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THE GENERALIZATION OF THE QUANTILE HEDGING PROBLEM FOR PRICE PROCESS MODEL INVOLVING FINITE NUMBER OF BROWNIAN AND FRACTIONAL BROWNIAN MOTIONS

The paper is devoted to the problem of quantile hedging of contingent claims in the framework of a model defined by the finite number of independent Brownian and fractional Brownian motions. The maximal success probability depending on initial capital is estimated.

1. Introduction

The problem of hedging of contingent claims is well known in the case of complete non-arbitrage models. Consider an investor, who wants to ensure that a claim H will be hedged and operates with an asset, whose price is modeled by a semimartingale. Necessary and sufficient condition for such successful hedging is the availability of capital $H_0 = E_{P^*}(H)$, where E_{P^*} is the expectation w.r.t. the unique martingale measure P^*, i.e. the measure w.r.t. which the price process X_t is a martingale. If investor is unwilling or unable to use the whole amount H_0 and wants or is able to use a certain amount $\nu < H_0$, it follows from the absence of arbitrage that he cannot supply the replication of claim H in all possible scenario, i.e. he cannot hedge the claim H with probability 1.

In this case the problem of quantile hedging for investor can be reduced to maximization of success probability, i.e. the probability to hedge the claim H.

In paper [4] the general principles concerning such type of hedging are considered in the case where the price process is a semimartingale.

In paper [6] the special case of jump-diffusion market is considered, the stochastic differential equation for hedging strategy is deduced, the hedging strategy and the price of European option are obtained.

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In paper [2] the model with so called long-range dependence, more exactly, with the unique fractional Brownian and independent Brownian motion is considered and maximal possible success probability on the available initial capital $\nu < H_0$ is estimated. The market is complete there.

In this paper our aim is to consider incomplete market with several fractional Brownian motions and independent Brownian motions and to construct the set on which the investor can hedge the contingent claim and find the dependence of the estimation of maximal success probability on the initial capital ν. For technical simplicity we restrict ourselves to the case of two pairs of such processes.

In Section 2 we sketch the solution of the quantile hedging problem on the complete non-arbitrage financial market.

In Section 3 we prove the incompleteness of the price process model defined by two Wiener processes and two fractional Brownian motions and estimate the successful probability for quantile hedging for this case.

In Section 4 we consider the results of Section 3 for European call option and illustrate the dependence of the probability of successful hedge on the initial capital ν.

Let B_t^i $(i = 1, 2)$ be Wiener processes and $B_t^{\mathrm{H_i}}$ $(i = 1, 2)$ be normalized fractional Brownian motions (fBm) with Hurst index $\mathrm{H_i}$ consequently on a probability space (Ω, F, P). The last means that $B_t^{\mathrm{H_i}}$ is a continuous Gaussian process with zero mean and stationary increments, which has the covariance function of the form

$$EB_t^{\mathrm{H_i}} B_s^{\mathrm{H_i}} = \frac{1}{2}\left(s^{2\mathrm{H_i}} + t^{2\mathrm{H_i}} - |s-t|^{2\mathrm{H_i}}\right), i = 1, 2.$$

we suppose everywhere that $H_i \in (1/2, 1)$, since exactly this case corresponds to the long-range dependence of the model. Consider the so-called mixed model, where the price process is defined as

$$\begin{aligned} X_t &= X_0 \exp\left\{mt + \sigma_1 B_t^1 + \sigma_2 B_t^2 + \mu_1 B_t^{\mathrm{H_1}} + \mu_2 B_t^{\mathrm{H_2}}\right\} = \\ &= X_0 \exp\left\{mt + \sigma_1\left(B_t^1 + \frac{\mu_1}{\sigma_1} B_t^{\mathrm{H_1}}\right) + \sigma_2\left(B_t^2 + \frac{\mu_2}{\sigma_2} B_t^{\mathrm{H_2}}\right)\right\} \\ &= X_0 \exp\left\{mt + \sigma_1\left(B_t^1 + \delta_1 B_t^{\mathrm{H_1}}\right) + \sigma_2\left(B_t^2 + \delta_2 B_t^{\mathrm{H_2}}\right)\right\}, \qquad (1) \end{aligned}$$

where $\delta_i = \frac{\mu_i}{\sigma_i}$, $i = 1, 2$, $X_0 > 0$ is some constant.

The processes B_t^1, B_t^2, $B_t^{\mathrm{H_1}}$, $B_t^{\mathrm{H_2}}$ are supposed to be mutually independent.

The absence of arbitrage in this model follows from existence of at least one martingale measure for the process (1), i.e. such measure w.r.t. which the discounted price process will be a martingale. The set of martingale measures for the price process model is constructed in Section 3.

2. Quantile hedging on the complete non-arbitrage market

We consider complete stochastic basis $(\Omega, F, F_t, t \geq 0, P)$. The problem of quantile hedging is to maximize the probability $P\{V_T \geq H\}$ over all self-financing strategies ξ with initial capital equal to fixed $V_0 \leq \nu$, admissible in the sense that the capital process

$$V_t = V_0 + \int_0^t \xi_s dX_s \tag{2}$$

for the strategy ξ is non-negative a.s. for all $0 \leq t \leq T$, and V_T evidently is a capital at maturity date.

In paper [4] this optimization problem is solved with the help of Neumann-Pearson lemma. Below we sketch the solution of this problem.

We call the set $A = \{V_T \geq H\}$ the success set corresponding to chosen strategy. Define such measure Q^* that

$$\frac{dQ^*}{dP^*} = \frac{H}{H_0}. \tag{3}$$

Then the optimization problem is reduced to maximization of probability $P\{A\}$ over all F_T-measurable sets A satisfying the condition

$$\frac{V_0}{H_0} = \frac{E_{P^*}[HI_A]}{H_0} = E_{Q^*}[I_A] \leq \alpha := \frac{\nu}{H_0}. \tag{4}$$

Put $\bar{a} = \inf\left\{a : Q^*\left[\frac{dP}{dP^*} > \bar{a} \cdot H\right] \leq \alpha\right\}$. Suppose that

$$Q^*\left[\frac{dP}{dP^*} = \bar{a} \cdot H\right] = 0. \tag{5}$$

Then by the Neumann-Pearson lemma the optimal set is

$$A = A_{\bar{a}} := \left\{\frac{dP}{dP^*} > \bar{a} \cdot H\right\}. \tag{6}$$

Moreover, due to the choice of $\bar{a}$ for initial capital ν there exists a strategy which permits to hedge almost surely the claim $\bar{H} = HI_{A_{\bar{a}}}$, i.e. to hedge H with probability $P(A_{\bar{a}})$. It is the strategy that maximizes the probability $P(V_T \geq H)$.

If we consider the simplest case, where the claim H depends only on the final asset price, i.e. $H = H_T = f(X_T)$, then in terms of the Neumann-Pearson lemma, the problem is reduced to finding such constant $\bar{a}$ that the probability P of the set $A = A_a := \left\{\frac{dP}{dP^*} > a \cdot H\right\}$ is maximal given that $Q^*(A) \leq \alpha = \frac{\nu}{H_0}$.

Unfortunately, as it is shown in [2], in case of a so-called mixed Brownian-fractional-Brownian motion model this set has rather complicated structure, in particular it depends on the trajectory of the Wiener process W_t on the whole interval $[0,T]$, and thus the probabilities $Q^*(A)$ and $P(A)$ are hardly computable, which does not permit to use directly the approach, introduced in [4] even for the simplest payoff functions $f(X_T)$ (e.g., European call option).

Using the procedure of re-discounting, another type of set was obtained in [2], and this permits us to avoid aforementioned difficulty. Put

$$\tilde{A}_a = \left\{ a < \frac{X_T}{X_0 \cdot f(X_T)} \right\}, \tag{7}$$

and let the claim H be hedged on this set, if the initial capital is equal to ν.

The set $\tilde{A}_{\bar{a}}$ is not necessarily the maximal success probability set for initial capital ν, nevertheless $P\left(\tilde{A}_{\bar{a}}\right)$ is lower estimate for the maximal probability. Therefore, one can guarantee successful hedge of the claim with probability not smaller than $P\left(\tilde{A}_{\bar{a}}\right)$.

3. Quantile hedging in incomplete case

In the incomplete case the equivalent martingale measure is no longer unique. The following lemma states the incompleteness of the market defined by price process (1). Note that the processes $B_t^i + \delta_i B_t^{\mathrm{H_i}}$, $i = 1,2$, consequently X_t, are not semimartingales w.r.t. the filtration generated by processes B_t^1, B_t^2, $B_t^{\mathrm{H_1}}$, $B_t^{\mathrm{H_2}}$. Nevertheless, as it is shown in [3], for $\mathrm{H_i} > \frac{3}{4}$, $i = 1,2$, the random process $Y_t^i = B_t^i + \delta_i B_t^{\mathrm{H_i}}$ is equivalent in distribution to Wiener process. Moreover, according to [5], there exists the representation

$$Y_t^i = W_t^i - \int_0^t \int_0^s r_{\delta_i}(s,u)\, dW_u^i ds, \tag{8}$$

where W_t^i is Wiener process w.r.t. P, $r = r_{\delta_i}$ is Volterra kernel, that is, the unique solution of equation

$$r(t,s) + \int_0^s r(t,x)\, r(s,x)\, dx = \delta_i^2 \mathrm{H_i}(2\mathrm{H_i} - 1) \cdot |t-s|^{2\mathrm{H_i}-2}, \tag{9}$$

$$0 \le s \le t \le T,$$

which satisfies $\int_0^t \int_0^s (r_{\delta_i}(s,u))^2\, duds < \infty$, $i = 1,2$. Thus, Y_t^i is a semimartingale w.r.t. the natural filtration F^Y, generated by the processes $Y_t^i, i = 1,2$. In what follows we fix this filtration and we will consider only it.

Lemma 3.1. *Let* $\mathrm{H_i} > \frac{3}{4}$, $i = 1, 2$, $\tilde{m}_1(s)$ *is predictable process with respect to natural filtration* F^Y , $\tilde{m}_2(s) = m - \tilde{m}_1(s)$ *and the following conditions hold:* $E\int_0^t m_i^2(s)ds < \infty$ *and*

$$E\exp\Big(-\int_0^T\Big(\frac{\tilde{m}_i(s)}{\sigma_i}+\frac{1}{2}\sigma_i-\int_0^s r_{\delta_i}(s,u)dW_u^i\Big)dW_s^i-$$

$$\frac{1}{2}\int_0^T\Big(\frac{\tilde{m}_i(s)}{\sigma_i}+\frac{1}{2}\sigma_i-\int_0^s r_{\delta_i}(s,u)dW_u^i\Big)^2ds\Big)=1,\ i=1,2.$$

Then the model for the process X_t *of the form (1) is incomplete since there exists a family of the martingale measures* P^* *depending on* $\tilde{m}_1$ *and* $\tilde{m}_2$*, with Radon-Nykodym derivatives of the form*

$$\frac{dP^*}{dP}\Big|_{F_t^Y}=\prod_{i=1}^2\exp\left(-\int_0^t\left(\frac{\tilde{m}_i(s)}{\sigma_i}+\frac{1}{2}\sigma_i-\int_0^s r_{\delta_i}(s,u)\,dW_u^i\right)dW_s^i-\right.$$

$$\left.-\frac{1}{2}\int_0^t\left(\frac{\tilde{m}_i(s)}{\sigma_i}+\frac{1}{2}\sigma_i-\int_0^s r_{\delta_i}(s,u)\,dW_u^i\right)^2ds\right). \tag{10}$$

Proof. The price of asset X_t can be rewritten in the form

$$X_t = X_0\exp\big(mt+\sigma_1Y_t^1+\sigma_2Y_t^2\big)=$$

$$=X_0\exp\left(mt+\sigma_1W_t^1-\sigma_1\int_0^t\int_0^s r_{\delta_1}(s,u)\,dW_u^1ds+\right.$$

$$\left.+\sigma_2W_t^2-\sigma_2\int_0^t\int_0^s r_{\delta_2}(s,u)\,dW_u^2ds\right)=$$

$$=X_0\prod_{i=1}^2\exp\left(\sigma_iW_t^i-\frac{1}{2}\sigma_i^2t+\int_0^t(\tilde{m}_i(s)+\frac{1}{2}\sigma_i^2)ds\right.$$

$$\left.-\sigma_i\int_0^t\int_0^s r_{\delta_i}(s,u)\,dW_u^ids\right).$$

Denote

$$\hat{W}_t^i=W_t^i+\int_0^t\left(\frac{\tilde{m}_i(s)}{\sigma_i}+\frac{1}{2}\sigma_i-\int_0^s r_{\delta_i}(s,u)\,dW_u^i\right)ds,\quad i=1,2. \tag{11}$$

By the Girsanov theorem this process is Wiener process w.r.t. the measure P^* of the form (10) and then w.r.t. P^* the process X_t has the form

$$X_t = X_0 \prod_{i=1}^{2} \exp\left(\sigma_i \hat{W}_t^i - \frac{1}{2}\sigma_i^2 t\right), \tag{12}$$

i.e. it is a martingale. (We suppose everywhere that non-risky asset identically equals 1.)

Correspondingly to the choice of $\tilde{m}_1$ and $\tilde{m}_2$ we obtain different martingale measures. Thus the martingale measure is not unique and the market is incomplete. □

Remark. Considering another filtration than F^Y, we can obtain another martingale measures for the process X_t, not defined in Lemma 3.1.

Example 3.1. Let we define the processes

$$\tilde{B}_t^1(\alpha) = B_t^1 \cos\alpha - B_t^2 \sin\alpha,$$

$$\tilde{B}_t^2(\alpha) = B_t^1 \sin\alpha + B_t^2 \cos\alpha.$$

These processes are uncorrelated and, thus, independent. We have

$$B_t^1(\alpha) = \tilde{B}_t^1 \cos\alpha + \tilde{B}_t^2 \sin\alpha,$$

$$B_t^2(\alpha) = -\tilde{B}_t^1 \sin\alpha + \tilde{B}_t^2 \cos\alpha,$$

and

$$X_t = X_0 \exp\left\{mt + \sigma_1\left(B_t^1 + \delta_1 B_t^{\mathrm{H}_1}\right) + \sigma_2\left(B_t^2 + \delta_2 B_t^{\mathrm{H}_2}\right)\right\} =$$

$$= X_0 \exp\left\{mt + \sigma_1\left(\tilde{B}_t^1(\alpha)\cos\alpha + \tilde{B}_t^2(\alpha)\sin\alpha + \delta_1 B_t^{\mathrm{H}_1}\right) + \right.$$

$$\left. + \sigma_2\left(-\tilde{B}_t^1(\alpha)\sin\alpha + \tilde{B}_t^2(\alpha)\cos\alpha + \delta_2 B_t^{\mathrm{H}_2}\right)\right\} =$$

$$= X_0 \exp\left\{mt + (\sigma_1\cos\alpha - \sigma_2\sin\alpha)\tilde{B}_t^1(\alpha) + (\sigma_1\sin\alpha + \sigma_2\cos\alpha)\tilde{B}_t^2(\alpha) + \right.$$

$$\left. + \mu_1 B_t^{\mathrm{H}_1} + \mu_2 B_t^{\mathrm{H}_2}\right\} =$$

$$= X_0 \exp\left\{mt + \tilde{\sigma}_1 \tilde{B}_t^1(\alpha) + \tilde{\sigma}_2 \tilde{B}_t^2(\alpha) + \mu_1 B_t^{\mathrm{H}_1} + \mu_2 B_t^{\mathrm{H}_2}\right\} =$$

$$= X_0 \exp\left\{mt + \tilde{\sigma}_1\left(\tilde{B}_t^1(\alpha) + \frac{\mu_1}{\tilde{\sigma}_1} B_t^{\mathrm{H}_1}\right) + \tilde{\sigma}_2\left(\tilde{B}_t^2(\alpha) + \frac{\mu_2}{\tilde{\sigma}_2} B_t^{\mathrm{H}_2}\right)\right\} =$$

$$= X_0 \exp\left\{mt + \tilde{\sigma}_1\left(\tilde{B}_t^1(\alpha) + \tilde{\delta}_1 B_t^{\mathrm{H}_1}\right) + \tilde{\sigma}_2\left(\tilde{B}_t^2(\alpha) + \tilde{\delta}_2 B_t^{\mathrm{H}_2}\right)\right\} =$$

$$= X_0 \exp\left\{mt + \tilde{\sigma}_1 \tilde{Y}_t^1 + \tilde{\sigma}_2 \tilde{Y}_t^2\right\}, \tag{13}$$

where $\tilde{\sigma}_1 = \sigma_1 \cos\alpha - \sigma_2 \sin\alpha$, $\tilde{\sigma}_2 = \sigma_1 \sin\alpha + \sigma_2 \cos\alpha$, $\tilde{\delta}_i = \frac{\mu_i}{\tilde{\sigma}_i}$, $\tilde{Y}_t^i = \tilde{B}_t^i(\alpha) + \tilde{\delta}_i B_t^{\mathrm{H_i}}$, $i = 1, 2$.

Note that in general case we have $\tilde{\delta}_i \neq \delta_i$, $i = 1, 2$. Thus, the processes $\tilde{Y}_t^i$ and Y_t^i have different Volterra kernels, which are the solutions of (9). It means that corresponding martingale measures for the process X_t, which satisfy (10), are different.

Remark. Although the martingale measures, mentioned above, are different, the following equality holds:

$$\sigma_1^2 + \sigma_2^2 = \tilde{\sigma}_1^2 + \tilde{\sigma}_2^2. \tag{14}$$

If the contingent claim in incomplete model is not attainable, it is possible to hedge the claim almost surely by means of superhedging strategy. As it is mentioned in [4], in such case the least amount of capital, which is needed to be on the safe side, is given by

$$\inf\left\{V_0 | \quad \exists \xi : (V_0, \xi) \quad admissible, \quad V_0 + \int_0^T \xi_s dX_s \geq H \quad P - a.s.\right\}. \tag{15}$$

Thus, the least needed amount is equal to the largest arbitrage-free price:

$$U_0 := \sup_{P^* \in \mathcal{P}} E^*[H] < \infty \tag{16}$$

where $\mathcal{P}$ is the set of all equivalent martingale measures P^* satisfying the conditions of lemma 3.1.

If investor is not able to use whole amount U_0 and is willing to use only amount $\nu < U_0$, then he cannot hedge the claim H with probability 1 because of the absence of arbitrage. But if the following inequality holds:

$$\sup_{P^* \in \mathcal{P}} E_{P^*}\left[H I_{\tilde{A}_a}\right] \leq \nu, \tag{17}$$

where $\tilde{A}_a$ has the form introduced in (7), then an investor with initial capital ν is able to hedge $H I_{\tilde{A}_a}$ almost surely, i.e., to hedge H with probability $P\left(\tilde{A}_a\right)$.

We make use of the form (7) of the set $\tilde{A}_a$.

Remark. When a increases, the sets $\tilde{A}_a$ decrease.
So, we look for minimal a, for which the inequality (17) holds. Let consider

any of martingale measures P^* of the form of (10) and find the minimal a, for which the following inequality holds:

$$E_{P^*}\left[HI_{\tilde{A}_a}\right] \le \nu. \tag{18}$$

In this case an investor with initial capital ν is able to hedge $HI_{\tilde{A}_a}$ almost surely, i.e., to hedge H with probability $P\left(\tilde{A}_a\right)$.

Theorem 3.1. *Let the function $f(x)$ satisfy the condition $\forall z \in R:$*

$$\lambda\left(\left\{\frac{x}{f(x)} = z\right\}\right) = 0, \tag{19}$$

where λ is the Lebesgue measure.

Then the probability of successful hedge of the claim $H = f(X_T)$ is at least $P\left(\tilde{A}_{\bar{a}}\right)$, where $\bar{a}$ is determined by the equation

$$\int_{C_{\bar{a}}} f\left(X_0 e^{\sqrt{T}\sigma y - \frac{1}{2}\sigma^2 T}\right)\frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = \nu \tag{20}$$

and

$$C_a = \left\{ y \,\middle|\, a < \frac{\exp\left(\sqrt{T}\sigma y - \frac{1}{2}\sigma^2 T\right)}{f\left(X_0 \exp\left(\sqrt{T}\sigma y - \frac{1}{2}\sigma^2 T\right)\right)} \right\}, \tag{21}$$

where

$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2}. \tag{22}$$

Proof. With respect to any of the martingale measures P^* of the form of (10) the price process X_T has the form (12) and it can be re-written in the following way:

$$X_T = X_0 \prod_{i=1}^{2} \exp\left(\sigma_i \hat{W}_T - \frac{\sigma_i^2}{2}\cdot T\right) =$$

$$= X_0 \exp(\sqrt{T}\sigma_1\xi_1 - \frac{\sigma_1^2}{2}T + \sqrt{T}\sigma_2\xi_1 - \frac{\sigma_2^2}{2}T),$$

$$= X_0 \exp(\sqrt{T}\sigma\xi - \frac{\sigma^2}{2}T), \tag{23}$$

where ξ_1, $\xi_2 \sim N(0,1)$ and $\xi = \sigma_1\xi_1 + \sigma_2\xi_2 \sim N(0, \sigma_1^2 + \sigma_2^2)$.

Remark. For another representation for X_t in the form of (13) the equality (14) holds and, thus, the representation (23) for X_T is true.

Then

$$\frac{X_T}{X_0 f(X_T)} = \frac{\exp\left(\sqrt{T}\sigma y - \frac{1}{2}\sigma^2 T\right)}{f\left(X_0 \exp\left(\sqrt{T}\sigma y - \frac{1}{2}\sigma^2 T\right)\right)}$$

and

$$E_{P^*}\left[HI_{\tilde{A}_a}\right] = \int\limits_{C_a} f\left(X_0 \exp\left(\sqrt{T}\sigma y - \frac{1}{2}\sigma^2 T\right)\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \quad . \tag{24}$$

Thanks to assumptions of the theorem for all $a \in R$ we have

$$\lambda\left(\left\{ y \,\middle|\, a < \frac{\exp\left(\sqrt{T}\sigma y - \frac{1}{2}\sigma^2 T\right)}{f\left(X_0 \exp\left(\sqrt{T}\sigma y - \frac{1}{2}\sigma^2 T\right)\right)} = a \right\}\right) = 0,$$

thus it follows from (24) that $E_{P^*}\left[HI_{\tilde{A}_a}\right]$ is continuous and non-decreasing function of a. This implies that for $\bar{a} = \inf\left\{a \,\middle|\, E_{P^*}\left[HI_{\tilde{A}_a}\right] \leq \nu\right\}$ one has $E_{P^*}\left[HI_{\tilde{A}_{\bar{a}}}\right] = \nu$, which is (20).

The set $\tilde{A}_{\bar{a}}$ is not necessarily the maximal success probability set for initial capital ν, nevertheless $P\left(\tilde{A}_{\bar{a}}\right)$ is lower estimate for the maximal probability. Therefore, one can guarantee successful hedge of the claim with probability not smaller than $P\left(\tilde{A}_{\bar{a}}\right)$. □

4. Quantile hedging for European call option

Let consider European call option

$$f(X_T) = (X_T - K)_+ , \tag{25}$$

where $K > 0$, and find the representation for the set $\tilde{A}_a = \left\{a < \frac{X_T}{X_0 \cdot f(X_T)}\right\}$ in this case.

Note that for $X_T \leq K$ the inequality $a < \frac{X_T}{X_0 \cdot f(X_T)} = \infty$ holds.

For $X_T \geq K$, the inequality $a < \frac{X_T}{X_0 \cdot f(X_T)}$ takes the form:

$$\frac{X_T}{X_0 (X_T - K)} > a,$$

$$X_T > (X_T - K)\, a X_0,$$

$$K a X_0 > X_T (a X_0 - 1). \tag{26}$$

If $aX_0 \leq 1$, then (26) is always true, i.e. $\tilde{A}_a = \Omega$. Thus, when $a \leq \frac{1}{X_0}$ we have that $\tilde{A}_a = \Omega$, but this cannot be true, because by the problem formulation it is impossible to hedge the claim almost surely, but on $\tilde{A}_a$ it will be hedged.

If $aX_0 > 1$, i.e. $a > \frac{1}{X_0}$, then $\tilde{A}_a = \left\{X_T < \frac{KaX_0}{aX_0-1}\right\}$, at that $K < \frac{KaX_0}{aX_0-1}$.

That is, the set $\tilde{A}_a$ can be presented in the following form:

$$\tilde{A}_a = \begin{cases} \Omega, & a \le \frac{1}{X_0}, \\ \left\{X_T < \frac{KaX_0}{aX_0-1}\right\}, & a > \frac{1}{X_0}. \end{cases} \tag{27}$$

Theorem 4.1. *For European call option (25) the maximal probability of successful hedge is at least $\Phi(U)$, where U is given by the equation*

$$X_0\left(\Phi\left(U-\sigma\sqrt{T}\right)-\Phi\left(L-\sigma\sqrt{T}\right)\right)-K\left(\Phi(U)-\Phi(L)\right)=\nu, \tag{28}$$

where

$$L=\frac{\ln\frac{K}{X_0}+\frac{\sigma^2}{2}T}{\sigma\sqrt{T}}. \tag{29}$$

and Φ - is standard normal distribution function.

Proof. The function $f(X_T) = (X_T - K)_+$, where $K > 0$, clearly satisfies the assumptions of Theorem 3.1. Then there exists $\bar{a}$, for which (20) holds or, equivalently, the inequality (18) turns to equality.

Since in this case the set $\tilde{A}_a$ has form (27), then the left-hand side (18) takes the form:

$$E_{P^*}\left[HI_{\tilde{A}}\right] = \int\limits_{\tilde{A}} (X_T-K)_+\,dP^* = \int\limits_{\left\{K<X_T<\frac{KaX_0}{aX_0-1}\right\}} (X_T-K)\,dP^*. \tag{30}$$

Accounting for (23) the inequality $K < X_T < \frac{KaX_0}{aX_0-1}$ can be rewritten as:

$$K < X_0 e^{\sigma\sqrt{T}\xi-\frac{\sigma^2}{2}T} < K\frac{aX_0}{aX_0-1};$$

$$\frac{\ln\frac{K}{X_0}+\frac{\sigma^2}{2}T}{\sigma\sqrt{T}} < \xi < \frac{\ln\frac{Ka}{aX_0-1}+\frac{\sigma^2 T}{2}}{\sigma\sqrt{T}}. \tag{31}$$

Denote:

$$U=U(a)=\frac{\ln\frac{Ka}{aX_0-1}+\frac{\sigma^2}{2}T}{\sigma\sqrt{T}}. \tag{32}$$

The inequality (31) takes the form $L < \xi < U(a)$, and equation (20) can be rewritten as:

$$E_{P^*}\left[HI_{\tilde{A}}\right] = \int\limits_{L}^{U(\bar{a})} \left(X_0 e^{\sigma\sqrt{T}y-\frac{\sigma^2}{2}T}-K\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\,dy =$$

$$= \int\limits_{L}^{U(\bar{a})} \frac{X_0 e^{\sigma\sqrt{T}y - \frac{\sigma^2}{2}T - \frac{y^2}{2}}}{\sqrt{2\pi}} dy - K\left(\Phi\left(U\left(\bar{a}\right)\right) - \Phi\left(L\right)\right) =$$

$$= X_0\left(\Phi\left(U\left(\bar{a}\right) - \sigma\sqrt{T}\right) - \Phi\left(L - \sigma\sqrt{T}\right)\right) - K\left(\Phi\left(U\left(\bar{a}\right)\right) - \Phi\left(L\right)\right) = \nu,$$

which means the equality (28).

Now compute $P\left(\tilde{A}_{\bar{a}}\right)$:

$$P\left(\tilde{A}_{\bar{a}}\right) = P\left(X_0 e^{\sigma\sqrt{T}\xi - \frac{\sigma^2}{2}T} < \frac{K\bar{a}X_0}{\bar{a}X_0 - 1}\right) = \tag{33}$$

$$= P\left(\xi < \frac{\ln\frac{K\bar{a}}{\bar{a}X_0 - 1} + \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}}\right) = \Phi\left(U\left(\bar{a}\right)\right)$$

If we find $U\left(\bar{a}\right)$ from equation (28), we can find $P\left(\tilde{A}_{\bar{a}}\right)$.

Note that to compute $P\left(\tilde{A}_{\bar{a}}\right)$ it is enough to find $U = U\left(\bar{a}\right)$, and it is not needed to evaluate $\bar{a}$. □

Example 4.1. For European call option (25) formulae (28), (33) allow for wishful value of $P\left(\tilde{A}_a\right)$ to compute $U = U_a$ and, correspondingly, the necessary capital ν, or vice versa: having ν to compute the success probability $P\left(\tilde{A}_a\right)$. Note that hedge always succeeds with probability at least $P\left(X_T < K\right) = \Phi\left(L\right)$, where L is given by formula (29), thus for $X_T < K$ there is nothing to pay for the claim, which means that it will be hedged with the void strategy. On the other hand, when the success probability $P\left(\tilde{A}_a\right)$ increases from $\Phi\left(L\right)$ to 1 the necessary capital ν increases from 0 to option's fair value H_0.

Let $H = 0,8$, $X_0 = 1$, $T = 10$, $m = 2$, $\sigma_1 = \sigma_2 = \mu_2 = \mu_2 = \frac{1}{2\sqrt{2}}$, then for different (depending on K) European calls, fixing $P\left(\tilde{A}_a\right)$, we can compute corresponding values ν.

5. Conclusion

The model defined by two independent Brownian and two fractional Brownian motions is considered. But all obtained results may be generalized for the model defined by the finite number of independent Brownian and fractional Brownian motions. The set of martingale measures for the price process model is constructed, which proves the absence of arbitrage and incompleteness of the financial market. The set $\tilde{A}_{\bar{a}}$ depending on the available initial investor's capital ν on which the contingent claim is hedged

K	$\Phi(L)$	$P\left(\tilde{A}_a\right)$	ν	H_0
5	0,964733	0,99	0,055677	0,23375
2	0,890456	0,9	0,000807	0,418561
		0,99	0,210488	
1	0,785402	0,9	0,053051	0,570805
		0,99	0,352732	
0,5	0,63765	0,9	0,141527	0,709281
		0,99	0,486208	
0,1	0,252797	0,9	0,305201	0,912955
		0,99	0,685882	
0,01	0,016919	0,9	0,373309	0,990063
		0,99	0,76209	

with probability 1 is constructed. The maximal success probability of contingent claim hedging is estimated from below by $P(\tilde{A}_{\bar{a}})$. For different parameters of the European call option the probability $P(\tilde{A}_{\bar{a}})$ is computed.

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