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STOCHASTIC FLOWS AND SIGNED MEASURE VALUED STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

Let N point particles be distributed over $\mathbb{R}^d$, $d \in \mathbb{N}$. The position of the i-th particle at time t will be denoted $r(t,q^i)$ where q^i is the position at $t=0$. m_i is the mass of the i-th particle. Let δ_r be the point measure concentrated in r and $\mathcal{X}_N(0) := \sum_{i=1}^N m_i \delta_{q^i}$ the initial mass distribution of the N point particles. The empirical mass distribution (also called the "empirical process") at time t is then given by [1]

$$\mathcal{X}_N(t) := \sum_{i=1}^N m_i \delta_{r(t,q^i)} = \int \delta_{r(t,q)} \mathcal{X}_N(0,dq),$$

i.e., by the $N-$particle flow. In Kotelenez (2008) the masses are positive and the motion of the positions of the point particles is described by a stochastic ordinary differential equation (SODE). Further, the resulting empirical process is the solution of a stochastic partial differential equation (SPDE) which, by a continuum limit, can be extended to an SPDE in smooth positive measures. Some generalizations to the case of signed measures with applications in 2D fluid mechanics have been made.[2] We extend some of those results and results of Kotelenez (2008), showing that the signed measure valued solutions of the SPDEs preserve the Hahn-Jordan decomposition of the initial distributions which has been an open problem for some time.

1. Introduction

$(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is a stochastic basis with right continuous filtration, and the measure P is complete. All our stochastic processes are assumed to live on Ω and to be $\mathcal{F}_t$-adapted, including all initial conditions in stochastic ordinary differential equations (SODEs) and stochastic partial differential equations (SPDEs). Moreover, the processes are assumed to be $dP \otimes dt$-measurable, where dt is the Lebesgue measure on $[0,\infty)$. [1]

The stochastic component of the displacement of $r(t,q^i)$ in a short time increment should be Brownian (multiplied by some diffusion coefficient, which may depend both on $r(t,q^i)$ and on $\mathcal{X}_N(t)$).

Following Kotelenez (1995a,b) we employ i.i.d. Gaussian standard white noise random fields $w_\ell(dq,dt)$ on $\mathbb{R}^d \times \mathbb{R}_+$, $\ell = 1,\dots,d$, as a stochastic perturbation for the positions of the particles.[2] We list some of the properties of $w_\ell(dr,dt)$:

Denote the Lebesgue measure of a Borel set $B \in \mathbb{R}^k$ by $|B|$. Let $A_i \in \mathcal{B}^d$ (the Borel sets in $\mathbb{R}^d$) with finite Lebesgue measure $|A_i|$, and $[s_i,t_i]$ be intervals in $\mathbf{R}_+$, $i=1,2$. Then $\int_u^v \int_B w_\ell(dr,d\tilde{u})$, $B \in \{A_i, i=1,2\}$ and $[u,v] \in \{[s_i,t_i], i=1,2\}$ are Gaussian random random variables with mean 0 and covariance $|A_1 \cap A_2||[s_1,t_1] \cap [s_2,t_2]|$. $\int_A w_\ell(dq,t) := \int_0^t \int_A w_\ell(dq,du)$ is $\mathcal{F}_t$-measurable for any $A \in \mathcal{B}^d$ with $|A| < \infty$ and any t. The stochastic integration with respect to $w_\ell(dr,dt)$ follows the pattern established by Walsh (1986),

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[1] Cf., e.g., Metivier and Pellaumail (1980), Ch. 1.2

[2] Note that $w_\ell(dq,dt)$ may be considered space-time differentials of properly defined Brownian sheets with parameters $\mathbb{R}^d \times \mathbb{R}_+$. Cf. Kotelenez (2008), Ch. 2, 15.

where $w_\ell(dr,dt)$ was used as a driving term for stochastic partial differential equations (SPDEs). Set $w(dr,t) := (w_1(dr,t),\dots,w_d(dr,t))^T$ where C^T denotes the transpose of a matrix C. Further, let $\mathcal{M}_{d\times d}$ denote the $d\times d$ matrices over $\mathbb{R}$. Let $\mathcal{J}_\varepsilon(r,q,\mu,t)$ be a "nice" $\mathcal{M}_{d\times d}$ valued function, jointly measurable in all arguments, depending on the position of the particle, the spatial noise coordinate, the (finite signed measure valued) μ and time t and a correlation parameter $\varepsilon>0$.[3] In addition to Lipschitz and measurability assumptions, "nice" means here that the one-dimensional components of $\mathcal{J}_\varepsilon(r,q,\dots,t)$ have to be square integrable in q with respect to the Lebesgue measure dq. Similar conditions are assumed for the one-dimensional component of the $\mathbb{R}^d$-valued function F. Consider the following system of SODEs driven by $w(dr,dt)$:[4]

$$\left.\begin{aligned} dr_\varepsilon^{i,\vec{N}} = {}& F(r_\varepsilon^{i,\vec{N}}(t),\mathcal{X}_{\varepsilon,N}(t),t)dt + \int \mathcal{J}_\varepsilon(r_\varepsilon^{i,\vec{N}}(t),p,\mathcal{X}_{\varepsilon,N}(t),t)w(dp,dt),\\ r_\varepsilon^{i,\vec{N}}(s) = {}& q_\varepsilon^i, i=1,\dots,N,\quad \mathcal{X}_{\varepsilon,N}(t) := \textstyle\sum_{i=1}^N m_i\delta_{r_\varepsilon^{i,\vec{N}}(t)},\ m_i\in\mathbb{R}\setminus\{0\}. \end{aligned}\right\} \tag{1}$$

Under appropriate Lipschitz conditions (cf. (2.3)) Kotelenez (1995b) shows that (1.1) has a unique strong Itô solution which is an $\mathbb{R}^{dN}$-valued diffusion process. The two-particle[5] and one-particle diffusion matrices of the noise are given by:

$$\left.\begin{aligned} \tilde{D}_\varepsilon(r^i,r^j,\mu,t) &:= \int \mathcal{J}_\varepsilon(r^i,p,\mu,t)\mathcal{J}_\varepsilon^T(r^j,p,\mu,t)dp \quad \forall\, i,j=1,\dots,N,\\ D(r,\mu,t) &:= \tilde{D}(r,r,\mu,t), \end{aligned}\right\} \tag{2}$$

where "AB" denotes the matrix multiplication of matrices A. Further, "$A_{k\ell}$" will denote the entries of the matrix A, and μ is a finite signed Borel measure on $\mathbb{R}^d$. For $m\in\mathbb{N}\cup\{0\}$ let $C^m(\mathbb{R}^d;\mathbb{R})$ be the space of m times continuously differentiable functions from $\mathbb{R}^d$ into $\mathbb{R}$. Further, let $C_0^m(\mathbb{R}^d;\mathbb{R})$ be the subspace of $C^m(\mathbb{R}^d;\mathbb{R})$, whose elements together with all their derivatives vanish at infinity. We denote by "$\bullet$" the inner product in $\mathbb{R}^d$ and by ∂_k and $\partial_{k\ell}^2$ the first and second partial derivatives w.r.t. the spatial coordinates r_k and r_k,r_ℓ, respectively. Itô's formula, applied to $\langle\mathcal{X}_{\varepsilon,N}(t),\varphi\rangle$ for $\varphi\in C_0^2(\mathbb{R}^d;\mathbb{R})$, yields the quasilinear SPDE associated with (1.1) where the derivatives are taken in the distributional sense.[6]

$$\left.\begin{aligned} d\mathcal{X}_\varepsilon = {}& \{\tfrac12 \textstyle\sum_{k,\ell=1}^d \partial_{k\ell}^2(D_{\varepsilon,k\ell}(\mathcal{X}_\varepsilon,t)\mathcal{X}_\varepsilon) - \nabla\bullet(\mathcal{X}_\varepsilon F(\cdot,\mathcal{X}_\varepsilon,t))\}dt\\ - {}& \nabla\bullet(\mathcal{X}_\varepsilon \textstyle\int \mathcal{J}_\varepsilon(\cdot,p,\mathcal{X}_\varepsilon,t)w(dp,dt)). \end{aligned}\right\} \tag{3}$$

For notational convenience we will in what follows suppress the dependence on ε in the coefficients and solutions.[7] Let us first assume that the weights m_i in (1.1) are positive and derive appropriate metrics for (1.1)/(1.3).[8] Let $|\cdot|$ denote the Euclidean metric on $\mathbb{R}^d$ and "$\wedge$" denote "minimum". Set

$$\rho(r-q) := |r-q|\wedge 1,\quad \tilde{\rho}(\cdot)\in\{|\cdot|,\rho(\cdot)\}. \tag{4}$$

[3]Cf. Kotelenez (2008), Ch. 5.

[4]We abbreviate $r_{\varepsilon,N}^i(t) := r_{\varepsilon,N}(t,q^i)$.

[5]The two-particle diffusion matrix, describing the pair correlations of the noise perturbations, is the time derivative of the mutual (tensor) quadratic variation of the noise.

[6]Cf. the following Lemma 1.3.

[7]If $\varepsilon\to 0$ Kotelenez and Kurtz (2010) show that positive solutions of (1.3) converge towards the solution of a deterministic quasilinear SPDE, provided the initial conditions converge appropriately. For fixed $\varepsilon>0$ Kotelenez, Leitman and Mann (2009) analyze the short and long time behavior of the flow of correlated Brownian motions and show that correlated Brownian motions exhibit the depletion effect which has been experimentally observed in colloids.

[8]For positive mass distributions in a one-dimensional domain the relation between flows and SPDEs has been investigated by Dorogovtsev (2007).

The space of all continuous Lipschitz functions f from $\mathbb{R}^d$ into $\mathbb{R}$ will be denoted $C_L(\mathbb{R}^d;\mathbb{R})$. Further, set

$$\mathbf{B}_{L,\infty} := C_{L,\infty}(\mathbb{R}^d;\mathbb{R})$$

which is the space of all uniformly bounded Lipschitz functions f from $\mathbb{R}^d$ into $\mathbb{R}$. We endow $\mathbf{B}_{L,\infty}$ with the norm[9]

$$\left.\begin{array}{c} \|f\|_{L,\infty} := \|f\|_L \vee |||f||| \\ \text{where } \; |||f||| := \sup_q |f(q)|; \quad \|f\|_L := \sup_{\{r\neq q, |r-q|\le 1\}} \frac{|f(r)-f(q)|}{\rho(r-q)}. \end{array}\right\} \tag{5}$$

Set

$$\mathbf{M}_f := \{\mu : \mu \text{ is a finite Borel measure on } \mathbb{R}^d\}. \tag{6}$$

We define a metric on $\mathbf{M}_f$ by [10]

$$\gamma_f(\mu-\nu) := \sup_{\|f\|_{L,\infty}\le 1} |\int f(r)(\mu-\nu)(dr)|, \tag{7}$$

and we conclude that $(\mathbf{M}_f, \gamma_f)$ is a complete separable metric space. [11]

We now define the space of finite signed Borel measures

$$\left.\begin{array}{l} \mathbf{M}_{f,s} := \{\mu : \mu = \mu_+ - \mu_-, \; \mu_\pm \in \mathbf{M}_f\}, \\ \text{denoting by } \mu^\pm \text{ the Hahn-Jordan decomposition of } \mu. \end{array}\right\} \tag{8}$$

The metric γ_f may obviously be extended to $\mathbf{M}_{f,s}$.

Lemma 1.1 $(\mathbf{M}_{f,s}, \gamma_f)$ is not complete.

Proof: Set $\mu_n : \sum_{k=1}^n (-1)^k \delta_{(\frac{1}{k+1},0,\cdots,0)}$ with 0 in the coordinates from 2 to d. Take an arbitrary $f \in \mathbf{B}_{L,\infty}$ such that $\|f\|_{L,\infty} \le 1$. Then for $n > m$

$$|\int f(r)(\mu_n-\mu_m)(dr)| = |\sum_{k=m+1}^n (-1)^k f(\tfrac{1}{k+1},0,\cdots,0)|$$

$$\le \sum_{k=m+1}^n |f(\tfrac{1}{k},0,\cdots,0) - f(\tfrac{1}{k+1},0,\cdots,0)| \le \sum_{k=m+1}^n |\tfrac{1}{k} - \tfrac{1}{k+1}|$$

$$\le \sum_{k=m+1}^n \tfrac{1}{k(k+1)} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty .$$

Hence,

$$\gamma_f(\mu_n-\mu_m) = \sup_{\|f\|_{L,\infty}\le 1} |\int f(r)(\mu_n-\mu_m)(dr)| \le \sum_{k=m+1}^n \frac{1}{k(k+1)}.$$

It follows that $\{\mu_n\}$ is a Cauchy sequence in $(\mathbf{M}_{f,s}, \gamma_f)$.

Suppose there is a (finite) signed measure μ such that

$$\gamma_f(\mu_n - \mu) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty .$$

Then for any $f \in \mathbf{B}_{L,\infty}$ with $\|f\|_{L,\infty} \le 1$ we must have

$$|\int f(r)[\mu_n-\mu](dr)| \le \gamma_f(\mu_n-\mu) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty .$$

[9] In the definition of the Lipschitz norm (1.5) we may, without loss of generality, restrict the quotient to $|r-q| \le 1$, since for values $|r-q| > 1$ the quotient is dominated by $2|||f|||$. This implies, in particular, that the Lipschitz norm is independent of the choice $\tilde{\rho}(\cdot)$.

[10] Since $\|\cdot\|_{L,\infty}$ does not depend on the choice $\tilde{\rho}(\cdot)$ from (1.1), the same holds for $\gamma_f(\cdot)$.

[11] Cf. Kotelenez (2008), Ch. 15.1.4, Th. 15.9.

However, choosing $f(r) \equiv 1$ we have[12] $\int 1_{\mathbb{R}^d}(r)\mu_n(dr) = -1$ if n is odd and 0 if n is even. $\int 1_{\mathbb{R}^d}(r)\mu(dr) =: c_\mu \in \mathbb{R}$, Therefore, for $f(r) \equiv 1$

$$|\int 1_{\mathbb{R}^d}(r)[\mu_n - \mu](dr)| := \begin{cases} |1 + c_\mu|, & n = 2k+1, \\ |c_\mu|, & n = 2k. \end{cases}$$

Therefore, this functional does not converge to 0, which completes the proof. □

We next proceed as in Kotelenez (1995a,b) and consider in the first step the product space[13]

$$\left.\begin{array}{c} \hat{\mathbf{M}} := \mathbf{M}_f \times \mathbf{M}_f = \{\hat{\mu} := (\mu_+, \mu_-) : \mu_\pm \in \mathbf{M}_f\} \\ \text{with metric} \\ \hat{\gamma}_f(\hat{\mu}_1, \hat{\mu}_2) := \sqrt{\gamma_f^2(\mu_{1+} - \mu_{2+}) + \gamma_f^2(\mu_{1-} - \mu_{2-})}. \end{array}\right\} \tag{9}$$

We obtain that $(\hat{\mathbf{M}}, \hat{\gamma}_f(\cdot,\cdot))$ is a complete separable metric space, and we may identify the Hahn-Jordan decomposition $\mu^\pm$ of a signed measure μ with the element $(\mu^+, \mu^-) \in \hat{\mathbf{M}}$. Define the following map Φ:

$$\left.\begin{array}{c} \Phi : (\hat{\mathbf{M}}, \hat{\gamma}_f) \to (\mathbf{M_{f,s}}, \gamma_f) \\ \hat{\mu} := (\mu_+, \mu_-) \mapsto \mu := \mu_+ - \mu_-. \end{array}\right\} \tag{10}$$

Lemma 1.2

$$\gamma_f(\Phi(\hat{\mu}), \Phi(\hat{\nu})) \le \sqrt{2}\hat{\gamma}_f(\hat{\mu}, \hat{\nu}), \tag{11}$$

whence Φ is uniformly continuous.

Proof: For $f \in \mathbf{B}_{L,\infty}$ and $\hat{\mu} := (\mu_+, \mu_-), \hat{\nu} := (\nu_+, \nu_-) \in \hat{\mathbf{M}}$

$$|\int f(r)(\mu_+ - \mu_- - \nu_+ + \nu_-)| \le |\int f(r)(\mu_+ - \nu_+)| + |\int f(r)(\mu_- - \nu_-)|.$$

□

Remark 1.3 For finite N the Hahn-Jordan decomposition of the original distribution $\mathcal{X}_\pm(0)$ is preserved through the flow of point measures for $t > 0$ because solutions starting at different locations do not coalesce in finite time.[14] However, convergence in the metric $\hat{\gamma}_f$ does not, in general, preserve the Hahn-Jordan decomposition in the limit.[15] To see this, it suffices to construct a convergent sequence of signed measures μ_n such that the Hahn-Jordan decompositions $\mu_n^\pm$ both converge toward the same positive measure μ with $\mu(\mathbb{R}^d) > 0$. E.g., choose $\mu_n^+ := \delta_{(\frac{1}{2n},0,\ldots,0)}$, $\mu_n^- := \delta_{(\frac{1}{2(n+1)},0,\ldots,0)}$ and $\mu := \delta_{(0,\ldots,0)}$. □

Hence, generalizing Kotelenez (2008), we show first that:

- (i) The pair of the positive and negative empirical processes $(\mathcal{X}_N^+, \mathcal{X}_N^-)(t)$ converges towards a pair of finite Borel measures $(\mathcal{X}_+, \mathcal{X}_-)(t)$ in the metric $\hat{\gamma}(\cdot,\cdot)$ for all $t > 0$.
- (ii) The resulting limit satisfies: $\mathcal{X}_\pm(t) \equiv \mathcal{X}^\pm(t)$, i.e., it is the Hahn-Jordan decomposition of the signed measure $\mathcal{X}(t)$ for all $t > 0$, provided $\mathcal{X}_\pm(0) = \mathcal{X}^\pm(0)$.

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[12] $1_A(r)$ is the indicator function of a Borel set A in $\mathbb{R}^d$.

[13] We use subscripts $\pm$ to indicate that we are not restricting the set to the Hahn-Jordan decomposition of $\mathbf{M}_{f,s}$. For more details cf. Seadler (2010).

[14] The proof of this statement follows from Theorem 2.1 in the next section.

[15] The author wants to thank T. Kurtz for pointing out this fact to him.

Since (i) has been solved in under fairly general assumptions the only problem remains (ii). We will solve (ii) as follows: The flows generated by the SODEs map disjoint sets into disjoint sets. We then use the following representation for the solutions of SPDEs, provided in Kotelenez (2008), p. 59, (4.1), and p. 187, (8.50)-(8.51):

$$\left.\begin{aligned} \mathcal{X}_+(\cdot) &= \int \delta_{r(\cdot,\mathcal{X},q)}\mathcal{X}^+(0,dq) \\ \mathcal{X}_-(\cdot) &= \int \delta_{r(\cdot,\mathcal{X},q)}\mathcal{X}^-(0,dq). \end{aligned}\right\} \tag{12}$$

Here

(13)

$$r(t,\mathcal{X},q) = q + \int_0^t F(r(s,\mathcal{X},q),\mathcal{X}(s),s)ds + \int_0^t \int \mathcal{J}(r(s,\mathcal{X},q),p,\mathcal{X}(s),s)w(dp,ds),$$

i.e., the coefficients in (1.13) are all the same (depending on the same $\mathcal{X}$) and the only difference is the starting point q, and we need to show

$$\mathcal{X}_\pm(\cdot) \equiv \mathcal{X}^\pm(\cdot). \tag{14}$$

2. Flows of SODEs

Hypothesis 2.1 Suppose

$$\tilde{F} : \mathbb{R}^d \times \Omega \times [0,\infty) \longrightarrow \mathbb{R}^d$$

and

$$\tilde{J} : \mathbb{R}^d \times \mathbb{R}^d \times \Omega \times [0,\infty) \longrightarrow \mathcal{M}_{d\times d}$$

such that both functions are adapted to a filtration $\tilde{F}_0^t \subset \mathcal{F}_t$. Further, suppose there is a sequence of bounded stopping times $\tau_n \uparrow \infty$ a.s. as $n \to \infty$ and a sequence of positive constants K_n with $K_n < \infty$ a.s. such that[16]

$$\left.\begin{aligned} &|\tilde{F}(r,t\wedge\tau_n) - \tilde{F}(q,t\wedge\tau_n| \le K_n\tilde{\rho}(r-q), \\ &\sum_{k,\ell=1}^d \int \left(\tilde{J}_{k,\ell}(r,p,t\wedge\tau_n) - \tilde{J}_{k,\ell}(q,p,t\wedge\tau_n)\right)^2 dp \le K_n^2\tilde{\rho}^2(r-q), \\ &|\tilde{F}(r,t\wedge\tau_n)| \le K_n(1+\tilde{\rho}(r)), \\ &|\sum_{k,\ell=1}^d \int \left(\tilde{J}_{k,\ell}(r,p,t\wedge\tau_n)\tilde{J}_{k,\ell}(q,p,t\wedge\tau_n)\right)dp| \le K_n^2(1+\tilde{\rho}(r))(1+\tilde{\rho}(q)). \end{aligned}\right\} \tag{15}$$

□

Consider the stochastic ordinary differential equation (SODE) with random coefficients

$$\begin{aligned} dr &= \tilde{F}(r,t)dt + \int \tilde{J}(r,p,t)w(dp,dt) \\ r(s) &= q. \end{aligned} \tag{16}$$

Theorem 2.1 Suppose (2.1). Then the following holds:

(i) (2.2) has a unique solution for all $q \in \mathbb{R}^d$ and also for all adapted initial conditions r_s.

(ii) Denote the solutions of (2.2) with starts in (adapted) r_s^i, $i = 1,2$, at time $s \ge 0$ by $r(\cdot, r_s^i)$. Then for any $T \ge 0$ and any $n \in \mathbb{N}$,

$$E\sup_{0\le t\le T\wedge\tau_n} \tilde{\rho}^2(r(t,r_s^1) - r(t,r_s^2)) \le K_n^2 E(\tilde{\rho}^2(r_s^1 - r_s^2)). \tag{17}$$

(iii) Suppose

$$P\{\omega : |r_s^1(\omega) - r_s^2(\omega)| = 0\} = 0.$$

[16] As customary, we will mostly suppress the dependence on ω in the following notation.

Then

$$P\{\cup_{t\geq 0}\{\omega : |r(t,\omega,r_s^1) - r(t,\omega,r_s^2)| = 0\}\} = 0. \tag{18}$$

Proof: Steps (i) and (ii) are standard and follow from the contraction mapping principle.[17] We will now prove (iii).[18] Suppose without loss of generality $s = 0$ and set

$$\tau := \inf\{t > 0 : |r(t,r_0^1) - r(t,r_0^2)| = 0\},$$

where, as customary, we set $\tau = \infty$ if the set in the right hand side is empty. τ is a stopping time. [19] We need to show $\tau = \infty$ a.s.

Considering a general twice continuously differentiable χ which is odd, concave monotone increasing such that[20]

$$\left.\begin{array}{l} \chi(u) = u \ \ \forall u \in [-1,1], \\[2ex] \chi'(u) > 0 \ \ \forall u \in \mathbb{R}, \\[2ex] |\chi''(u)u^2| \leq c_1|\chi'(u)u| \leq c_2|\chi(u)| \ \ \forall u \in \mathbb{R}, \\[2ex] \tilde{\rho}^2(r) \leq \chi(|r|^2) \leq 3\tilde{\rho}^2(r). \end{array}\right\} \tag{19}$$

where χ' and χ'' are the first and second derivatives of χ, respectively, and where $0 < c_1 \leq c_2 < \infty$.

Set for $t < \tau$

$$f_\chi(r(t,r_0^1), r(t,r_0^2), t) :=$$

$$\frac{\chi'(|r(t,r_0^1)-r(t,r_0^2)|^2)2(r(t,r_0^1)-r(t,r_0^2))\bullet(\tilde{F}(r(t,r_0^1),t)-\tilde{F}(r(t,r_0^2),t))}{\chi(|r(t,r_0^1)-r(t,r_0^2)|^2)}$$

$$+\frac{\chi'(|r(t,r_0^1)-r(t,r_0^2)|^2)\sum_{k,\ell=1}^d\int\left(\tilde{J}_{k,\ell}(r(t,r_0^1),p,t)-\tilde{J}_{k,\ell}(r(t,r_0^2),p,t)\right)^2dp}{\chi(|r(t,r_0^1)-r(t,r_0^2)|^2)},$$

$$+2\sum_{\ell=1}^d\sum_{i,j=1}^d\chi''(|r(t,r_0^1)-r(t,r_0^2)|^2)(r_i(t,r_0^1)-r_i(t,r_0^2))(r_j(t,r_0^1)-r_j(t,r_0^2))\times$$

$$\times\int(\mathcal{J}_{i,\ell}(r(t,r_0^1),p,t)-\mathcal{J}_{i,\ell}(r(t,r_0^2),p,t))(\mathcal{J}_{j,\ell}(r(t,r_0^1),p,t)-\mathcal{J}_{j,\ell}(r(t,r_0^2),p,t))dpdt$$

and

$$g_\chi(r(t,r_0^1), r(t,r_0^2), p, t) :=$$

$$\frac{\chi'(|r(t,r_0^1)-r(t,r_0^2)|^2)2\left(r(t,r_0^1)-r(t,r_0^2)\right)^T\left(\tilde{J}(r(t,r_0^1),p,t)-\tilde{J}(r(t,r_0^2),p,t)\right)}{\chi(|r(t,r_0^i)-r(t,r_0^j)|^2)}.$$

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[17] Cf. Kotelenez (2008), Ch. 4.

[18] The proof is an adaptation of a proof provided by Krylov (2005).

[19] Cf. Liptser and Shiryayev (1974), Ch. 1.3, Lemma 1.11.

[20] For the bounded metric $\rho(\cdot)$ we take, e.g.,

$$\chi(x) := \begin{cases} x, & |x| \leq 1, \\[1ex] 1 + \arctan(x-1), & x > 1, \\[1ex] -1 - \arctan(x+1), & x < -1, \end{cases}$$

and we have $\rho^2(r) \leq \chi(|r|^2) \leq 3\rho^2(r)$.

For the unbounded (Euclidean) metric $|\cdot|$ we choose $\chi(x) := |x|$.

For $t \geq \tau$ and $\tau < \infty$ we set

$$f_\chi(r(t,r_0^1),r(t,r_0^2),t) := 0 =: g_\chi(r(t,r_0^1),r(t,r_0^2),p,t).$$

Hence, the Itô formula yields

$$\begin{aligned} & d\chi(|r(t,r_0^1)-r(t,r_0^2)|^2) \\ (20) \qquad & = \chi(|r(t,r_0^1)-r(t,r_0^2)|^2)\big(\int g_\chi(r(t,r_0^1),r(t,r_0^2),p,t)\bullet w(dp,dt) \\ & + f_\chi(r(t,r_0^1),r(t,r_0^2,t))dt\big). \end{aligned}$$

Employing the properties of χ from (2.5) and the Lipschitz assumptions there are finite positive constants $c_i,\ i=1,2$ such that

$$\int_0^t \int |g_\chi(r(s\wedge\tau\wedge\tau_n,r_0^i),r(s\wedge\tau\wedge\tau_n,r_0^j),p,s\wedge\tau\wedge\tau_n)|^2 dpds \leq c_1 K_n^2 t \ \forall t.$$

and

$$|f_\chi(r(t\wedge\tau\wedge\tau_n,r_0^1),r(t\wedge\tau\wedge\tau_n,r_0^2),t\wedge\tau\wedge\tau_n)| \leq c_2 K_n^2.$$

Consequently, $\chi(|r(t\wedge\tau,r_0^1)-r(t\wedge\tau,r_0^2)|^2)$ is the solution of a bilinear SODE, driven by a "nice" semi-martingale. Letting $\tau_n \to \infty$, we verify the following representation by Itô's formula:

$$(21) \qquad \left.\begin{aligned} & \chi(|r(t\wedge\tau,r_0^1)-r(t\wedge\tau,r_0^2)|^2) = \chi(|r_0^1-r_0^2|^2)\exp[\varphi_\chi(t,r_0^1,r_0^2)] \\ & \text{where} \\ & \varphi_\chi(t,r_0^1,r_0^2) := \int_0^{t\wedge\tau}\int g_\chi(r(s,r_0^1),r(s,r_0^2),p,s)\bullet w(dp,ds) \\ & + \int_0^{t\wedge\tau}(f_\chi(r(s,r_0^1),r(s,r_0^2),s) - \tfrac{1}{2}\int |g_\chi(r(s,r_0^1),r(s,r_0^2),p,s)|^2 dp)ds. \end{aligned}\right\}$$

Therefore, for any choice of $\tilde\rho \in \{|\cdot|,\rho(\cdot)\}$

$$(22) \qquad \tilde\rho^2(r(t\wedge\tau,r_0^1)-r(t\wedge\tau,r_0^2))^2 \geq \tfrac{1}{3}\tilde\rho(r_0^1-r_0^2)^2\exp[\varphi_\chi(t,r_0^1,r_0^2)].\ \}$$

We obtain from (2.8) and the assumption that $\tau = \infty$ a.s. □

$\mathcal{B}^k$ denotes the Borel sets in $\mathbb{R}^k$ and $r^{\vec N}, q^{\vec N}$ elements of $\mathbb{R}^{dN}$.

Theorem 2.2 Suppose (2.1). Then, for any $N \in \mathbb{N}$ there is a $\mathbb{R}^{dN}$-valued map in the variables $(t,\omega,r^{\vec N},s)$, $0 \leq s \leq t < \infty$ such that for any fixed $s \geq 0$

$$\overline{r}^{\vec N}(\cdot,\cdot,s) : \Omega\times\mathbb{R}^{dN} \to C([s,T];\mathbb{R}^{dN}),$$

and the following holds:

(i) For any $t \in [s,T]$ $\overline{r}^{\vec N}(t,\cdot,s)$ is $\tilde{\mathcal{F}}_0^t \otimes \mathcal{B}^{dN} - \mathcal{B}^{dN}$-measurable.

(ii) The i-th d-vector of $\overline{r}^{\vec N} = (\overline{r}^1,\ldots,\overline{r}^i,\ldots\overline{r}^N)$, denoted $\overline{r}^{i,\vec N} =: \overline{r}^i$, depends only on the i-th d-vector initial condition $r_s^i \in L_{2,\mathcal{F}_s}(\mathbf{R}^d)$ in addition to its dependence on $w(dq,dt)$, and with probability 1 (uniformly in $t \in [s,\infty)$)

$$(23) \qquad \overline{r}^{i,\vec N}(t,r_s^i,s) \equiv r^i(t,r_s^i,s),$$

where the right hand side of (2.9) is the ith d-dimensional component of the solution of (2.2).

(iii) If $u \geq s$ is fixed, then with probability 1 (uniformly in $t \in [u,\infty)$)

$$(24) \qquad \overline{r}^{\vec N}(t,\cdot,\overline{r}^{\vec N}(u,\cdot,r_s^{\vec N},s),u) \equiv \overline{r}^{\vec N}(t,\cdot,r_s^{\vec N},s).$$

□

If $N = 1$ we will just write $\overline{r}(\cdot,\cdot,\cdot,\cdot)$, and since for $N > 1$ the corresponding system is a system of N identical equations, indexed by possibly different initial conditions it suffices in what follows to formulate conditions and results for $N = 1$.

The *proof* is provided in the Appendix. □

Denote for $m \in \mathbb{N}$ and sufficiently often differentiable $f : \mathbb{R}^d \to \mathbb{R}$:

$$|||f|||_m := \sum_{k=0}^{m} \max_{|\mathbf{j}|\le k} |||\partial_r^{\mathbf{j}} f|||$$

where $\mathbf{j} = (j_{\ell_1}, \dots, j_{\ell_m})$ is a multiindex of non-negative integers with $|\mathbf{j}| = \sum_{i=1}^m j_{\ell_i}$ and $\partial_r^{\mathbf{j}}$ is the corresponding partial differential operator with respect to the space variable r.

Hypothesis 2.2 Suppose that for some $m \ge 1$

(25)

$$\max_{1\le k,\ell\le d} \; ess \sup_{\omega\in\Omega,\ 0\le u\le T} \{|||\tilde{F}_k(\cdot,u,\omega)|||_m + \max_{|\mathbf{j}|\le m+1} |||\sum_{\ell=1}^{d} \int (\partial_r^{\mathbf{j}} \tilde{\mathcal{J}})^2_{k\ell}(\cdot,p,u,\omega)dp|||\} < \infty.$$

□

Remark 2.3 Hypothesis 2.2 obviously implies 2.1. □

Let $L_0(\Omega; C(\mathbb{R}^d \times [0,T]; \mathbb{R}^d)$ denote the set of all $C(\mathbb{R}^d \times [0,T]; \mathbb{R}^d)$-valued random variables.

Proposition 2.4 Assume Hypothesis 2.2. Then for any multiindex $\mathbf{j}$ with $|\mathbf{j}| \le m-1$

$$(\partial_q^{\mathbf{j}} \overline{r})(\cdot,\cdot,\cdot) \in L_0(\Omega; C(\mathbb{R}^d \times [0,T]; \mathbb{R}^d) \quad \forall T > 0. \tag{26}$$

Proof: Cf. Kotelenez, loc. cit. Ch. 6, Corollary 6.11. □

In what follows we wish to show that $\forall t > 0$ $\overline{r}(t,\omega,\cdot)$ maps disjoint Borel sets $A(\omega)$ and $B(\omega)$ onto disjoint Borel sets $A_t(\omega)$ and $B_t(\omega)$ a.s. To this end we introduce the following notation:

$$D_{2d} := \{(q_1,\dots,q_d,q_{d+1},\dots.q_{2d}) : (q_1,\dots,q_d) = (q_{d+1},\dots.q_{2d})\}. \tag{27}$$

We then have for $A, B \in \mathcal{B}^d$:

$$A \cap B = \emptyset \Leftrightarrow A \times B \subset D_{2d}^c,$$

where the latter set is the complement of D_{2d} in $\mathbb{R}^{2d}$. Let

$$\Omega_{t,cts} := \{\omega : \overline{r}(t,\omega,\cdot) \text{ is continuous as a function of } q\}$$

where by (2.12) $P(\Omega_{t,cts}) = 1$.

Theorem 2.5 Assume Hypothesis 2.2 with $m = 1$. Then

$$\Omega_t := \{\omega : (\overline{r}(t,\omega,a), \overline{r}(t,\omega,b)) \in D_{2d}^c \ \forall (a,b) \in D_{2d}^c\} \cap \Omega_{t,cts} \in \tilde{\mathcal{F}}_0^t$$

and

$$P(\Omega_t) = 1. \tag{28}$$

Proof: (i) We first define a convenient partition of D_{2d}, setting

$$S_1^+ := \{(q_1,\dots,q_{2d}) : q_1 > q_{d+1}\}, \quad S_1^- := \{(q_1,\dots,q_{2d}) : q_1 < q_{d+1}\},$$
$$S_2^+ := \{(q_1,\dots,q_{2d}) : q_1 = q_{d+1}, q_2 > q_{d+2}\},$$
$$S_2^- := \{(q_1,\dots,q_{2d}) : q_1 = q_{d+1}, q_2 < q_{d+2}\},$$

. . . .

$$S_d^+ := \{(q_1,\dots,q_{2d}) : q_i = q_{d+i}, i = 1,..,d-1, q_d > q_{2d}\},$$
$$S_d^- := \{(q_1,\dots,q_{2d}) : q_i = q_{d+i}, i = 1,..,d-1, q_d < q_{2d}\}.$$

Let $n \in \mathbb{N}$ and define approximations of the above partition by compact sets for $i = 1, \ldots, d$:

$$S_{i,n}^{+} := \{(q_1, \ldots, q_{2d}) \in S_i^{+} : q_i \geq q_{d+i} + \tfrac{1}{n},\ |(q_1, \ldots, q_{2d})| \leq n\},$$

$$S_{i,n}^{-} := \{(q_1, \ldots, q_{2d}) \in S_i^{-} : q_{d+i} \geq q_i + \tfrac{1}{n},\ |(q_1, \ldots, q_{2d})| \leq n\}.$$

We then have

$$D_{2d,c,n} := \cup_{i=1}^{n} \cup_{\pm} S_{i,n}^{\pm} \uparrow D_{2d}^{c}, \quad \text{as } n \to \infty, \tag{29}$$

and $D_{2d,c,n}$, as a finite union of compact sets, is compact.

(ii) Abbreviate

$$r(t,\omega,q) := \overline{r}(t,\omega,q), \quad f(\omega,a,b) := |\overline{r}(t,\omega,a) - \overline{r}(t,\omega,b)|$$

and employ the notation:

$$a, b \in \mathbb{R}^d, \quad \tilde{a}, \tilde{b} \in \mathbb{Q}^d.$$

By the continuity of the field $r(t,\omega,\cdot)$ and the compactness of $D_{2d,c,n}$, for each $n \in \mathbb{N}$ and each $\omega \in \Omega_{t,cts}$ there is an $\eta(n,\omega) > 0$ such that

$$\left.\begin{aligned} &\Omega_{t,n} := \{\omega \in \Omega_{t,cts} : f(\omega,a,b) \geq \eta(n,\omega) > 0\ \forall a,b \in D_{2d,c,n}\}, \\ &= \tilde{\Omega}_{t,n} := \{\omega \in \Omega_{t,cts} : f(\omega,\tilde{a},\tilde{b}) \geq \eta(n,\omega) > 0\ \forall \tilde{a},\tilde{b} \in D_{2d,c,n}\}. \end{aligned}\right\} \tag{30}$$

By the countability $\{(\tilde{a},\tilde{b}) \in \mathbb{Q}^d : (\tilde{a},\tilde{b}) \in D_{2d,c,n}\}$, $\tilde{\Omega}_{t,n} \in \tilde{\mathcal{F}}_0^t$, whence by (2.16) we also have $\Omega_{t,n} \in \tilde{\mathcal{F}}_0^t$. Now Proposition 2.1 in addition to (2.16) implies

$$P\{\Omega_{t,n}\} = 1 \quad \forall n.$$

Further, $\Omega_{t,n} \downarrow \Omega_t$, as $n \to \infty$. Hence,

$$1 = \lim_{n\to\infty} P\{\Omega_{t,n}\} = P\{\Omega_t\}. \tag{31}$$

□

Corollary 2.6 Assume Hypothesis 2.2 with $m \geq 1$. Let $S^{\pm} \in \mathcal{B}^d \otimes \tilde{\mathcal{F}}_0^0$ be the random supports of $\mathcal{X}_0^{\pm}$ and $A(\omega), B(\omega)$ the $\omega-$sections of S^+ and S^-, respectively such that $A(\omega) \times B(\omega) \in D_{2d}^c$ a.s. Then $\forall t > 0$

$$(\overline{r}(t,\omega,A(\omega)), \overline{r}(t,\omega,B(\omega))) \in D_{2d}^{d} \quad \text{a.s.} \tag{32}$$

Proof: Fix $\omega \in \Omega_t$ from (2.14). Then, $\overline{r}(t,\omega,a), \overline{r}(t,\omega,b)) \in D_{2d}^c\ \forall (a,b) \in D_{2d}^c$. Hence, we must also have $(\overline{r}(t,\omega,a), \overline{r}(t,\omega,b)) \in D_{2d}^c\ \forall (a,b) \in A \times B$ if $A \times B \subset D_{2d}^d$. In particular, it follows for the $\omega-$sections of S^+ and S^- which are in $\mathcal{B}^d$.[21] Setting

$$\Omega_{t,A,B} := \{\omega \in \Omega_{t,cts} : (\overline{r}(t,\omega,a), \overline{r}(t,\omega,b)) \in D_{2d}^c\ \forall (a,b) \in A(\omega) \times B(\omega)\},$$

it follows from the previous argument that $\Omega_{t,A,B} \supset \Omega_t$. Hence, by (2.14) and the completeness of P, $P\{\Omega_{t,A,B}\} = 1$, which implies (2.18). □

Let $\mathbf{K}$ be a metric space with metric d_K. If f is a stochastic process on $[s,\infty)$ with values in $\mathbf{K}$, we set for $t \geq s$

$$(\pi_{s,t} f)(u) := f(u \wedge t), \quad (u \geq s).$$

$L_{0,\mathcal{F}_s}(\mathbf{K})$ is the space of $\mathbf{K}-$valued $\mathcal{F}_s$-adapted random variables ξ, and $L_{2,\mathcal{F}_s}(\mathbf{K}) \subset L_{0,\mathcal{F}_s}(\mathbf{K})$ such that for $\xi \in L_{2,\mathcal{F}_s}(\mathbf{K})$ $Ed_k^2(\xi,\eta) < \infty$ where $\eta \in \mathbf{K}$ is an arbitrary fixed element. Similarly, $L_{0,\mathcal{F}}(C([s,T];\mathbf{K}))$ is the space of random variables with values in $C([s,T];\mathbf{K})$ which as processes are adapted to the filtration $\mathcal{F}_t$, and $L_{2,\mathcal{F}}(C([s,T];\mathbf{K})) \subset L_{0,\mathcal{F}}(C([s,T];\mathbf{K}))$ is the space of square integrable random variables with values in $C([s,T];\mathbf{K})$. $L_{loc,2,\mathcal{F}}(C([s,T];\mathbf{K}))$ is the space of processes $\xi(\cdot)$ such that there are

[21] Cf. Bauer (1968), Section 22, Lemma 22.1.

localizing stopping times τ with $\xi(\cdot\wedge\tau)\in L_{2,\mathcal{F}}(C([s,T];\mathbf{K}))$. Similar is the definition of $L_{loc,2,\mathcal{F}}(C((s,T];\mathbf{K}))$.

The coefficients for (1.1) are

$$F:\mathbb{R}^d\times\mathbf{M}_{f,s}\times[0,\infty)\to\mathbb{R}^d;$$

$$\mathcal{J}:\mathbb{R}^d\times\mathbb{R}^d\times\mathbf{M}_{f,s}\times[0,\infty)\to\mathcal{M}_{d\times d}.$$

In the following stochastic ordinary differential equation (2.19) the empirical process from (1.1) has been replaced by the signed measure valued process $\tilde{\mathcal{Y}}(\cdot)$: [22].

$$\left.\begin{array}{rl} dr^i(t)= & F(r^i(t),\tilde{\mathcal{Y}}(t),t)dt+\int\mathcal{J}(r^i(t),p,\tilde{\mathcal{Y}}(t),t)w(dp,dt)\\ r^i(s)= & r^i_s\in L_{2,\mathcal{F}_s}(\mathbb{R}^d),\quad \tilde{\mathcal{Y}}\in L_{loc,2,\mathcal{F}}(C((s,T];\mathbf{M}_{f,s})),\quad i=1,\dots,N, \end{array}\right\}\tag{33}$$

(2.19) is a special case of (2.2) if $N=1$,[23] setting

$$\tilde{F}(r,t):=F(r,\tilde{\mathcal{Y}}(t),t),\quad \tilde{J}(r,p,t):=\mathcal{J}(r,p,\tilde{\mathcal{Y}}(t),t).$$

Hypothesis 2.3 Suppose $(r_\ell,\mu_\ell,t)\in\mathbb{R}^d\times\mathbf{M}_{f,s}\times\mathbb{R},\ \ell=1,2$. Let $c_{F,\mathcal{J}}\in(0,\infty)$. Working, if necessary, with a sequence of stopping times τ_n as in (2.1) we assume global Lipschitz and boundedness conditions, working with the metric $\rho(\cdot)$:

(34)

$$\left.\begin{array}{l} (a)\quad |F(r_1,\mu_1,t)-F(r_2,\mu_2,t)|\le c_{F,\mathcal{J}}\{(\hat{\gamma}_f(\hat{\mu}_1)\vee\hat{\gamma}_f(\hat{\mu}_2))\rho(r_1-r_2)+\hat{\gamma}_f(\hat{\mu}_1-\hat{\mu}_2)\},\\ \sum_{k,\ell=1}^d[\int(\mathcal{J}_{k\ell}(r_1,p,\mu_1,t)-\mathcal{J}_{k\ell}(r_2,p,\mu_2,t))^2dp\\ \le c^2_{F,\mathcal{J}}\{(\hat{\gamma}^2_f(\hat{\mu}_1)\vee\hat{\gamma}^2_f(\hat{\mu}_2))\rho^2(r_1-r_2)+\hat{\gamma}^2_f(\hat{\mu}_1-\hat{\mu}_2)\};\\ (b)\quad |F(r,\mu,t)|^2+\sum_{k,\ell=1}^d\{\int\mathcal{J}^2_{k\ell}(r,p,\mu,t)dp\}\le c_{F,\mathcal{J}}. \end{array}\right\}$$

The constant $c_{F,\mathcal{J}}$ in (2.20) may also depend on the space dimension d. Alternatively, if we drop the boundedness assumption on the coefficients in (2.20), we need to impose a linear growth condition in addition to corresponding Lipschitz conditions from (2.20) in terms of the Euclidean metric:

(35)

$$\left.\begin{array}{l} (a)\quad |F(r_1,\mu_1,t)-F(r_2,\mu_2,t)|\le c_{F,\mathcal{J}}\{\hat{\gamma}_f(\hat{\mu}_1)\vee\hat{\gamma}_f(\hat{\mu}_2)|r_1-r_2|+\hat{\gamma}_f(\hat{\mu}_1-\hat{\mu}_2)\},\\ \sum_{k,\ell=1}^d[\int(\mathcal{J}_{k\ell}(r_1,p,\mu_1,t)-\mathcal{J}_{k\ell}(r_2,p,\mu_2,t))^2dp\\ \le c^2_{F,\mathcal{J}}\{(\hat{\gamma}^2_f(\hat{\mu}_1)\vee\hat{\gamma}^2_f(\hat{\mu}_2))|r_1-r_2|^2+\hat{\gamma}^2_f(\hat{\mu}_1-\hat{\mu}_2)\};\\ (b)\quad |F(r,\mu,t)|\le c_{F,\mathcal{J}}(1+|r|)\ ,\\ |\sum_{k,\ell=1}^d\{\int\mathcal{J}^2_{k\ell}(r,p,\mu,t)dp\}\le c^2_{F,\mathcal{J}}(1+|r|^2)\ . \end{array}\right\}.$$

□

A solution of the system (2.19), if it exists, is denoted by $r^{\vec{N}}(t,\tilde{\mathcal{Y}},r^{\vec{N}}_s,s)$. As for (2.2) we may formulate the results for the $d-$dimensional components of $r^{\vec{N}}$ which will be denoted r^k, $k=1,\dots,N$. The proofs of the following existence and uniqueness theorems

[22] (2.19) describes the motion of a system of diffusing particles in a random environment (represented by $\tilde{\mathcal{Y}},\ w_\ell,\ \ell=1,\dots,d$

[23] This is obvious for $N=1$. But for $N>1$ the system (2.19) is a system of identical equations, indexed by possibly different initial conditions.

are essentially identical to the proofs provided by Kotelenez (2008), Ch. 4, for the case of positive measures and may be omitted here.

Theorem 2.7 Assume either (2.20) or (2.21) and $\tilde{\rho}(\cdot) \in \{\rho(\cdot), |\cdot|\}$. Then:

1) To each $s \geq 0$, $r^k_s \in L_{2,\mathcal{F}_s}(\mathbf{R}^d)$, $\tilde{\mathcal{Y}} \in L_{loc,2,\mathcal{F}}(C((s,T];\mathbf{M}_{f,s}))$ (2.19) has a unique solution $r^k(\cdot,\tilde{\mathcal{Y}},r^k_s,s) \in L_{loc,2,\mathcal{F}}(C([s,T];\mathbb{R}^d))$.

2) Let $\tilde{\mathcal{Y}}_i \in L_{loc,2,\mathcal{F}}(C((s,T];\mathbf{M}_{f,s}))$ and $r^k_{s,i} \in L_{2,\mathcal{F}_s}(\mathbb{R}^d)$, $i = 1,2$. Then for any $T \geq s$ and any stopping time $\tau \geq s$, which is localizing for $\tilde{\mathcal{Y}}_i$, $i = 1,2$,

(36)
$$\left.\begin{array}{rl} & E\sup_{s\le t\le T\wedge\tau}\tilde{\rho}^2(r^k(t,\tilde{\mathcal{Y}}_1,r^k_{s,1},s)-r^k(t,\tilde{\mathcal{Y}}_2,r^k_{s,2},s))1_{\{\tau>s\}} \\ \le & c_{T,F,\mathcal{J},\tilde{\mathcal{Y}},\tau}\{E(\tilde{\rho}^2(r^k_{s,1}-r^k_{s,2})1_{\{\tau>s\}})+E\int_s^{T\wedge\tau}(\hat{\gamma}_f^2(\hat{\tilde{\mathcal{Y}}}_1(u)-\hat{\tilde{\mathcal{Y}}}_2(u))1_{\{\tau>s\}}du\}. \end{array}\right\}$$

Further, with probability 1 uniformly in $t \in [s,\infty)$

$$r^k(t,\tilde{\mathcal{Y}},r^k_{s,1},s) \equiv r^k(t,\pi_{s,t}\tilde{\mathcal{Y}},r^k_{s,1},s). \qquad \square \tag{37}$$

Next, we consider the $\mathbb{R}^{dN}$-valued system of coupled SODEs (1.1). Since for each ω the initial measure is a finite sum of point measures, it is finite. Therefore,

$$\mathcal{X}_N(s) := \sum_{i=1}^N m_i\delta_{r^i(s)}, \in L_{0,\mathcal{F}_s}(\mathbf{M}_{f,s}).$$

Further, a solution of (1.1), if it exists, preserves the initial positive and negative mass, i.e., $\mathcal{X}^+_N(\cdot,\mathbb{R}^d) \equiv \sum_{i=1}^N 1_{m_i>0}m_i$ and $\mathcal{X}^-_N(\cdot,\mathbb{R}^d) \equiv -\sum_{i=1}^N 1_{m_i<0}m_i$, where $\mathcal{X}_N(t) := \sum_{i=1}^N m_i\delta_{r^i(t)}$. [24] Therefore, we may take $\tilde{\mathcal{Y}}(t) := \mathcal{X}_N(t) := \sum_{i=1}^N m_i\delta_{r^i(t)}$ in Theorem 2.7. We endow $\mathbb{R}^{dN}$ with the metric

$$\tilde{\rho}_N(r^{\vec{N}},q^{\vec{N}}) := \max_{1\le i\le N}\tilde{\rho}(r_i,q_i),$$

where $r^{\vec{N}} := (r^1,\ldots,r^N)$, $q^{\vec{N}} := (q_1,\ldots,q_N) \in \mathbb{R}^{dN}$.

Theorem 2.8 Assume (2.20) or (2.21) in addition to $\mathcal{X}_N(s) \in L_{0,\mathcal{F}_s}(\mathbf{M}_f)$. Then, to each initial condition $r^{\vec{N}}(s) \in L_{0,\mathcal{F}_s}(\mathbb{R}^{dN})$ (1.1) has a unique solution $r^{\vec{N}}_\varepsilon(\cdot,r_N(s)) \in L_{0,\mathcal{F}}(C([s,\infty);\mathbb{R}^{dN}))$ which is a Markov process on $\mathbb{R}^{dN}$.

Proof: Cf. Kotelenez (2008). Ch. 4, Theorem 4.7. $\square$

Remark 2.9 We finally provide a useful representation for the perturbation by Gaussian white noises. Let $\mathbf{H}_0$ be the space of measurable functions on $\mathbb{R}^d$ which are square integrable with respect to the Lebesgue measure with norm $|\cdot|_0$ and scalar product $<\cdot,\cdot>_0$. Let $\{\phi_n\}_{n\in\mathbb{N}}$ be a complete orthonormal system (CONS) in $\mathbf{H}_0$ and define an $\mathcal{M}_{d\times d}$-valued function $\widehat{\phi}_n(\cdot)$ by

$$\widehat{\phi}_n := \begin{pmatrix} \phi_n & 0 & \ldots\ldots & 0 \\ 0 & \phi_n & \ldots. & 0 \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ & & & \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots\ldots & \phi_n \end{pmatrix}, \tag{38}$$

[24]This follows from Theorem 2.1 since by Hypothesis 2.3 $d-$dimensional solutions of (2.19) with different starts do not coalesce. Cf. also our Remark 1.3.

i.e., $\widehat{\phi}_n(\cdot)$ is a $d \times d$ matrix-valued function whose entries on the main diagonal are all $\phi_n(\cdot)$ and whose other entries are all 0. Set

$$\beta^n(t) := \int_0^t \int \widehat{\phi}_n(p) w(dp, ds). \tag{39}$$

Then the $\beta^n(\cdot)$ are i.i.d. standard $\mathbb{R}^d$-valued Brownian motions (or Wiener processes). Moreover, for any $\tilde{\mathcal{Y}} \in L_{loc,2,\mathcal{F}}(C((s,T];\mathbf{M}_{f,s}))$ and $r(\cdot) \in L_{0,\mathcal{F}}(C([0,\infty);\mathbb{R}^d))$ (the space of $\mathbb{R}^d$-valued adapted continuous processes)

$$\left.\begin{array}{l} \int \mathcal{J}(r(t), p, \tilde{\mathcal{Y}}(t), t) w(dp, dt) = \sum_{n=1}^{\infty} \sigma_n(r(t), \tilde{\mathcal{Y}}(t), t) d\beta^n(dt) \\ \qquad\qquad \text{where} \\ \sigma_n(r, \mu, t) := \int \mathcal{J}(r, p, \mu, t) \widehat{\phi}_n(p) dp. \end{array}\right\} \tag{40}$$

The right hand side of the first line in (2.26) defines the increment of an $\mathbb{R}^d$-valued square integrable continuous martingale $m(\cdot)$.[25] In particular, Hypothesis 2.3 implies for the mutual quadratic variation of the one-dimensional components of

$$m(t) := m(r(\cdot), \tilde{\mathcal{Y}}(\cdot), t)$$

the following estimate:

$$[m_k(t), m_\ell(t)] \le c_{\mathcal{J}} t, \quad \text{with } 0 < c_{\mathcal{J}} < \infty. \tag{41}$$

□

Remark 2.10 We wish to compare our results to results on homeomorphisms of stochastic flows proved by Kunita (1990), Ch. 4.5, pp. 154-164. To this end we need to rewrite our equation (2.2) into the notation of Kunita. We define the $\mathbb{R}^d$-valued semi-martingale $H(r,t) := (H_1(r,t), \dots, H_d(r,t))$, depending on the spacial parameter r by

$$H(r,t) := \int_0^t \tilde{F}(r,s) ds + \int_0^t \int \tilde{J}(r,p,s) w(dp, ds).$$

The local characteristics of $H(x,t)$ are given by[26]

$$\tilde{a}(r,q,s) := \int \tilde{J}(r,p,s) \tilde{J}^T(q,p,s) dp$$

$$\tilde{b}(r,s) := \tilde{F}(r,s).$$

Let K be a compact set in $\mathbb{R}^d$. Define the metrics

$$\left.\begin{array}{l} \tilde{\|}\tilde{a}(t)\tilde{\|}_{1,K} := \sup_{r,q \in K} \frac{|\tilde{a}(r,q,t)|}{(1+|r|)(1+|q|)} + \sum_{k=1}^d \sup_{r,q \in K} |\frac{\partial}{\partial r_k} \frac{\partial}{\partial q_k} \tilde{a}(r,q,t)| \\ \\ \|\tilde{b}(t)\|_{1,K} := \sup_{r \in K} \frac{\tilde{b}(r,t)}{1+|r|} + \sum_{k=1}^d \sup_{r \in K} |\frac{\partial}{\partial r_k} \tilde{b}(r,t)| \end{array}\right\}$$

where $|\tilde{a}(r,q,t)| := \sum_{k,\ell}^d |\tilde{a}_{k,\ell}(r,q,t)|$ is the sum of the absolute values of all entries of a $d \times d$-matrix $\tilde{a}(r,q,t)$.

Our Lipschitz assumption in (2.1) on the diffusion coefficients is, in the terminology of Kunita, equivalent to

$$|\tilde{a}(r,r,t \wedge \tau_n) + \tilde{a}(q,q,t \wedge \tau_n) - 2\tilde{a}(r,q,t \wedge \tau_n)| \le K_n^2 \tilde{\rho}^2(r-q).$$

which is weaker than Kunita's Lipschitz assumption.[27]

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[25]The statement follows from Doob's inequality and the fact that the terms in the right hand side of (2.26) are uncorrelated martingales,.

[26]Cf. Kunita, pp. 79, 85, 101.

[27]Cf. also Kotelenez and Kurtz (2010), Section 4.4, for a general comparison with Kunita's approach.

Apart from the fact that in Kotelenez (2008) a direct proof of the homeomorphism has been provided, it appears to be not completely trivial to apply the homeomorphism property to obtain (2.14). □

3. Stochastic Partial Differential Equations (SPDEs)

The general conditions of Theorem 2.7 on initial values and input processes will be assumed throughout this Section.

Let $\tilde{\mathcal{Y}}(\cdot)$ be as in (2.13) and

$$\bar{r}(t,\omega,\tilde{\mathcal{Y}}(\omega),q) := \bar{r}(t,\omega,q,0),$$

where $\bar{r}(t,\omega,q,0)$ is the flow solution of (2.2) with

$$\tilde{F}(r,\omega,t) := F(t,\tilde{\mathcal{Y}}\omega,t), \quad \tilde{\mathcal{J}}(r,p,\omega,t) := \mathcal{J}(r,p,\tilde{\mathcal{Y}},t).$$

Recall that by Theorem 2.2 $\bar{r}(t,\omega,\tilde{\mathcal{Y}}(\omega),q)$ is measurable in (t,ω,q).[28] The map

$$(\omega,q) \mapsto \delta_{\{\bar{r}(t,\omega,\tilde{\mathcal{Y}}(\omega),q)\}},$$

$$\Omega \times \mathbb{R}^d \mapsto \mathbf{M}_f,$$

is $\mathcal{F}_t \otimes \mathcal{B}^d - \mathcal{B}_{\mathbf{M}_f}$ measurable, where $\mathcal{B}_{\mathbf{M}_f}$ is the Borel $\sigma-$ algebra on $\mathbf{M}_f$.

Next, let $\mathcal{X}_0$ an $\mathcal{F}_0$ measurable $\mathbf{M}_{f,s}$-valued random measure. Define the "*flow of particles*" governed by the flow of SODEs (2.19) and with initial distribution $\mathcal{X}_0$ by:

$$\mathcal{Y}(t,\omega) := \mathcal{Y}(t,\omega,\tilde{\mathcal{Y}}(\omega),\mathcal{X}_0(\omega)) := \int \delta_{\{\bar{r}(t,\omega,\tilde{\mathcal{Y}}(\omega),q)\}}\mathcal{X}_0(dq,\omega). \tag{42}$$

Lemma 3.1 (i) $\mathcal{Y}(\cdot)$, given by the first line in (3.1), is a weak solution[29] of the following (bilinear) stochastic partial differential equation (SPDE) with random coefficients:

$$\left.\begin{aligned} d\mathcal{Y} = \quad & \big(\tfrac{1}{2}\textstyle\sum_{k,\ell=1}^d \partial^2_{k\ell}\big(\mathcal{Y}D_{k\ell}(\cdot,\tilde{\mathcal{Y}},t)\big) - \bigtriangledown \bullet (\mathcal{Y}F(\cdot,\tilde{\mathcal{Y}},t))dt \\ & - \bigtriangledown \bullet(\mathcal{Y}\textstyle\int \mathcal{J}(\cdot,\tilde{\mathcal{Y}},p,t)w(dp,dt)) \end{aligned}\right\} \tag{43}$$

with initial condition $\mathcal{X}_0$ at $s=0$ and Hahn-Jordan decomposition $\mathcal{X}_0^{\pm}$. Further,

$$D_{k\ell}(r,\tilde{\mathcal{Y}},t) := \tilde{D}_{k\ell}(r,r,\tilde{\mathcal{Y}},t), \tag{44}$$

where $\tilde{D}_{k\ell}(r,r,\tilde{\mathcal{Y}},t)$ denotes the two-particle diffusion matrix.

(ii) In addition to the conditions of Theorem 2.7 assume Hypothesis 2.2 with $m \geq 1$. Then

$$\mathcal{Y}^{\pm}(t,\omega) := \mathcal{Y}^{\pm}(t,\omega,\tilde{\mathcal{Y}}(\omega),\mathcal{X}_0(\omega)) = \int \delta_{\{\bar{r}(t,\omega,\tilde{\mathcal{Y}}(\omega),q)\}}\mathcal{X}_0^{\pm}(dq,\omega), \tag{45}$$

i.e., $\int \delta_{\{\bar{r}(t,\omega,\tilde{\mathcal{Y}}(\omega),q)\}}\mathcal{X}_0^{\pm}(dq,\omega)$ is the Hahn-Jordan decomposition $\mathcal{Y}^{\pm}(t)$ of $\mathcal{Y}(t)$ for all $t>0$.

Proof: Take a test function $\varphi \in C_0^2(\mathbb{R}^d,\mathbb{R})$ and let $<\cdot,\cdot>$ denote the duality between measures and continuous test functions with compact supports (extending the $\mathbf{H}_0$ inner product), i.e.,

$$<\varphi,\mathcal{Y}(t)> := \int \varphi(r)\mathcal{Y}(t,dr).$$

[28] The measurability in all three parameters, including t is a consequence of the continuity of the sample paths.

[29] "Weak solution" is here to be understood in the sense of partial differential equations (PDEs), not in the sense of stochastic differential equations.

To simplify the notation we abbreviate

$$\bar{r}(t,q) := \bar{r}(t,\tilde{\mathcal{Y}},q), \quad m(\bar{r}(s,q),ds) := \int \mathcal{J}_\varepsilon(\bar{r}(s,p),\tilde{\mathcal{Y}}(s),p,s)w(dp,ds).$$

The incremental mutual quadratic variations now satisfy

$$[m_k(\bar{r}(s,q)),m_\ell(\bar{r}(s,q))](ds) = D_{k\ell}(\bar{r}(s,q),\tilde{\mathcal{Y}}(s),s)ds.$$

Hence, we obtain

$$< \varphi,\mathcal{Y}(t) >= \int \varphi(r)\int \delta_{\bar{r}(t,q)}(dr)\mathcal{X}_0(dq)$$

(by Itô's formula)

$$= \int \varphi(q)\mathcal{X}_0(dq) + \int \int_0^t (\nabla\varphi)(\bar{r}(s,q)) \bullet F(\bar{r}(s,q),\tilde{\mathcal{Y}}(s),s)ds\mathcal{X}_0(dq)$$

$$+ \int \int_0^t (\nabla\varphi)(\bar{r}(s,q)) \bullet m(\bar{r}(s,q),ds)\mathcal{X}_0(dq)$$

$$+\tfrac{1}{2}\sum_{k,\ell=1}^d \int_0^t (\partial_{k\ell}^2\varphi)(\bar{r}(s,q))D_{k\ell}(\bar{r}(s,q),\tilde{\mathcal{Y}}(s),s)ds\mathcal{X}_0(dq)$$

$$= I + II(t) + III(t) + IV(t).$$

Note that by (2.26)

$$m_k(\bar{r}(s,q),ds) = \sum_{n=1}^\infty\sum_{\ell=1}^d \sigma_{n,k\ell}(\bar{r}(s,q),\tilde{\mathcal{Y}}(s),s)\beta_{n,\ell}(ds).$$

Hence,

$$III(t) = \int\int_0^t \sum_{k=1}^d (\partial_k\varphi)(r)m_k(r,ds)\int \delta_{\bar{r}(s,q)}(dr)\mathcal{X}_0(dq)$$

$$= \sum_{n=1}^\infty\sum_{k,\ell=1}^d \int\int_0^t (\partial_k\varphi)(r)\sigma_{n,k\ell}(r,\tilde{\mathcal{Y}}(s),s)\beta_{n,\ell}(ds)\int \delta_{\bar{r}(s,q)}(dr)\mathcal{X}_0(dq)$$

$$= \sum_{n=1}^\infty\sum_{k,\ell=1}^d \int_0^t \int (\partial_k\varphi)(r)\sigma_{n,k\ell}(r,\tilde{\mathcal{Y}}(s),s)\mathcal{Y}(s,dr)\beta_{n,\ell}(ds)$$

$$= \sum_{n=1}^\infty\sum_{k,\ell=1}^d \int_0^t < (\partial_k\varphi)(\cdot),\sigma_{n,k\ell}(\cdot,\tilde{\mathcal{Y}}(s),s)\mathcal{Y}(s) > \beta_{n,\ell}(ds)$$

$$= -\sum_{n=1}^\infty\sum_{k,\ell=1}^d \int_0^t < \varphi(\cdot),\partial_k\big(\sigma_{n,k\ell}(\cdot,\tilde{\mathcal{Y}}(s),s)\mathcal{Y}(s)\big) > \beta_{n,\ell}(ds)$$

(integrating by parts in the sense of distributions)

$$=< \varphi(\cdot),\int_0^t \sum_{k=1}^d \partial_k\big\{-\mathcal{Y}(s)\sum_{n=1}^\infty\sum_{\ell=1}^d \big(\sigma_{n,k\ell}(\cdot,\tilde{\mathcal{Y}}(s),s)\big)\big\}\beta_{n,\ell}(ds) >$$

$$=< \varphi(\cdot),\int_0^t \nabla\bullet\big\{-\mathcal{Y}(s)\int \mathcal{J}(\cdot,\tilde{\mathcal{Y}}(s),p,s)w(dp,ds)\big\} > .$$

Similarly, we may rewrite $II(t)$ and $IV(t)$. Consequently,

$$(46)\quad \left.\begin{aligned} &< \varphi,\mathcal{Y}(t) > \\ &=< \varphi,\mathcal{X}_0 > - < \varphi,\int_0^t \nabla\bullet\big(\mathcal{Y}(s)F(\cdot,\tilde{\mathcal{Y}}(s),s)ds\big) > \\ &- < \varphi,\int_0^t \nabla\bullet\big(\mathcal{Y}(s)\int \mathcal{J}(\cdot,\tilde{\mathcal{Y}}(s),p,s)w(dp,ds)\big) > \\ &+ < \varphi,\tfrac{1}{2}\sum_{k,\ell=1}^d \int_0^t \partial_{k\ell}^2\big(\mathcal{Y}(s)D_{k\ell}(\cdot,\tilde{\mathcal{Y}}(s),s)ds\big) > . \end{aligned}\right\}$$

Since the measures are uniquely determined by the duality $< \cdot, \cdot >$ with test functions $\varphi \in C_0^2(\mathbb{R}^d, \mathbb{R})$, (3.5) implies (3.2).

(ii) is a consequence of Corollary 2.6. □

Observe that $\mathcal{Y}(t)$ given by (3.1) depends on the measure valued input process $\tilde{\mathcal{Y}}$ in addition to the initial distribution $\mathcal{X}_0$.

Lemma 3.2 Suppose $\mathcal{Y}(t, \tilde{\mathcal{Y}}_1)$ and $\mathcal{Y}(t, \tilde{\mathcal{Y}}_2)$ are two solutions of (3.2) with representations (3.1). Then $\forall T > 0$ there is a positive $c_T < \infty$ such that

$$E \sup_{0 \le t \le T} \hat{\gamma}_f^2\big(\hat{\mathcal{Y}}(t, \tilde{\mathcal{Y}}_1), \hat{\mathcal{Y}}(t, \tilde{\mathcal{Y}}_2)\big) \le c_T \int_0^T E\hat{\gamma}_f^2\big(\hat{\tilde{\mathcal{Y}}}_1(s), \hat{\tilde{\mathcal{Y}}}_2(s)\big) ds. \tag{47}$$

Proof: Truncating the initial distribution $\mathcal{X}^{\pm}(0, \omega)$ if necessary, we may without loss of generality assume that

$$ess\ sup_{\omega} \sum_{\pm} \gamma_f(\mathcal{X}^{\pm}(0, \omega)) \le c < \infty. \tag{48}$$

Hence,

$$E \sup_{0 \le t \le T} \hat{\gamma}_f^2\big(\hat{\mathcal{Y}}(t, \tilde{\mathcal{Y}}_1), \hat{\mathcal{Y}}(t, \tilde{\mathcal{Y}}_2)\big)$$

$$= E \sup_{0 \le t \le T} \sup_{\|f\|_{L,\infty} \le 1} \sum_{\pm} [\int (f(r(t, \tilde{\mathcal{Y}}_1, q)) - f(r(t, \tilde{\mathcal{Y}}_2, q)))\mathcal{X}_0^{\pm}(dq)]^2$$

$$\le E \sup_{0 \le t \le T} \sum_{\pm} | \int (|r(t, \tilde{\mathcal{Y}}_1, q) - r(t, \tilde{\mathcal{Y}}_2, q))\mathcal{X}_0^{\pm}(dq)|^2$$

$$\le c_T \sum_{\pm} \int E\mathcal{X}_0(\mathbb{R}^d) \sup_{0 \le t \le T} |(r(t, \tilde{\mathcal{Y}}_1, q) - r(t, \tilde{\mathcal{Y}}_2, q)|^2 \mathcal{X}_0^{\pm}(dq)$$

(by the Cauchy Schwarz inequality)

$$\le \tilde{c}_T E \int_0^T \hat{\gamma}_f^2(\hat{\tilde{\mathcal{Y}}}_1(u), \hat{\tilde{\mathcal{Y}}}_2(u)) du$$

(by (2.22) and the assumption on the boundedness of all measures.)

□

Theorem 3.3 In addition to the conditions of Theorem 2.7 assume Hypothesis 2.2 with $m \ge 1$. Then the following holds:

(i) There is a weak solution of the following quasi-linear SPDE (1.3) with initial condition $\mathcal{X}_0$ and Hahn-Jordan decomposition $\mathcal{X}_0^{\pm}$.

(ii) This solution, $\mathcal{X}(t, \mathcal{X}, \mathcal{X}_0)$, has the representation

$$\mathcal{X}(t) := \mathcal{X}(t, \mathcal{X}, \mathcal{X}_0) = \int \delta_{\{\overline{r}(t, \omega, \mathcal{X}(\omega), q)\}} \mathcal{X}_0(dq). \tag{49}$$

Further,

$$\mathcal{X}^{\pm}(t) := \mathcal{X}^{\pm}(t, \mathcal{X}, \mathcal{X}_0) = \int \delta_{\{\overline{r}(t, \omega, \mathcal{X}(\omega), q)\}} \mathcal{X}_0^{\pm}(dq), \tag{50}$$

i.e., $\int \delta_{\{\overline{r}(t, \omega, \mathcal{X}(\omega), q)\}} \mathcal{X}_0^{\pm}(dq)$ is the Hahn-Jordan decomposition $\mathcal{X}^{\pm}(t)$ of $\mathcal{X}(t)$ for all $t > 0$.

Proof: (i) Define recursively

$$\mathcal{Y}_0(t) \equiv \mathcal{X}_0, \quad \mathcal{Y}_n(t) := \int \delta_{\{\overline{r}(t, \mathcal{Y}_{n-1}, q)\}} \mathcal{X}_0(dq).$$

By possibly truncating the initial measure we may without loss of generality assume that the total variation of initial distribution is bounded uniformly in ω[30] and, consequently by mass conservation, the same holds for the measures $\mathcal{Y}_n(t)$. Therefore, by (3.6)

[30] Cf. (3.7).

in Lemma 3.2

$$E\sup_{0\le t\le T}\hat{\gamma}_f^2\big(\hat{\mathcal{Y}}_n(t),\hat{\mathcal{Y}}_m(t)\big)\le c_T\int_0^T E\hat{\gamma}_f^2\big(\hat{\mathcal{Y}}_{n-1}(s),\hat{\mathcal{Y}}_{m-1}(s)\big)ds$$

$$\le c_T\int_0^T E\sup_{0\le s\le t}\hat{\gamma}_f^2\big(\hat{\mathcal{Y}}_{n-1}(s),\hat{\mathcal{Y}}_{m-1}(s)\big)ds.$$

The contraction mapping principle yields a unique adapted $\hat{\mathbf{M}}$-valued process $\hat{\mathcal{X}}(\cdot)\in C([0.\infty);\hat{\mathbf{M}})$ a.s. such that

$$E\sup_{0\le t\le T}\hat{\gamma}_f^2\big(\hat{\mathcal{Y}}_n(t),\hat{\mathcal{X}}(t)\big)\to 0,\quad \text{as } n\to\infty. \tag{51}$$

Further, observe that the Hahn-Jordan decomposition for $\mathcal{Y}_n(t)$ is given by (3.4) with $\mathcal{Y}(t):=\mathcal{Y}_n(t)$ and $\tilde{\mathcal{Y}}(t):=\mathcal{Y}_{n-1}(t)$.

(ii) Setting $\mathcal{X}:=\hat{\mathcal{X}}^+-\hat{\mathcal{X}}^-$, we now define

$$\tilde{\mathcal{X}}(t):=\int\delta_{\{\overline{r}(t,\mathcal{X},q)\}}\mathcal{X}_0(dq),$$

and by (3.4) we have the Hahn-Jordan decomposition

$$\tilde{\mathcal{X}}^{\pm}(t)=\int\delta_{\{\overline{r}(t,\mathcal{X},q)\}}\mathcal{X}_0^{\pm}(dq). \tag{52}$$

Again by Lemma 3.2

$$E\sup_{0\le t\le T}\hat{\gamma}_f^2\big(\hat{\mathcal{Y}}_n(t),\hat{\tilde{\mathcal{X}}}(t)\big)\le c_T\int_0^T E\hat{\gamma}_f^2\big(\hat{\mathcal{Y}}_{n-1}(s),\hat{\mathcal{X}}(s)\big)ds$$

$$\to 0,\quad \text{as } n\to\infty.$$

By the uniqueness of limit in (3.10) we have $\hat{\tilde{\mathcal{X}}}(t)\equiv\hat{\mathcal{X}}(t)$. Hence, by (3.11) we obtain (3.8) and (3.9). Finally, Lemma 3.1 implies that $\mathcal{X}(\cdot)$ is a weak solution of (1.3). □

Remark 3.4 Under additional smoothness assumptions on the coefficients F and $\mathcal{J}$[31] smoothness of the initial conditions implies smoothness of the solutions, where smoothness is derived in appropriate Sobolev spaces of functions. Moreover, if the smoothness of the coefficients is sufficiently large then the solution of (1.3) is unique with continuous paths in $\mathbf{H}_0$.[32] □

4. Appendix - Proof of Theorem 2.2

For this proof we may, without loss of generality, assume $N=1$.

(i) We next adjust the classical proof of the Markov property for certain SODEs to our setting.[33] Let "diameter" be defined as usual for metric spaces, i.e., for a Borel set $A\subset\mathbb{R}^d$ we set

$$diam\ (A):=\sup_{r,\tilde{r}\in A}|b-\tilde{b}|.$$

By the separability of $\mathbb{R}^d$, there is a sequence of countable decompositions $\{E_k^m\}_{k\in\mathbb{N}}$, where E_k^m are non-empty Borel sets of diameter $\le 3^{-m}$ for all $k\in\mathbb{N}$ and $m\in\mathbb{N}$. In each E_k^m we choose an arbitrary but fixed element $r_k^m, k, m\in\mathbb{N}$. Now we define maps:

$$f_m:\mathbb{R}^d\to\mathbb{R}^d,\ f_m(r)=r_k^m,\ \text{if } r\in E_k^m,\quad k,=1,2,\ldots \tag{53}$$

(ii) Let $r(t,\omega,f_m(q),s)$ be the solution of (2.2)[34] with start in $f_m(q)$ at time s. Set

$$\bar{r}_m(t,\omega,q,s):=r(t,\omega,f_m(q),s). \tag{54}$$

[31] Cf. Hypothesis 2.2.
[32] Cf. Kotelenez (1995b) and Kotelenez (2008), Chapter 8.
[33] Cf., e.g., Dynkin (1965), Ch. VI, §2 as well as Arnold, Curtain and Kotelenez (1980).
[34] We will always use continuous versions of solutions of (2.2).

We claim that for fixed $0 \le s < t$ $\bar{r}_m(t,\cdot,\cdot,s)$ is $\tilde{\mathcal{F}}_0^t \otimes \mathcal{B}^d - \mathcal{B}^d$-measurable. To this end take $A \in \mathcal{B}^d$ and set

$$C := \{(\omega, q) : \bar{r}_m(t,\omega,q,s) \in A\}.$$

If $C = \emptyset$ we are done; otherwise take $(\omega_0, q_0) \in C$ and observe

$$C_{q_0} := \{\omega : (\omega, q_0) \in C\} = \{\omega : r(t,\omega,f_m(q_0),s) \in A\} \in \tilde{\mathcal{F}}_0^t$$

since $r(t,\omega,f_m(q_0),s)$ is an Itô solution of (2.2) with start in $f_m(q)$ at time s. There is a k_0 such that $q_0 \in E^m_{k_0}$. Hence,

$$\bar{r}_m(t,\omega,q,s) = \bar{r}_m(t,\omega,q_0,s) \ \ \forall q \in E^m_{k_0}.$$

Thus,

$$E^m_{k_0} \times C_{q_0} = \{(\omega,q) : \ q \in E^m_{k_0}, \ (\omega,q) \in C\} \subset C.$$

In the same way we find for any $(\omega,q) \in C$ a k_q such that

$$E^m_{k_q} \times C_q \subset C, \ C_q \in \tilde{\mathcal{F}}_0^t.$$

Hence, there is a subsequence of positive integers $\{k_p, \ p \in \mathbb{N}\}$ such that

$$C = \cup_{q:\ (\omega,q)\in C} E^m_{k_q} \times C_q = \cup_{p=1}^{\infty} E^m_{k_p} \times C_{q_{k_p}},$$

since every q is contained in some $E^m_{k_p}$. Since $E^m_{k_p} \times C_{q_{k_p}} \in \tilde{\mathcal{F}}_0^t \otimes \mathcal{B}^d$ we obtain

$$C \in \tilde{\mathcal{F}}_0^t \otimes \mathcal{B}^d. \tag{55}$$

(iii) For fixed q $\bar{r}_m(t,\cdot,q,s)$ solves (2.2) for the initial value $(f_m(q),s)$. Comparing this solution with the solution of (2.2) for the initial value (q,s), $r(t,\cdot,q,s)$, we obtain from (2.3) that for any $T > s$

$$\left.\begin{aligned} &P\{\omega : \sup_{s\le t\le T} \tilde{\rho}(\bar{r}_m(t,\omega,q,s) - r(t,\omega,q,s)) < 2^{-\frac{m}{2}}\} \\ &\le 2^m E \sup_{s\le t\le T} \tilde{\rho}^2(\bar{r}_m(t,\cdot,q,s) - r(t,\cdot,q,s)) \ \le const\Big(\frac{2}{3}\Big)^m. \end{aligned}\right\} \tag{56}$$

Thus, by the Borel-Cantelli Lemma $\sup_{s\le t\le T} \tilde{\rho}(\bar{r}_m(t,\cdot,q,s) - r(t,\cdot,q,s))$ a.s., as $n \to \infty$. Set

$$\left.\begin{aligned} &D := \{(\omega,q) : \exists \lim_{m\to\infty} \bar{r}_m(t,\omega,q,s) \text{ uniformly for } t \in [s,T]\} \\ &= \cap_\ell \cup_m \cap_{\tilde{m}} \{(\omega,q) : \sup_{s\le t\le T} \tilde{\rho}(\bar{r}_m(t,\omega,q,s) - \bar{r}_{m+\tilde{m}}(t,\omega,q,s)) < \tfrac{1}{\ell}\}. \end{aligned}\right\} \tag{57}$$

By the $\tilde{\mathcal{F}}_0^t \otimes \mathcal{B}^d - \mathcal{B}^d$ measurability of $\bar{r}_m(t,\cdot,\cdot,s)$

$$D_t := \cap_\ell \cup_m \cap_{\tilde{m}} \{(\omega,q) : \tilde{\rho}(\bar{r}_m(t,\omega,q,s) - \bar{r}_{m+\tilde{m}}(t,\omega,q,s)) < \frac{1}{\ell}\} \in \tilde{\mathcal{F}}_0^t \otimes \mathcal{B}^d. \tag{58}$$

Define

$$\bar{r}(t,\omega,q,s) := \begin{cases} \lim_{m\to\infty} \bar{r}_m(t,\omega,q,s), & \forall (\omega,q) \in D \text{ uniformly for } t \in [s,T], \\ 0 & \text{otherwise.} \end{cases} \tag{59}$$

By (4.6) $\bar{r}(t,\cdot,\cdot,s)$ is $\tilde{\mathcal{F}}_0^t \otimes \mathcal{B}^d - \mathcal{B}^d$.

(iv) For fixed q step (iii) implies the existence of an $\Omega_{t,q,s} \in \tilde{\mathcal{F}}_0^t$ with $P(\Omega_{t,q,s}) = 1$ such that

$$\bar{r}(t,\cdot,q,s) = r(t,\cdot,q,s) \ \forall \omega \in \Omega_{t,q,s}. \tag{60}$$

Set

$$\tilde{r}(t) := q + \int_s^t \tilde{F}(\bar{r}(u,\cdot,q,s),u)du + \int_s^t \int \tilde{\mathcal{J}}(\bar{r}(u,\cdot,q,s),p,u)w(dp,du).$$

An Itô-Riemann approximation to the right hand side of the previous equation may be written as

$$\sum_{i=1}^{n} \tilde{G}(\bar{r}(u_i^n,\cdot,q,s), u_{i+1}^n - u_i^n, w(d\cdot,u_{i+1}^n) - w(d\cdot,u_i^n), \tag{61}$$

where the function $\tilde{G}$ is an abbreviation for the Itô-Riemann approximations of the deterministic and the stochastic integrals. The same partition $\{u_i^n\}$ yields an Itô-Riemann approximation of the solution of (2.2), $r(t,\cdot,q,s)$:

$$\sum_{i=1}^{n} \tilde{G}(r(u_i^n,\cdot,q,s), u_{i+1}^n - u_i^n, w(d\cdot,u_{i+1}^n) - w(d\cdot,u_i^n)). \tag{62}$$

At least for all $\omega \in \cap_{i=1}^n \Omega_{u_i^m,q,s}$ the values in (4.9) and (4.10) are identical. Hence, the limits are the same for all $\omega \in \cap_{n=1}^\infty \cap_{i=1}^n \Omega_{u_i^m,q,s}$ with $P(\cap_{n=1}^\infty \cap_{i=1}^n \Omega_{u_i^m,q,s}) = 1$. However, the limit of (4.10) equals a.s. $r(t,\cdot,q,s)$ which itself equals a.s. $\bar{r}(t,\cdot,q,s)$. Recall that our processes have continuous sample paths. Hence, with probability 1, uniformly in $t \in [s,T]$ for all $T > s$

$$\bar{r}(t,\cdot,q,s) = q + \int_s^t \tilde{F}(\bar{r}(u,\cdot,q,s),u)du + \int_s^t \int \tilde{\mathcal{J}}(\bar{r}(u,\cdot,q,s),p,u)w(dp,du), \tag{63}$$

i.e., $\bar{r}(t,\cdot,q,s)$ is another version of the solutions of (2.2) with initial value (q,s).

(v) Next, we show that $\bar{r}(t,\cdot,f_m(r_s),s)$ is a version of the unique solution of (2.2), $r(t,\cdot,f_m(r_s),s)$ for the initial valued (r_s,s) where r_s is $\mathcal{F}_s$-measurable and square integrable. Abbreviate

$$\bar{r}(t) := \bar{r}(t,\cdot,f_m(r_s),s), \quad r(t) := r(t,\cdot,f_m(r_s),s), \;\; \bar{r}_k(t) := \bar{r}(t,\cdot,r_k^m,s).$$

Set $A_k := r_s^{-1}(E_k^m)$. Further, let $\tilde{G}$ have the same meaning as in (4.10) and denote by $\bar{r}_k^n(t)$, $r^n(t)$ the processes used in the Picard-Lindelöf approximation of the solutions $\bar{r}_k(t)$ and $r(t)$, respectively. We then have

$$\left.\begin{aligned} r(t) &= f_m(r_s) + \textstyle\int_s^t \tilde{G}(r(u),du,w(\cdot,du)) \\ &= \textstyle\sum_{k=1}^\infty \{r_k^m 1_{A_k} + \int_s^t 1_{A_k}\tilde{G}(r(u),du,w(d\cdot,du))\} \\ &= \textstyle\sum_{k=1}^\infty \{r_k^m 1_{A_k} + \lim_{n\to\infty} \int_s^t 1_{A_k}\tilde{G}(r^n(u),du,w(d\cdot,du))\}. \end{aligned}\right\} \tag{64}$$

Further, by (4.11) in addition to the properties of the approximations we have with probability 1

$$\left.\begin{aligned} \bar{r}(t) &= \textstyle\sum_{k=1}^\infty \{r_k^m 1_{A_k} + \int_s^t 1_{A_k}\tilde{G}(\bar{r}_k(u),du,w(d\cdot,du)\} \\ &= \textstyle\sum_{k=1}^\infty \{r_k^m 1_{A_k} + \lim_{n\to\infty} \int_s^t 1_{A_k}\tilde{G}(\bar{r}_k^n(u),du,w(d\cdot,du))\}. \end{aligned}\right\} \tag{65}$$

We show by induction that with probability 1 for all k

$$1_{A_k} r(t) = 1_{A_k} \bar{r}_k(t).$$

The equality holds for $n = 0$. Assume that it also holds for $n-1$, $n \geq 1$. Then, with probability 1,

$$1_{A_k} r^n(t) = r^n_k + \int_s^t 1_{A_k} \tilde{G}(r^{n-1}(u), du, w(d\cdot, du))$$

$$= r^n_k + \int_s^t 1_{A_k} \tilde{G}(1_{A_k} r^{n-1}(u), du, w(d\cdot, du))$$

$$= r^n_k + \int_s^t 1_{A_k} \tilde{G}(1_{A_k} \bar{r}^{n-1}_k(u), du, w(d\cdot, du))$$

by induction hypothesis

$$= r^n_k + \int_s^t 1_{A_k} \tilde{G}(\bar{r}^{n-1}_k(u), du, w(d\cdot, du))$$

$$1_{A_k} \bar{r}^n_k(t).$$

Thus, $r^n(t) = \bar{r}^n(t) := \sum_{k=1}^\infty 1_{A_k} \bar{r}^n_k(t)$ a.s., whence

$$\bar{r}(t, \cdot, f_m(r_s), s) = r(t, \cdot, f_m(r_s), s) \quad \text{a.s.} \tag{66}$$

(vi) As in (4.4), comparing $\bar{r}(t, \cdot, f_m(r_s), s)$ with $r(t, \cdot, r_s, s)$ we obtain an $\tilde{F}_0^T$-measurable set $\tilde{\Omega}_{T,s}$ with $P(\tilde{\Omega}_{T,s}) = 1$ such that

$$\lim_{m\to\infty} \sup_{s \leq t \leq T} \tilde{\rho}(\bar{r}(t, \omega, f_m(r_s(\omega)), s) - r(t, \omega, r_s(\omega), s)) = 0 \ \forall \omega \in \tilde{\Omega}_{T,s}. \tag{67}$$

Note that

$$f_m(r_s(\omega)) \to r_s(\omega) \ \forall \omega.$$

Therefore, by (4.2) and (4.5),

$$\{(\omega, r_s(\omega)) : \omega \in \tilde{\Omega}_{T,s}\} \subset D. \tag{68}$$

Altogether, we obtain with probability 1

$$\left.\begin{aligned} \bar{r}(t, \cdot, r_s, s) = \lim_{m\to\infty} \bar{r}(t, \cdot, f_m(r_s), s) = r(t, \cdot, r_s, s), \\ \text{uniformly in } t \in [s, T],\ T > s, \end{aligned}\right\} \tag{69}$$

whence, as in step (iv), $\bar{r}(\cdot, \cdot, r_s, s)$ is a version of the the unique solution of (2.2) with continuous sample paths for $t \in [s, T]$ and all $T > s$. □

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