Theory of Stochastic Processes
Vol. 16 (32), no. 2, 2010, pp. 120–125

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SOME PROPERTIES OF STRONGLY AND WEAKLY DIVISIBLE STATISTICAL STRUCTURES IN A BANACH SPACE OF MEASURES

Some properties of divisible statistical structures in a Banach space of measures are considered. Linear functionals of the integral type on the Banach space of measures are defined, and the criteria of weak and strong divisibilities of statistical structures are given in terms of such sets of functionals.

It is well known that the divisible statistical structures play a decisive role in many problems of the statistics of random processes (see, e.g, [1]). As it turned out ([2]–[3]), these problems can be investigated using the Banach space theory. In particular this can be accomplished by considering the linear shells of alternating countably additive finite measures which are stretched onto the initial family of probability measures. The choice of such a family allows us to represent a Banach space of measures as a direct sum of subspaces. Also, linear functionals of the integral type on a Banach space of measures are defined in this paper, and the criteria of weak and strong divisibilities of statistical structures are given in terms of such sets of functionals.

Let $\{E, S\}$ be a measurable space, I the set of indices, $\{\mu_i,\, i \in I\}$ the family of probability measures on S.

Definition 1. An object $\{E, S, \mu_i, i \in I\}$ is called a statistical structure.

Definition 2. A statistical structure $\{E, S, \mu_i, i \in I\}$ is called orthogonal (singular) if a family of probability measures $\{\mu_i, i \in I\}$ consists of pairwise singular measures.

Simple examples convince us that the notion of measure orthogonality is not transitive.

Definition 3. A statistical structure $\{E, S, \mu_i, i \in I\}$ is called weakly divisible if there exists a family of S-measurable sets $\{X_i,\, i \in I\}$, such that the following relations are fulfilled:

$$\mu_i(X_j) = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Definition 4. A statistical structure $\{E, S, \mu_i, i \in I\}$ is called divisible if there exists a family of S-measurable sets $\{X_i, i \in I\}$, such that the following relations are fulfilled:

$$1)\ \ \mu_i(X_j) = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

$$2)\ \ \forall i \neq j, \quad \mathrm{Card}(X_i \cap X_j) < 2^{\aleph_0}\,.$$

Definition 5. A statistical structure $\{E, S, \mu_i, i \in I\}$ is called strongly divisible if there exists a disjunctive family of S-measurable sets $\{X_i, i \in I\}$, such that the following relations are fulfilled:

$$\forall i \in I, \quad \mu_i(X_i) = 1.$$

2000 *Mathematics Subject Classification.* 62H05 ; 62H12.

Key words and phrases. Statistical structures, measure, Banach space.

Remark 1. Strong divisibility yields divisibility, divisibility yields weak divisibility, and weak divisibility yields orthogonality. But not *vice versa*. It is well known (see [2]) that if $\operatorname{Card} E = c$, then we can construct an orthogonal statistical structure having maximal possible power 2^{2^c}, a weakly divisible structure having maximal possible structure 2^c, and a strongly divisible structure having maximal possible power c.

Example 1. Let $E = [0,1] \times [0,1]$, S be a Borel σ-algebra of parts of E. Take the measurable sets

$$X_i = \{0 \le x \le 1,\ \ y = i,\ \ i \in [0,1]\}$$

and assume that l_i are linear Lebesgue probability measures on X_i. Then a statistical structure $\{[0,1] \times [0,1], S, l_i, i \in [0,1]\}$ is strongly divisible.

Example 2. Let $E = [0,1] \times [0,1]$, S be a Borel σ-algebra of parts of E. Take the measurable sets

$$X_i = \begin{cases} 0 \le x \le 1, & y = i, \quad i \in [0,1], \\ x = i - 2, & 0 \le y \le 1, \quad i \in [2,3]. \end{cases}$$

Let l_i be linear Lebesgue probability measures on X_i. Then a statistical structure $\{[0,1] \times [0,1], S, l_i, i \in [0,1] \cup [2,3]\}$ is divisible, but not strongly divisible.

Example 3. Let $E = [0,1] \times [0,1]$, S be a Borel σ-algebra of parts of E. Take the measurable sets

$$X_i = \begin{cases} 0 \le x \le 1, & 0 \le y \le 1, \quad z = i, \quad i \in [0,1], \\ x = i - 2, & 0 \le y \le 1, \quad 0 \le z \le 1, \quad i \in [2,3], \\ 0 \le x \le 1, & y = i - 4, \quad 0 \le z \le 1, \quad i \in [4,5], \end{cases}$$

and assume that l_i are plane Lebesgue measures on X_i. Then a statistical structure $\{E, S, l_i, i \in [0,1] \cup [2,3] \cup [4,5]\}$ is weakly divisible, but not divisible.

Example 4. Let $E = [0,1] \times [0,1]$, S be a Borel σ-algebra of parts of E. Consider the sets

$$X_i = \begin{cases} 0 \le x \le 1, & y = i, \quad i \in (0,1], \\ 0 \le x \le 1, & 0 \le y \le 1, \quad i = 0, \end{cases}$$

and assume that l_i, $i \in (0,1]$, are linear Lebesgue measures on $\{0 \le x \le 1,\ y = i\}$, and l_0 is a plane Lebesgue measure on $[0,1] \times [0,1]$. Then a statistical structure $\{E, S, l_i, i \in [0,1]\}$ is orthogonal, but not weakly divisible.

Let M^σ be a linear real space of all alternating finite measures on S.

Definition 6. A linear subset $M_\beta \subset M^\sigma$ is called a Banach space of measures if

1) a norm can be defined on M_β so that M_β will be a Banach space with respect to this norm, and, for any orthogonal measures $\mu, \nu \in M_\beta$ and any real number $\lambda \neq 0$, the following inequality is fulfilled:

$$\|\mu + \lambda\nu\| \ge \|\mu\|;$$

2) if $\mu \in M_\beta$ and $|f(x)| \le 1$, then

$$\nu_f(A) = \int_A f(x)\nu(dx) \in M_\beta,$$

where $f(x)$ is a S-measurable real function and $\|\nu_f\| \le \|\nu\|$.

Example 5. Let $\{\mu_\alpha,\ \alpha \in A\}$ be pairwise orthogonal probability measures on $\{E, S\}$, and let $g_\alpha(x)$ be real S-measurable functions. M_β is the set of measures of the form

$$\nu(B) = \sum_{\alpha \in A_\nu} \int_B g_\alpha(x)\mu_\alpha(dx),$$

where $A_\nu \subset A$ is a countable set from A and

$$\sum_{\alpha \in A_\nu} \int_E |g_\alpha(x)|\mu_\alpha(dx) < \infty.$$

Let

$$\|\nu\| = \sum_{\alpha \in A_\nu} \int_E |g_\alpha(x)|\mu_\alpha(dx).$$

Then M_β is Banach space of measures. Let $\nu(B) = \sum_{\alpha \in A_\nu} \int_B g_\alpha(x)\mu_\alpha(dx)$, $\mu(B) = \sum_{\alpha \in A_\mu} \int_B f_\alpha(x)\mu_\alpha(dx)$, and $\mu \perp \nu$, $\overline{A} = A_\nu \cap A_\mu$. Then

$$\sum_{\alpha \in \overline{A}} g_\alpha(x) f_\alpha(dx) = 0 \quad \text{a.s. } \mu_{\overline{A}}.$$

We have

$$\begin{aligned}
\|\nu + \lambda\mu\| &= \sum_{\alpha \in \overline{A}} \int_E |g_\alpha(x) + \lambda f_\alpha(x)|\mu_\alpha(dx) \\
&= \sum_{\alpha \in \overline{A}} \left\{ \int_{\{x : \sum f_\alpha(x) = 0\}} |g_\alpha(x)|\mu_\alpha(dx) + |\lambda| \int_{E \setminus \{x : \sum f_\alpha(x) = 0\}} |f_\alpha(x)|\mu_\alpha(dx) \right\} \\
&= \sum_{\alpha \in \overline{A}} \int_{\{x : \sum f_\alpha(x) = 0\}} |g_\alpha(x)|\mu_\alpha(dx) + |\lambda| \sum_{\alpha \in \overline{A}} \int_{E \setminus \{x : \sum f_\alpha(x) = 0\}} |f_\alpha(x)|\mu_\alpha(dx) \\
&\geq \|\nu\|.
\end{aligned}$$

Theorem 1 ([3]). *Let M_β be a Banach space of measures. Then there exists a family of pairwise orthogonal probability measures $\{\mu_i, i \in I\}$ from this space such that*

$$M_\beta = \underset{i \in I}{\oplus} M_\beta(\mu_i),$$

where $M_\beta(\mu_i)$ is a Banach space of elements ν of the form

$$\nu(B) = \int_B f(x)\mu_i(dx), \quad B \in S, \qquad \int_E |f(x)|\mu_i(dx) < \infty$$

with the norm

$$\|\nu\|_{M_\beta(\mu_i)} = \int_E |f(x)|\mu_i(dx).$$

Let M_β be a Banach space of measures. Fix the set of measures $\{\mu_i, i \in I\}$, for which $M_\beta = \underset{i \in I}{\oplus} M_\beta(\mu_i)$. The following lemmas are easy to be proved.

Lemma 1. *If $\int_E f(x)\nu(dx)$ is defined for all $\nu \in M_\beta$, then there exists at most a countable set $I_0 \subset I$, for which*

$$\int_E f(x)\mu_i(dx) \neq 0, \quad i \in I_0.$$

In addition,

$$\sum_{i \in I_0} \int_E |f(x)|\mu_i(dx) < \infty.$$

For any measure ν of the form $\nu(C) = \sum_{i \in I_1} \int_C g_i(x)\mu_i(dx)$, we have $\nu \in M_\beta$ and

$$\int_E f(x)\nu(dx) = \sum_{i \in I_0 \cap I_1} \int_E g_i(x) f(x)\mu_i(dx).$$

Lemma 2. *If $\int_E f(x)\nu(dx)$ is defined for all $\nu \in M_\beta$, then $\int_E f(x)\nu(dx)$ is a continuous functional with respect to ν (in the metric M_β).*

Denote, by $F(M_\beta)$, the set of those f, for which $\int_E f(x)\nu(dx)$ is defined for all $\nu \in M_\beta$.

Theorem 2. *Let $M_\beta = \bigoplus_{i \in I} M_\beta(\mu_i)$. For an orthogonal statistic structure $\{E, S, \mu_i, i \in I\}$ to be weakly divisible, it is necessary and sufficient that the correspondence $f \to l_f$ given by the equality*

$$\int_E f(x)\nu(dx) = l_f(\nu), \quad \forall \nu \in M_\beta,$$

be one-to-one. Here, $l_f(\nu)$ is a linear continuous functional on M_β, $f \in F(M_\beta)$.

Proof. Sufficiency. For $f \in F(M_\beta)$, we define a linear continuous functional l_f by the equality

$$\int_E f(x)\nu(dx) = l_f(\nu).$$

Denote, by I_f, the countable subset I, for which

$$\int_E f(x)\mu_i(dx) = 0 \quad \text{for} \quad i \notin I_f,$$

This is possible by virtue of Lemma 1. Let us consider the functional l_{f_ψ} on $M_\beta(\mu_i)$, to which f_ψ corresponds. Then, for $\psi_1, \psi_2 \in M_\beta(\mu_i)$,

$$\int_E f_{\psi_1}\psi_2(dx) = l_{f_{\psi_1}}(\psi_2) = \int_E f_1(x) f_2(x)\mu_i(dx) = \int_E f_{\psi_1}(x) f_2(x)\mu_i(dx).$$

Therefore, $f_{\psi_1} = f_1$ a.e. with respect to the measure μ_i. Let $f_i(x) > 0$ a.e. with respect to the measure μ_i and

$$\int_E f_i(x)\mu_i(dx) < \infty, \quad \mu_i^*(C) = \int_C f_i(x)\mu_i(dx).$$

Then

$$\int_E f_{\mu_i^*}(x)\mu_j(dx) = l_{f_{\mu_i^*}}(\mu_j) = 0, \quad \forall j \neq i.$$

Denote $C_i = \{x : f_{\mu_i^*}(x) > 0\}$. Then

$$\int_E f_{\mu_i^*}(x)\mu_j(dx) = l_{f_{\mu_i^*}}(\mu_j) = 0, \quad \forall j \neq i.$$

Hence, it follows that

$$\mu_j(C_i) = 0, \quad \forall j \neq i.$$

On the other hand,

$$\mu_i^*(E - C_i) = \int_{E - C_i} f_{\mu_i}(x)\mu_i(dx) = \int_E f_{\mu_i}(x) I_{(E - C_i)}(x)\mu_i(dx)$$
$$= \int_E f_{\mu_i^*}(x) I_{(E - C_i)}(x)\mu_i(dx) = 0$$

since $f_{\mu_i^*} = f_{\mu_i}(x)$ a.e. with respect to the measure μ_i and $f_{\mu_i^*}(x) I_{(E - C_i)}(x) \equiv 0$. The sufficiency is proved.

Necessity. Since the statistical structure $\{E, S, \mu_i, i \in I\}$ is weakly divisible, there exist S-measurable sets C_i such that $\mu_i(E - C_i) = 0$ and $\mu_j(C_i) = 0$, $j \neq i$. We put the linear continuous functional l_{C_i} into correspondence to a function $I_{C_i}(x) \in F(M_\beta)$ by the formula

$$\int_E I_{C_i}(x)\mu_i(dx) = l_{C_i}(\mu_i) = \|\mu_i\|_{M_\beta(\mu_i)}.$$

We put the linear continuous functional $l_{f_{\psi_1}}$ into correspondence to the function $f_{\psi_1}(x) = f_1(x)I_{C_i}(x) \in F$. Then, for any $\psi_2 \in M_\beta(\mu_i)$,

$$\int_E f_{\psi_1}(x)\psi_2(dx) = \int_E f_1(x)I_{C_i}(x)\psi_2(dx) = \int_E f(x)f_1(x)I_{C_i}(x)\mu_i(dx)$$
$$= l_{f_{\psi_1}}(\psi_2) = \|\psi_2\|_{M_\beta(\mu_i)}.$$

Let $\mathcal{E}$ be the collection of extensions of functionals l satisfying the condition $l_f \le p(x)$ on those subspaces, where they are defined. Let us introduce, on $\mathcal{E}$, a partial ordering, having assumed $l_{f_1} < l_{f_2}$, if l_{f_2} is defined on a set larger than l_{f_1}, and $l_{f_2}(x) = l_{f_1}(x)$ there, where both of them are defined. Let $\{l_{f_i}\}_{i\in I}$ be a linear ordered subset in $\mathcal{E}$. Let $M_\beta(\mu_i)$ be the subspace, on which l_{f_i} is defined. Define l_f on $\bigcup\limits_{i\in I} M_\beta(\mu_i)$, having assumed $l_f(x) = l_{f_i}(x)$ if $x \in M_\beta(\mu_i)$. It is obvious that $l_{f_i} < l_f$. Since any linearly ordered subset in $\mathcal{E}$ has an upper bound, by virtue of Horn's lemma, $\mathcal{E}$ contains a maximal element Λ defined on some set X' and satisfying the condition $\Lambda(x) \le p(x)$ for $x \in X'$. But X' must coincide with the entire space M_β, because, otherwise, we could extend Λ to a wider space by adding, as above, one more dimension. This contradicts the maximality of Λ. Hence, $X' = M_\beta$. Therefore, the extension of the functional is defined everywhere.

If we put the linear continuous functional l_f into correspondence to the function $f(x) = \sum\limits_{i\in I} g_i(x)I_{C_i}(x) \in F(M_\beta)$, then we obtain $\int\limits_E f(x)\nu(dx) = \|\nu\| = \sum\limits_{i\in I_0} \|\mu_i\|_{M_\beta(\mu_i)}$, where $\nu = \sum\limits_{i\in I_0} \int\limits_E g_i(x)\mu_i(dx)$.

Theorem 2 is thereby proved. □

Remark 2. From the proved theorem, it follows that the above-indicated correspondence puts some function $f \in F(M_\beta)$ into correspondence to each linear continuous functional l_f. If, in $F(M_\beta)$, we identify the functions coinciding with respect to the measure $\{\mu_i, i \in I\}$, then the correspondence will be bijective.

It is also well known that, in the $(ZFC)\&(CH)\&(MA)$ theory, there exists a continual weakly divisible structure that is not strongly divisible. Here and in the sequel, we denote, by (MA), Martin's axiom ([4]).

Theorem 3. *Let $M_\beta = \bigoplus\limits_{i\in I} M_\beta(\mu_i)$, E be a total metric space, and $\{\mu_i, i \in I\}$ be the family of pairwise orthogonal Borel probability measures on the space E. Let* $\operatorname{Card} I < 2^{\aleph_0}$*. In the $(ZFC \,\&\, MA)$ theory, for an orthogonal Borel statistical structure $\{E, S, \mu_i, i \in I\}$ to be strongly divisible, it is necessary and sufficient that the correspondence $f \to l_f$ given by the equality*

$$\int_E f(x)\nu(dx) = l_f(\nu), \quad \forall \nu \in M_\beta,$$

be one-to-one, and $l_f(\nu)$ be a linear continuous functional.

The necessity is proved in the same manner as the necessity in Theorem 2. We will show the sufficiency.

According to Theorem 2, the Borel orthogonal statistical structure $\{E, S, \mu_i, i \in I\}$, $\operatorname{Card} I < 2^{\aleph_0}$, is weakly divisible. We represent $\{\mu_i, i \in I\}$ as an inductive sequence $\{\mu_i, i < \omega_\alpha\}$, where ω_α denotes the first ordinal number of the power of the set I. Since

$\{E, S, \mu_i, i \in I\}$ is weakly divisible, there exists a family of measurable parts $\{X_i\}_{i<\omega_\alpha}$ of the space E, such that the following relation is fulfilled:

$$(\forall i)(\forall j)\,(i \in [0, \omega_\alpha) \& j \in [0, \omega_\alpha) \Rightarrow \mu_i(X_j)) = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

We define the ω_α-sequence of parts of the space E, so that the following relations are fulfilled:

1) $(\forall i)\,(i < \omega_\alpha \Rightarrow B_i$ is a Borel subset in $E)$;
2) $(\forall i)\,(i < \omega_\alpha \Rightarrow B_i \subset X_i)$;
3) $(\forall i_1)(\forall i_2)(i_1 < \omega_\alpha)\&(i_2 < \omega_\alpha)\&(i_1 \neq i_2) \Rightarrow B_{i_1} \cap B_{i_2} = \varnothing$;
4) $(\forall i)(i < \omega_\alpha \Rightarrow \mu_i(B_i) = 1)$.

Assume that $B_0 = X_0$. Let the partial sequence $(B_j)_{j<i}$ be already defined for $i < \omega_\alpha$. It is clear that $\mu^*\Big(\bigcup\limits_{j<i} B_j\Big) = 0$. Thus, there exists a Borel subset Y_i of the space E such that the following relations are valid:

$$\bigcup_{j<i} B_j \subset Y_i \quad \text{and} \quad \mu_i(Y_i) = 0.$$

Assume $B_i = X_i - Y_i$.

Thereby, the ω_α-sequence of $(B_i)_{i<\omega_\alpha}$-disjunctive measurable subsets of the space E is constructed. Therefore, $(\forall i)(i < \omega_\alpha \Rightarrow \mu_i(B_i) = 1)$. The theorem is proved.

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