space of φ-sub-Gaussian random variables), if there exists a constant $r_\xi \geq 0$ such that the inequality

$$E \exp(\lambda\xi) \leq \exp(\varphi(r_\xi \lambda))$$

holds $\forall \lambda \in R$.

Proposition 1. [2]-[4] *The space $Sub_\varphi(\Omega)$ is a Banach space with respect to the norm*

$$\tau_\varphi(\xi) = \inf\{a \geq 0 : E \exp(\lambda\xi) \leq \exp(\varphi(a\lambda)), \lambda \in R\}.$$

Definition 5. [2] Let T be a parametric space. A random process $\{X(t), t \in T\}$ belongs to the space $Sub_\varphi(\Omega)$ if for all $t \in T$ $X(t) \in Sub_\varphi(\Omega)$ and $\sup_{t\in T} \tau_\varphi(X(t)) < \infty$.

Definition 6. [5] A family Λ of random variables $\xi \in Sub_\varphi(\Omega)$ is called strictly $Sub_\varphi(\Omega)$ if there exists a constant $C_\Lambda > 0$ such that for any finite set I, $\xi_i \in \Lambda$, $i \in I$, and for $\forall \lambda_i \in R$ the inequality holds:

$$\tau_\varphi\left(\sum_{i\in I} \lambda_i \xi_i\right) \leq C_\Lambda \left(E\left(\sum_{i\in I} \lambda_i \xi_i\right)^2\right)^{1/2}.$$

C_Λ is called a determinative constant.

Definition 7. [5] φ-sub-Gaussian random process $\{X(t), t \in T\}$ is called strictly $Sub_\varphi(\Omega)$ if the family of random variables $\{X(t), t \in T\}$ is strictly $Sub_\varphi(\Omega)$.

Definition 8. [2] Let $f = (f(x), x \in R)$ be a real-valued function. The function $f^* = (f^*(x), x \in R)$ defined by the formula $f^*(x) = \sup_{y\in R}(xy - f(y))$ is called the Young-Fenchel transform of the function f.

2. Estimations of the deviation process

Denote by $M_j := S''_\Delta(t_j), j = \overline{0,N}$ – the so-called "moments" of spline $S_\Delta(t)$; $h_j = t_j - t_{j-1}, j = \overline{1,N}$, $\lambda_j = \frac{h_{j+1}}{h_j + h_{j+1}}$, $\mu_j = 1 - \lambda_j$, $j = \overline{1, N-1}$.

As the spline $S_\Delta(t)$ is a cubic polynomial on every segment of partition and as $S_\Delta, S'_\Delta, S''_\Delta$ are continuous we obtain the system of $N-2$ equations for the moments of spline $M_j, j = \overline{0,N}$ ([1]):

$$\mu_j M_{j-1} + 2M_j + \lambda_j M_{j+1} = 6 \cdot \frac{(y_{j+1} - y_j)/h_{j+1} - (y_j - y_{j-1})/h_j}{h_j + h_{j+1}} \qquad (1)$$

For unique solution of this system we specify 2 additional – boundary conditions – in the next form:

$$2M_0 + \lambda_0 M_1 = d_0 \qquad \mu_N M_{N-1} + 2M_N = d_N, \qquad (2)$$

where $\lambda_0, d_0, \mu_N, d_N$ are given values.

The equalities (1) and (2) defining the spline can be written in the matrix form:

$$A\overrightarrow{M} = \overrightarrow{d}, \qquad (3)$$

where

$$A := \begin{pmatrix} 2 & \lambda_0 & 0 & \dots\dots\dots\dots\dots & \\ \mu_1 & 2 & \lambda_1 & 0 & \dots\dots\dots \\ 0 & \mu_2 & 2 & \lambda_2 & \dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots & & & & \\ \dots\dots & 0 & \mu_{N-1} & 2 & \lambda_{N-1} \\ \dots\dots\dots\dots\dots\dots & & & \mu_N & 2 \end{pmatrix}, \quad \overrightarrow{M} := \begin{pmatrix} M_0 \\ \dots \\ \dots \\ \dots \\ \dots \\ M_N \end{pmatrix}, \quad \overrightarrow{d} = \begin{pmatrix} d_0 \\ \dots \\ \dots \\ \dots \\ \dots \\ d_N \end{pmatrix},$$

$d_j, j = \overline{1, N-1}$ – right sides of (1).

Denote by $|\Delta| = \max_j h_j$ the diameter of the partition Δ, and β such finite number that $\max_{1 \le j \le N} \frac{|\Delta|}{h_j} \le \beta$.

Theorem 1. *Let $X(t)$ be $L_2(\Omega)$-process which satisfies the inequality*

$$\sup_{t \in T} E|X(t+h) - X(t)|^2 \le \sigma^2(h), \qquad (4)$$

where $\sigma(h), h > 0$ is a known function such that $\sigma(h), h > 0$ increases monotonically, $\frac{h}{\sigma(h)}, h > 0$ is non-decreasing and $\sigma(h) \downarrow 0, h \downarrow 0$. Suppose $S_\Delta(t)$ is the spline which interpolates this process and satisfies the boundary conditions (2). Then $\forall t \in T$

$$(EY_N^2(t))^{1/2} \le \left(\frac{2}{9\sqrt{3}} \parallel A^{-1} \parallel [6\beta^2 + c_0|d_0| + c_0|d_N|] + \frac{3}{2} \right) \sigma(|\Delta|), \qquad (5)$$

where $c_0 = \frac{(b-a)^2}{\sigma(b-a)}$.

Remark 2. By the norm of matrix A we mean $\| A \| = \max_i \sum_j |a_{ij}|$.

Proof. For $t \in [t_{j-1}, t_j], j = \overline{1, N}$ we receive the equalities ([1]):

$$\begin{aligned} S_\Delta(t) &= M_{j-1}(t_j - t)\frac{(t_j - t)^2 - h_j^2}{6h_j} + M_j(t - t_{j-1})\frac{(t - t_{j-1})^2 - h_j^2}{6h_j} + \\ &+ \frac{y_{j-1} + y_j}{2} + \frac{y_j - y_{j-1}}{2h_j}(2t - (t_j + t_{j-1})). \end{aligned} \tag{6}$$

From (3) we obtain: $M_j = \sum_{i=1}^{N-1} A_{ij}^{(-1)} d_i + A_{j0}^{(-1)} d_0 + A_{jN}^{(-1)} d_N$, where $A_{ij}^{(-1)}$ are the elements of the matrix A^{-1}. Coefficients before M_{j-1}, M_j

$$\left|(t_j - t)\frac{(t_j - t)^2 - h_j^2}{6h_j}\right| \le \frac{h_j^2}{9\sqrt{3}}, \quad \left|(t - t_{j-1})\frac{(t - t_{j-1})^2 - h_j^2}{6h_j}\right| \le \frac{h_j^2}{9\sqrt{3}}.$$

Besides,

$$\frac{h_j^2 (Ed_i^2)^{1/2}}{6} = \frac{h_j^2}{h_i + h_{i+1}}\left(E\left(\frac{y_{i+1} - y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i}\right)^2\right)^{1/2} \le \beta^2 \sigma(|\Delta|),$$

$$\left(E\left(\frac{y_{j-1} + y_j}{2} - X(t) + \frac{y_j - y_{j-1}}{2h_j}(2t - (t_j + t_{j-1}))\right)^2\right)^{1/2} \le \frac{3}{2}\sigma(|\Delta|).$$

We get:

$$h_j^2((EM_{j-1}^2)^{1/2} + (EM_j^2)^{1/2}) \le 2 \| A^{-1} \| [6\beta^2\sigma(|\Delta|) + |\Delta|^2(|d_0| + |d_N|)]. \tag{7}$$

Thus

$$\begin{aligned} (EY_N^2(t))^{1/2} &\le \left(E(M_{j-1}^2)^{1/2} + E(M_j^2)^{1/2}\right)\frac{h_j^2}{9\sqrt{3}} + \frac{3}{2}\sigma(|\Delta|) \le \\ &\le \frac{2}{9\sqrt{3}} \| A^{-1} \| [6\beta^2\sigma(|\Delta|) + |\Delta|^2(|d_0| + |d_N|)] + \frac{3}{2}\sigma(|\Delta|). \end{aligned}$$

As $|\Delta| \le b - a$ and $\frac{h}{\sigma(h)}$ decreases on h, then

$$\frac{|\Delta|^2}{\sigma(|\Delta|)} \le (b-a)\frac{|\Delta|}{\sigma(|\Delta|)} \le \frac{(b-a)^2}{\sigma(b-a)} =: c_0,$$

so

$$(EY_N^2(t))^{1/2} \leq \left(\frac{2}{9\sqrt{3}} \parallel A^{-1} \parallel [6\beta^2 + c_0|d_0| + c_0|d_N|] + \frac{3}{2} \right) \sigma(|\Delta|).$$

Corollary 1. *In the case of uniform partition of T and the next boundary conditions $\lambda_0 = d_0 = \mu_N = d_N = 0$, i.e. $M_0 = M_N = 0$, the inequality (5) becomes*

$$(E((S_\Delta(t) - X(t))^2)^{1/2} \leq \left(\frac{4}{3\sqrt{3}} + \frac{3}{2} \right) \sigma \left(\frac{1}{N} \right).$$

Proof. The corollary results from Theorem 1 and the next proposition.

Proposition 2. *[1] If for the matrix A $|\lambda_0| < 2, |\mu_N| < 2$, then $\parallel A^{-1} \parallel \leq \max[(2-\lambda_0)^{-1}, (2-\mu_N)^{-1}, 1]$.*

Theorem 2. *Let $X(t)$ be $L_2(\Omega)$-process which satisfies (4), $S_\Delta(t)$ be the spline which interpolates this process and satisfies boundary conditions (2), $A, Y_N(t), c_0, \beta$ are defined above. Then $\forall t, t+h \in [a,b], \forall h > 0$ the estimation holds true:*

$$(E(Y_N(t+h) - Y_N(t))^2)^{1/2} \leq$$
$$\leq \sigma(h) \left(4 + \frac{4\beta}{3} \parallel A^{-1} \parallel (6\beta^2 + c_0(|d_0| + |d_N|)) \right). \tag{8}$$

Proof. Consider 2 possible cases.

Case 1. Let $t, t+h \in [t_{j-1}, t_j], j = \overline{1, N}$, then

$$(E(Y_N(t+h) - Y_N(t))^2)^{1/2} \leq (E(S_\Delta(t+h) - S_\Delta(t))^2)^{1/2} + \sigma(h).$$

From (6) we get

$$I := (E(S_\Delta(t+h) - S_\Delta(t))^2)^{1/2} = (E(M_{j-1}I_1 + M_j I_2 - I_3)^2)^{1/2},$$

$$|I_1| = \left| \frac{(t_j - t)(h^2 - 2h(t_j - t)) - h((t_j - (t+h))^2 - h_j^2)}{6h_j} \right| \leq h_j^2 \cdot \frac{h}{3h_j},$$

$$|I_2| = \left| \frac{(t - t_{j-1})(h^2 + 2h(t - t_{j-1})) + h((t + h - t_{j-1})^2 - h_j^2)}{6h_j} \right| \leq h_j^2 \cdot \frac{h}{3h_j},$$

$$(EI_3^2)^{1/2} = (E((y_j - y_{j-1})\frac{h}{h_j})^2)^{1/2} \leq \sigma(h_j)\frac{h}{h_j} \leq \sigma(h)$$

From (7) and the inequalities above we obtain:

$$\begin{aligned} I &\le \frac{h}{3h_j}\cdot 2 \parallel A^{-1} \parallel [6\beta^2\sigma(|\Delta|)+c_0\sigma(|\Delta|)(|d_0|+|d_N|)]+\sigma(h)\le \\ &\le \sigma(h)\left(1+\frac{2\beta}{3}\parallel A^{-1}\parallel (6\beta^2+c_0(|d_0|+|d_N|))\right) \end{aligned}$$

(as $\frac{|\Delta|}{\beta}\le h_j, \beta\ge 1$, then $\frac{1}{h_j}\le\frac{\beta}{|\Delta|}$; as $h\le|\Delta|$ then $\frac{h}{|\Delta|}\sigma(|\Delta|)\le\sigma(h)$).

$$(E(Y_N(t+h)-Y_N(t))^2)^{1/2}\le\sigma(h)\left(2+\frac{2\beta}{3}\parallel A^{-1}\parallel (6\beta^2+c_0(|d_0|+|d_N|))\right).$$

Case 2. Now suppose $t\in[t_{i-1},t_i], i=\overline{1,N-1},\quad t+h\in[t_{j-1},t_j], j=\overline{2,N}, i<j$, then again $(E(Y_N(t+h)-Y_N(t))^2)^{1/2}\le I+\sigma(h)$, where

$$\begin{aligned} I &= (E(S_\Delta(t+h)-S_\Delta(t))^2)^{1/2}=(E(S_\Delta(t+h)-S_\Delta(t_{j-1}))^2)^{1/2}+ \\ &+ (E(S_\Delta(t_{j-1})-S_\Delta(i))^2)^{1/2}+(E(S_\Delta(i)-S_\Delta(t))^2)^{1/2}\le \\ &\le \sigma(h)\left(3+\frac{4\beta}{3}\parallel A^{-1}\parallel (6\beta^2+c_0(|d_0|+|d_N|))\right). \end{aligned}$$

$$(E(Y_N(t+h)-Y_N(t))^2)^{1/2}\le\sigma(h)\left(4+\frac{4\beta}{3}\parallel A^{-1}\parallel (6\beta^2+c_0(|d_0|+|d_N|))\right).$$

Thus from cases 1,2 we obtain (8).

Corollary 2. *In the case of uniform partition of the segment T and boundary conditions $\lambda_0=d_0=\mu_N=d_N=0$, i.e. $M_0=M_N=0$, the inequality (8) becomes*

$$(E(Y_N(t+h)-Y_N(t))^2)^{1/2}\le 12\sigma(h),\quad t,t+h\in T, h>0.$$

Denote $m_j:=S'_\Delta(t_j), j=\overline{0,N}$, then for $t\in[t_{j-1},t_j], j=\overline{1,N}$, the next equality is satisfied ([1]):

$$\begin{aligned} S_\Delta(t) &= m_{j-1}\frac{(t_j-t)^2(t-t_{j-1})}{h_j^2}-m_j\frac{(t-t_{j-1})^2(t_j-t)}{h_j^2}+ \\ &+ y_{j-1}\frac{(t_j-t)^2(2(t-t_{j-1})+h_j)}{h_j^2}+y_j\frac{(t-t_{j-1})^2(2(t_j-t)+h_j)}{h_j^2}, \end{aligned}$$

whence the system of $N-2$ equations follows:

$$\lambda_j m_{j-1}+2m_j+\mu_j m_{j+1}=3\lambda_j\cdot\frac{y_j-y_{j-1}}{h_j}+3\mu_j\frac{y_{j+1}-y_j}{h_{j+1}}, j=\overline{1,N-1}. \quad (9)$$

We set the next boundary conditions:

$$2m_0+\mu_0 m_1=c_0 \qquad \lambda_N m_{N-1}+2m_N=c_N, \quad (10)$$

where $\mu_0, c_0, \lambda_N, c_N$ are the specified values.

The equalities (9) and (10), which define the spline, can be written in the matrix form:

$$\begin{pmatrix} 2 & \mu_0 & 0 & \dots\dots\dots\dots & & \\ \lambda_1 & 2 & \mu_1 & 0 & \dots\dots\dots & \\ 0 & \lambda_2 & 2 & \mu_2 & \dots\dots\dots & \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots\dots & & 0 & \lambda_{N-1} & 2 & \mu_{N-1} \\ \dots\dots\dots\dots & & & & \lambda_N & 2 \end{pmatrix}\cdot\begin{pmatrix} m_0 \\ \dots \\ \dots \\ \dots \\ \dots \\ m_N \end{pmatrix}=\begin{pmatrix} c_0 \\ \dots \\ \dots \\ \dots \\ \dots \\ c_N \end{pmatrix}, \quad (11)$$

where $c_j, j=\overline{1,N-1}$ are the right sides of (9).

Theorem 3. *Let $X(t)$ be $L_2(\Omega)$-process, $S_\Delta(t)$ – spline, which interpolates it and satisfies the boundary conditions:*

$$2m_0=2y_0', 2m_N=2y_N', \quad (12)$$

where $y_0'=X'(t_0), y_N'=X'(t_N)$. Assume there $\exists X'(t), t\in T$, and

$$(E(X'(t)-X'(s))^2)^{1/2}\leq\omega_1(h), \forall t,s\in T:|t-s|\leq h, \quad (13)$$

where $\omega_1(h)$ is a positive monotonically increasing function, $\frac{h}{\omega_1(h)}\uparrow$, $\omega_1(h)\downarrow 0, h\downarrow 0$. Then

$$(E(Y_N(t))^2)^{1/2}\leq\frac{21}{4}|\Delta|\cdot\omega_1(|\Delta|), \forall t\in T,$$

$$(E(Y_N'(t))^2)^{1/2}\leq\frac{21}{2}\omega_1(|\Delta|), \forall t\in T.$$

Proof. Denote the matrix of coefficients in (11) by B under boundary conditions (12) and equations $3m_0=3y_0'=c_0, 3m_N=3y_N'=c_N$ instead of the first and the last equation of the system (11).

From (11) we get: $B(\overrightarrow{m}-\frac{1}{3}\overrightarrow{c})=(I-\frac{1}{3}B)\overrightarrow{c}$, where $\overrightarrow{m}=(m_0,...,m_N)^T$, $\overrightarrow{c}=(c_0,...,c_N)^T$. But

$$(I-\frac{1}{3}B)\overrightarrow{c}=\begin{pmatrix} 0\\ \lambda_1(c_1-c_0)-\mu_1(c_2-c_1)\\ \ldots\\ \ldots\\ \lambda_{N-1}(c_{N-1}-c_{N-2})-\mu_{N-1}(c_N-c_{N-1})\\ 0\end{pmatrix},$$

whence

$$(E\parallel(I-\frac{1}{3}B)\overrightarrow{c}\parallel^2)^{1/2}\le\max_{k=\overline{1,N-1}}(\lambda_k\omega_1(3|\Delta|)+\mu_k\omega_1(3|\Delta|))\le 3\omega_1(|\Delta|),$$

$$(E\parallel\overrightarrow{m}-\frac{1}{3}\overrightarrow{c}\parallel^2)^{1/2}\le\parallel B^{-1}\parallel\cdot(E\parallel(I-\frac{1}{3}B)\overrightarrow{c}\parallel^2)^{1/2}\le 3\omega_1(|\Delta|).$$

Then $\forall j=\overline{1,N-1}\quad\exists\xi_j\in[t_{j-1},t_j]:\quad\frac{c_j}{3}=X'(\xi_j)$ and $\frac{c_0}{3}=X'(a)$, $\frac{c_N}{3}=X'(b)$, so denoting $\overrightarrow{X'}=(X'(t_0),...,X'(t_N))^T=(y'_0,...,y'_N)^T$ we obtain:

$$(E\parallel\overrightarrow{m}-\overrightarrow{X'}\parallel^2)^{1/2}\le(E\parallel\overrightarrow{m}-\frac{1}{3}\overrightarrow{c}\parallel^2)^{1/2}+(E\parallel\frac{1}{3}\overrightarrow{c}-\overrightarrow{X'}\parallel^2)^{1/2}\le 4\omega_1(|\Delta|).$$

Let's use the next representation and inequalities: for $t\in[t_{j-1},t_j], j=\overline{1,N}$

$$\begin{aligned} S'_\Delta(t) &- \frac{y_j-y_{j-1}}{h_j}=\left[\frac{3}{h_j^2}\left(t-\frac{t_{j-1}+t_j}{2}\right)^2-\frac{1}{4}\right][(m_{j-1}-y'_{j-1})+\\ &+ (m_j-y'_j)\left(y'_{j-1}+y'_j-2\cdot\frac{y_j-y_{j-1}}{h_j}\right)]+\frac{1}{h_j}(t-\frac{t_{j-1}+t_j}{2})\times\\ &\times [(m_j-y'_j)-(m_{j-1}-y'_{j-1})+(y'_j-y'_{j-1})], \end{aligned}\tag{14}$$

$$\left|\frac{3}{h_j^2}\left(t-\frac{t_j+t_{j-1}}{2}\right)^2-\frac{1}{4}\right|\le\frac{1}{2},\quad\frac{1}{h_j}\left|t-\frac{t_j+t_{j-1}}{2}\right|\le\frac{1}{2},$$

$$\left(E\left(m_{j-1}-y'_{j-1}\right)^2\right)^{1/2}\le 4\omega_1(|\Delta|),\left(E\left(y'_{j-1}-\frac{y_j-y_{j-1}}{h_j}\right)^2\right)^{1/2}\le\omega_1(|\Delta|),$$

$$\left(E\left(m_j-y_j'\right)^2\right)^{1/2} \le 4\omega_1(|\Delta|), \left(E\left(y_j'-\frac{y_j-y_{j-1}}{h_j}\right)^2\right)^{1/2} \le \omega_1(|\Delta|),$$

then from (14) for $t\in[t_{j-1},t_j], j=\overline{1,N}$:

$$\begin{aligned}\left(E\left(S_\Delta'(t)-\frac{y_j-y_{j-1}}{h_j}\right)^2\right)^{1/2} &\le \frac{1}{2}(4+4+1+1)\omega_1(|\Delta|)+\\ &+ \frac{1}{2}(4+4+1)\omega_1(|\Delta|)=\frac{19}{2}\omega_1(|\Delta|),\end{aligned}$$

whence $\forall t\in T$ (t^* is the closest of ends t_{j-1},t_j to t):

$$(E(X'(t)-S_\Delta'(t))^2)^{1/2}\le \omega_1(|\Delta|)+\frac{19}{2}\omega_1(|\Delta|)=\frac{21}{2}\omega_1(|\Delta|),$$

$$(E(X(t)-S_\Delta(t))^2)^{1/2}=(E(\int_{t^*}^t(X'(u)-S_\Delta(u))du)^2)^{1/2}\le\frac{21}{4}|\Delta|\omega_1(|\Delta|).$$

Theorem 4. *Let $X(t)$ be a $L_2(\Omega)$- process, $S_\Delta(t)$ – spline, which interpolates it under the boundary conditions:*

(i) $2M_0+M_1=\frac{6}{h_1}\left(\frac{y_1-y_0}{h_1}-y_0'\right), 2M_N+M_{N-1}=\frac{6}{h_N}\left(y_N'-\frac{y_N-y_{N-1}}{h_N}\right)$ *or*

(ii) $2M_0=2y_0'', 2M_N=2y_N''$,

where $y_0'=X'(t_0), y_N'=X'(t_N), y_0''=X''(t_0), y_N''=X''(t_N)$. *Assume there* $\exists X''(t), t\in T$, *and*

$$(E(X''(t)-X''(s))^2)^{1/2}\le\omega_2(h), \forall t,s\in T: |t-s|\le h, \tag{15}$$

where $\omega_2(h)$ is a positive monotonically increasing function, $\frac{h}{\omega_2(h)}$ is non-decreasing, $\omega_2(h)\downarrow 0, h\downarrow 0$. Then

$$(E(Y_N(t))^2)^{1/2}\le\frac{5}{2}|\Delta|^2\cdot\omega_2(|\Delta|), \forall t\in T,$$

$$(E(Y_N'(t))^2)^{1/2}\le 5|\Delta|\omega_2(|\Delta|), \forall t\in T,$$
$$(E(Y_N''(t))^2)^{1/2}\le 5\omega_2(|\Delta|), \forall t\in T.$$

Proof. From (3) under (i) boundary conditions we obtain:

$$A(\overrightarrow{M}-\frac{1}{3}\overrightarrow{d})=(I-\frac{1}{3}A)\overrightarrow{d},$$

$$(I - \frac{1}{3}A)\overrightarrow{d} = \frac{1}{3}\begin{pmatrix} d_0 - d_1 \\ \mu_1(d_1 - d_0) - \lambda_1(d_2 - d_1) \\ \mu_2(d_2 - d_1) - \lambda_2(d_3 - d_2) \\ \cdots \\ d_N - d_{N-1} \end{pmatrix}. \tag{16}$$

Note that for (ii) the arguments will be the same as for (i) except that the first and the last elements of the vector (16) will be equal to zero, but all the inequalities will remain true.

Notice that $\frac{d_j}{6} = \frac{(y_{j+1}-y_j)/h_{j+1}-(y_j-y_{j-1})/h_j}{h_j+h_{j+1}} = y[t_{j-1}, t_j, t_{j+1}]$ – the difference quotient. So $\frac{d_j}{6} = \frac{1}{2}X''(\xi_j)$ for some point $\xi_j \in (t_{j-1}, t_{j+1})$. From Taylor's formulae with the remainder term in the Lagrange form we get that for some point $\xi_0, t_0 < \xi_0 < t_1$, the equality is satisfied: $\frac{d_0}{6} = \frac{(y_1-y_0)/h_1-y_0'}{h_1} = \frac{1}{2}X''(\xi_0)$. Similar statement holds true for d_N. Thus as

$$\frac{1}{3}(E(\mu_k(d_k - d_{k-1}) - \lambda_k(d_{k+1} - d_k))^2)^{1/2} \le 3\omega_2(|\Delta|), k = \overline{1, N-1},$$

$$\frac{1}{3}(E(d_0 - d_1)^2)^{1/2} \le 3\omega_2(|\Delta|), \quad \frac{1}{3}(E(d_N - d_{N-1})^2)^{1/2} \le 3\omega_2(|\Delta|),$$

we receive the estimation:

$(E \parallel (I - \frac{1}{3}A)d \parallel^2)^{1/2} \le 3\omega_2(|\Delta|)$, whence

$$(E \parallel \overrightarrow{M} - \frac{1}{3}\overrightarrow{d} \parallel^2)^{1/2} \le \|A^{-1}\|(E(\parallel (I - \frac{1}{3}A)\overrightarrow{d} \parallel)^2)^{1/2} \le 3\omega_2(|\Delta|).$$

But from the equalities above for $d_i, i = \overline{1, N-1}, d_0, d_N$ we get:

$$(E \parallel \overrightarrow{X''} - \frac{1}{3}\overrightarrow{d} \parallel^2)^{1/2} \le \omega_2(|\Delta|),$$

so

$$\begin{aligned}(E \parallel \overrightarrow{M} - \overrightarrow{X''} \parallel^2)^{1/2} &\le (E \parallel \overrightarrow{M} - \frac{1}{3}\overrightarrow{d} \parallel^2)^{1/2} + (E \parallel \frac{1}{3}\overrightarrow{d} - \overrightarrow{X''} \parallel^2)^{1/2} \le \\ &\le 4\omega_2(|\Delta|), \quad \overrightarrow{X''} = (X''(t_0), ..., X''(t_N))^T.\end{aligned}$$

$S_\Delta(t)$ is a piecewise-linear function, then $\forall t \in [t_{j-1}, t_j], j = \overline{1, N}$

$$\begin{aligned}(E|X''(t) - S''_\Delta(t)|^2)^{1/2} &\le (E|X''(t) - X''(t_j)|^2)^{1/2} + \\ &+ (E|X''(t_j) - S_\Delta(t)|^2)^{1/2} \le 5\omega_2(|\Delta|).\end{aligned}$$

Since $S_\Delta(t_j) = X(t_j), j = \overline{0, N}$, then from Rolle's theorem for any interval $(t_{j-1}, t_j) \quad \exists \xi_j : \quad X'(\xi_j) = S'_\Delta(\xi_j)$, so $\forall t \in [t_{j-1}, t_j], j = \overline{1, N}$

$$\begin{aligned}(E(|X'(t)-S'_\Delta(t)|)^2)^{1/2} &= (E(|\int_{\xi_j}^t (X''(u)-S_\Delta(u))du|)^2)^{1/2} \leq \\ &\leq 5|t-\xi_j|\cdot\omega_2(|\Delta|) \leq 5|\Delta|\omega_2(|\Delta|),\end{aligned}$$

$$\begin{aligned}(E(|X(t)-S_\Delta(t)|)^2)^{1/2} &= (E(|\int_{t^*}^t (X''(u)-S_\Delta(u))du|)^2)^{1/2} \leq \\ &\leq |t-t^*|\cdot 5|\Delta|\omega_2(|\Delta|) \leq \frac{5}{2}|\Delta|^2\omega_2(|\Delta|)\end{aligned}$$

(t^* is the closest of ends t_{j-1}, t_j to t).

3. Examples of approximation (space $L_p(T)$)

Theorem 5. *i) Let $\{X(t), t\in T=[a,b]\}$ be a $SSub_\varphi(\Omega)$-process with determinative constant C_Λ and which satisfies (4), $S_\Delta(t)$ is the spline interpolating this process under boundary conditions $2M_0+\lambda_0 M_1 = d_0,\quad \mu_N M_{N-1} + 2M_N = d_N$, where $\lambda_0, d_0, \mu_N, d_N$ are given values. Then $\forall\varepsilon>0:$*

$$p^{(-1)}\left(\frac{\varepsilon}{c_3 C_\Lambda \sigma(|\Delta|)}\right)\frac{1}{c_3 C_\Lambda\sigma(|\Delta|)}\frac{\varepsilon}{(b-a)^{1/p}} \geq p,$$

where $p^{(-1)}(t), t>0$ is the inverse function for $p(t)$, $p(t)$ is the density of $\varphi(x), \varphi(x) = \int_0^x p(t)dt$, $c_3 = \frac{2}{9\sqrt{3}} \parallel A^{-1}\parallel [6\beta^2 + c_0|d_0| + c_0|d_N|] + \frac{3}{2}$, A, β, c_0 are defined above, the next inequality holds:

$$P\left\{\|X(t)-S_\Delta(t)\|_{L_p} > \varepsilon\right\} \leq 2\exp\left\{-\varphi^*\left(\frac{\varepsilon}{c_3 C_\Lambda\sigma(|\Delta|)(v-u)^{1/p}}\right)\right\},$$

where φ^* is the Young-Fenchel transform of the function $\varphi, p\geq 1$.

ii) Let $\{X(t), t\in T=[a,b]\}$ be a separable $SSub_\varphi(\Omega)$-process with determinative constant C_Λ, satisfying (4) and assume $\int_0^\nu \psi(\ln(\sigma^{(-1)}(u)))du < \infty$ for all z and for sufficiently small $\nu>0$ where $\psi(u) = u/\varphi^{(-1)}(u)$. For $\forall t,s\in T$ put $EX(t)X(s) = R(t,s)$ and suppose $\frac{\partial^2 R(t,s)}{\partial t\partial s}$ exists and satisfies

$$\sup_{|t-s|\leq h}\left(\left.\frac{\partial^2 R(u,v)}{\partial u\partial v}\right|_{u=v=t} + \left.\frac{\partial^2 R(u,v)}{\partial u\partial v}\right|_{u=v=s} - \left.\frac{2\partial^2 R(u,v)}{\partial u\partial v}\right|_{u=t,v=s}\right) \leq \omega_1^2(h),$$

where $\omega_1(h)$ is a positive monotonically increasing function, $\frac{h}{\omega_1(h)}$ is nondecreasing, $\omega_1(h)\downarrow 0, h\downarrow 0$. Then for the spline with boundary conditions $2m_0 = 2y'_0, 2m_N = 2y'_N$, where $y'_0 = X^{(1)}(t_0), y'_N = X^{(1)}(t_N)$, $\forall\varepsilon>0:$

$$p^{(-1)}\left(\frac{\varepsilon}{\frac{21}{4}C_\Lambda|\Delta|\omega_1(|\Delta|)}\right)\frac{1}{\frac{21}{4}C_\Lambda|\Delta|\omega_1(|\Delta|)}\frac{\varepsilon}{(b-a)^{1/p}}\geq p,$$

the inequality holds true:

$$P\left\{\|X(t)-S_\Delta(t)\|_{L_p}>\varepsilon\right\}\leq 2\exp\left\{-\varphi^*\left(\frac{\varepsilon}{\frac{21}{4}C_\Lambda|\Delta|\omega_1(|\Delta|)(b-a)^{1/p}}\right)\right\}.$$

iii) Let $\{X(t), t\in T=[a,b]\}$ *be a separable* $SSub_\varphi(\Omega)$*-process with determinative constant* C_Λ*, satisfying (4) and assume* $\int_0^\nu \psi(\ln(\sigma^{(-1)}(u)))du<\infty$ *for all* z *and for sufficiently small* $\nu>0$ *where* $\psi(u)=u/\varphi^{(-1)}(u)$*. For* $\forall t,s\in T$ *put* $EX(t)X(s)=R(t,s)$ *and suppose* $\frac{\partial^4 R(t,s)}{\partial t^2\partial s^2}$ *exists and satisfies*

$$\sup_{|t-s|\leq h}\left(\frac{\partial^4 R(u,v)}{\partial u^2\partial v^2}\bigg|_{u=v=t}+\frac{\partial^4 R(u,v)}{\partial u^2\partial v^2}\bigg|_{u=v=s}-\frac{2\partial^4 R(u,v)}{\partial u^2\partial v^2}\bigg|_{u=t,v=s}\right)\leq\omega_2^2(h),$$

where $\omega_2(h)$ *is a positive monotonically increasing function,* $\frac{h}{\omega_2(h)}$ *is non-decreasing,* $\omega_2(h)\downarrow 0, h\downarrow 0$*. Then for the spline with boundary conditions* $2M_0+M_1=\frac{6}{h_1}\left(\frac{y_1-y_0}{h_1}-y_0'\right), 2M_N+M_{N-1}=\frac{6}{h_N}\left(y_N'-\frac{y_N-y_{N-1}}{h_N}\right)$ *or* $2M_0=2y_0'', 2M_N=2y_N''$*, where* $y_0''=X^{(2)}(t_0), y_N''=X^{(2)}(t_N)$*,* $\forall\varepsilon>0:$

$$p^{(-1)}\left(\frac{\varepsilon}{\frac{5}{2}C_\Lambda|\Delta|^2\cdot\omega_2(|\Delta|)}\right)\frac{1}{\frac{5}{2}C_\Lambda|\Delta|^2\cdot\omega_2(|\Delta|)}\frac{\varepsilon}{(b-a)^{1/p}}\geq p,$$

the inequality holds:

$$P\left\{\|X(t)-S_\Delta(t)\|_{L_p}>\varepsilon\right\}\leq 2\exp\left\{-\varphi^*\left(\frac{\varepsilon}{\frac{5}{2}C_\Lambda|\Delta|^2\omega_2(|\Delta|)(b-a)^{1/p}}\right)\right\}.$$

Proof. The theorem comes from theorems 1,3,4, theorem 3.1 from [6] (inequality for norm in $L_p(T)$ of $Sub_\varphi(\Omega)$ process) and theorems 3.7-3.9 from [7] (conditions for the existence of continuous partial derivatives of a random field of the space $SSub_\varphi(\Omega)$).

Example 1. Assume $\sigma(h)=ch^\alpha, 0\leq\alpha\leq 1, c>0, p=2, \varphi(x)=x^2/2, T=[0,1], c=\alpha=1$, the partition Δ is uniform, $C_\Lambda=1$.

Applying theorem 5, let's obtain the inequalities for the order of partition N, which should be satisfied in order that the corresponding spline with given boundary conditions approximates the original process with given accuracy ε and reliability $1-\delta$ in the norm of L_2.

a) Let the boundary conditions for spline be $S_\Delta''(0)=S_\Delta''(0)=0$. Then for $\varepsilon=0.03, \delta=0.01$ we get $N\geq 244$, for $\varepsilon=0.1, \delta=0.1\Rightarrow N\geq 55$.

b) assume that the conditions of theorem 5 ii) are satisfied, $\omega_1(h) = h$. Then for $\varepsilon = 0.03, \delta = 0.01$ for the corresponding spline we derive $N \geq 24$; for $\varepsilon = 0.1, \delta = 0.1$ we have: $N \geq 12$.

c) assume that the conditions of theorem 5 iii) are satisfied, $\omega_2(h) = h$. We get: for $\varepsilon = 0.03, \delta = 0.01$ for the corresponding spline the order of the partition should meet $N \geq 7$, for $\varepsilon = 0.1, \delta = 0.1 \Rightarrow N \geq 4$.

4. Application of splines for simulation of gaussian processes

Consider the problem of simulation of centered gaussian process $\{\xi(t), t \in T = [0,1]\}$, with known covariance function $B(t,s) = E\xi(t)\xi(s), t, s \in T$, in $L_p(T)$ with given accuracy $\varepsilon > 0$ and reliability $1 - \delta, 0 < \delta < 1$ by spline with corresponding boundary conditions (2).

Then the algorithm of such simulation is following:

1) for given $B(t,s)$ choose such function $\sigma(h) = ch^\alpha, c > 0, 0 < \alpha \leq 1$, that $\sup_{s,t \in T: |t-s| \leq h} (B(t,t) - 2B(t,s) + B(s,s)) \leq \sigma^2(h)$.

2) for given ε, δ and obtained $\sigma(h)$ find the order of spline N :

$$P\left(\| \xi(t) - S_\Delta(t) \|_{L_p(T)} > \varepsilon\right) \leq \delta.$$

(using theorem 5 i)). Calculate the points of the partition $t_k = \frac{k}{N}, k = \overline{0,N}$ and nonnegatively definite matrix $R = (R_{ij})_{i,j=0}^N, R_{ij} = B(t_i, t_j)$.

3) get eigenvalues $b_0^2, ..., b_N^2$ and eigenvectors $\overrightarrow{x_0}, ..., \overrightarrow{x_N}$ of R.

Then as it is known the matrix $A = \begin{pmatrix} \overrightarrow{x_0} \\ ... \\ \overrightarrow{x_N} \end{pmatrix}$ reduces R to diagonal form i.e. $D = ARA^T$, where $D = \begin{pmatrix} b_0^2 & 0 & \cdots & \cdots \\ 0 & b_2^2 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & 0 & b_N^2 \end{pmatrix}$

4) simulate vector $\overrightarrow{\zeta} = (\zeta_0, ..., \zeta_N,), \zeta_i \sim N(0,1)$ are independent, calculate $\overrightarrow{\eta} = (\eta_0, ..., \eta_N)$, where $\eta_i = b_i \zeta_i$.

5) find vector $\overrightarrow{\widetilde{\xi}} = A\overrightarrow{\eta}$, construct corresponding spline with boundary conditions (2) – desired model of the process $\xi(t)$.

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