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V. S. KOROLIUK, N. LIMNIOS, AND I. V. SAMOILENKO

LÉVY APPROXIMATION OF IMPULSIVE RECURRENT PROCESS WITH SEMI-MARKOV SWITCHING

The weak convergence of an impulsive recurrent process with semi-Markov switching in the scheme of the Lévy approximation is proved. The singular perturbation problem for the compensating operator of an extended Markov renewal process is used to prove the relative compactness.

1. Introduction

The Lévy approximation is still an active area of research in several theoretical and applied directions. Since Lévy processes are now standard, the Lévy approximation is quite useful to analyze complex systems (see, e.g., [1, 9]). Moreover, they are involved in many applications, e.g., risk theory, finance, queueing, physics, *etc.* For a background on the Lévy process see, e.g., [1, 9, 3].

In particular, in [5, Chapter 7], the following impulsive process was studied as partial sums in a series scheme:

$$\xi^\varepsilon(t)=\xi_0^\varepsilon+\sum_{k=1}^{\nu(t)}\alpha_k^\varepsilon(x_{k-1}^\varepsilon),\quad t\ge 0,\varepsilon>0. \tag{1}$$

Here, the random variables $\alpha_k^\varepsilon(x), k\ge 1$ are supposed to be independent and perturbed by a jump Markov process $x(t), t\ge 0$. The embedded Markov chain $x_n, n\ge 0$ is defined by $x_n=x(\tau_n), n\ge 0$, where $0=\tau_0\le\tau_1\le ...\le\tau_n\le ...$ are the jump times of $x(t), t\ge 0$. The corresponding counting process of jumps $\nu(t):=\max\{k\ge 0:\tau_k\le t\}$.

We propose to study a generalization of problem (1):

$$\xi^\varepsilon(t)=\xi_0^\varepsilon+\sum_{k=1}^{\nu(t)}\alpha_k^\varepsilon(\xi_{k-1}^\varepsilon,x_{k-1}^\varepsilon),\quad t\ge 0,\varepsilon>0. \tag{2}$$

Here, the random variables $\alpha_k^\varepsilon(u,x), k\ge 1$ depend on the process $\xi^\varepsilon(t)$, and the switching process $x(t), t\ge 0$ is a semi-Markov one.

We propose to study the convergence of (2) using a combination of two methods. The one, based on the theory of semimartingales, is combined with a singular perturbation problem for the compensating operator of an extended Markov renewal process. So, the method includes two steps.

At the first step, we prove the relative compactness of the semimartingale representation of a family ξ^ε, $\varepsilon>0$, by proving the following two facts (see [2, Chapter 3]):

$$\lim_{c\to\infty}\sup_{\varepsilon\le\varepsilon_0}\mathbf{P}\{\sup_{t\le T}|\xi^\varepsilon(t)|>c\}=0,\forall\varepsilon_0>0,T>0$$

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known as the compact containment condition, and

$$\mathbf{E}|\xi^\varepsilon(t) - \xi^\varepsilon(s)|^2 \leq k|t-s|,$$

for some positive constant k independent of ε.

At the second step, we prove the convergence of the extended Markov renewal process $\xi_n^\varepsilon, x_n^\varepsilon, \tau_n^\varepsilon, n \geq 0$, by using the singular perturbation technique as presented in [5, Chapter 3].

Finally, we apply Theorem 6.3 from [5].

Similar results in case of the Poisson approximation of processes with locally independent increments with Markov switching are presented in [6].

The paper is organized as follows. In Section 2, we present the time-scaled impulsive process (2) and the switching semi-Markov process. In the same section, we present the main results of the Lévy approximation. In Section 3, we present the proof of the theorem.

2. Main results

Let us consider the space $\mathbb{R}^d$ endowed with a norm $|\cdot|$ $(d \geq 1)$, and $(E, \mathcal{E})$, a *standard phase space*, (i.e., E is a Polish space, and $\mathcal{E}$ is its Borel σ-algebra). For a vector $v \in \mathbb{R}^d$ and a matrix $c \in \mathbb{R}^{d\times d}$, v^* and c^* denote their transpose, respectively. Let $C_3(\mathbb{R}^d)$ be a measure-determining class of real-valued bounded functions such that $g(u)/|u|^2 \to 0$ as $|u| \to 0$ for $g \in C_3(\mathbb{R}^d)$ (see p.354 in [4] and [5, Chapter 7]).

For any $\varepsilon > 0$ and any sequence $z_k, k \geq 0$ of elements from $\mathbb{R}^d \times E$, the random variables $\alpha_k^\varepsilon(z_{k-1}), k \geq 1$ are supposed to be independent. Let us denote, by $G_{u,x}^\varepsilon$, the distribution function of $\alpha_k^\varepsilon(x)$, that is,

$$G_{u,x}^\varepsilon(dv) := \mathbf{P}(\alpha_k^\varepsilon(u,x) \in dv), k \geq 0, \varepsilon > 0, x \in E, u \in \mathbb{R}^d.$$

The switching semi-Markov process $x(t), t \geq 0$ on the standard phase space $(E, \mathcal{E})$ is defined by the semi-Markov kernel

$$Q(x, B, t) = P(x, B)F_x(t), x \in E, B \in \mathcal{E}, t \geq 0,$$

which defines the associated Markov renewal process $x_n, \tau_n, n \geq 0$:

$$Q(x, B, t) = \mathbf{P}(x_{n+1} \in B, \tau_{n+1} - \tau_n \leq t | x_n = x) = \mathbf{P}(x_{n+1} \in B | x_n = x) \times$$

$$\mathbf{P}(\tau_{n+1} - \tau_n \leq t | x_n = x) =: P(x, B)F_x(t).$$

The corresponding counting process of jumps $\nu(t) := \max\{k \geq 0 : \tau_k \leq t\}$.

We make the following assumption for the counting process $\nu(t)$:

$$\int_0^t \mathbf{E}[\varphi(s) d\nu(s)] < l_1 \int_0^t \mathbf{E}(\varphi(s)) ds \tag{3}$$

for any nonnegative, increasing $\varphi(s)$ and $l_1 > 0$ that does not depend on $\varphi(s)$.

The impulsive processes $\xi^\varepsilon(t), t \geq 0, \varepsilon > 0$ on $\mathbb{R}^d$ in the series scheme with small series parameter $\varepsilon \to 0$, $(\varepsilon > 0)$ are defined by the sum ([5, Section 9.2.1])

$$\xi^\varepsilon(t) = \xi_0^\varepsilon + \sum_{k=1}^{\nu(t/\varepsilon^2)} \alpha_k^\varepsilon(\xi_{k-1}^\varepsilon, x_{k-1}^\varepsilon), \quad t \geq 0. \tag{4}$$

Here,

$$\xi_n^\varepsilon := \xi(\varepsilon^2 \tau_n) = \xi_0^\varepsilon + \sum_{k=1}^{n} \alpha_k^\varepsilon(\xi_{k-1}^\varepsilon, x_{k-1}^\varepsilon).$$

It is worth noticing that the coupled process $\xi^\varepsilon(t), x^\varepsilon(t), t \geq 0$, is a Markov additive process (see, e.g., [5, Section 2.5]).

The Lévy approximation of the Markov impulsive process (4) is considered under the following conditions.

C1: The semi-Markov process $x(t), t \geq 0$ is uniformly ergodic with the stationary distribution

$$\pi(dx)q(x) = q\rho(dx), q(x) := 1/m(x), q := 1/m,$$

$$m(x) := \mathbf{E}\theta_x = \int_0^\infty \overline{F}_x(t)dt, m := \int_E \rho(dx)m(x),$$

$$\rho(B) = \int_E \rho(dx)P(x, B), \rho(E) = 1.$$

C2: *Lévy approximation.* The family of impulsive processes $\xi^\varepsilon(t), t \geq 0$ satisfies the Lévy approximation conditions [5, Section 9.2].

L1: Initial-value condition:

$$\sup_{\varepsilon>0} \mathbf{E}|\xi_0^\varepsilon| \leq C < \infty$$

and

$$\xi_0^\varepsilon \Rightarrow \xi_0.$$

L2: Approximation of the mean values:

$$a^\varepsilon(u; x) = \int_{\mathbb{R}^d} vG_{u,x}^\varepsilon(dv) = \varepsilon a_1(u; x) + \varepsilon^2[a(u; x) + \theta_a^\varepsilon(u; x)],$$

and

$$c^\varepsilon(u; x) = \int_{\mathbb{R}^d} vv^* G_{u,x}^\varepsilon(dv) = \varepsilon^2[c(u; x) + \theta_c^\varepsilon(u; x)],$$

where $a_1(u; x), a(u; x)$ and $c(u; x)$ are bounded functions.

L3: Poisson approximation condition for the intensity kernel (see [5, Chapter 9])

$$G_g^\varepsilon(u; x) = \int_{\mathbb{R}^d} g(v)G_{u,x}^\varepsilon(dv) = \varepsilon^2[G_g(u; x) + \theta_g^\varepsilon(u; x)]$$

for all $g \in C_3(\mathbb{R}^d)$, and the kernel $G_g(u; x)$ is bounded for all $g \in C_3(\mathbb{R}^d)$, that is,

$$|G_g(u; x)| \leq G_g \quad \text{(a constant depending on } g).$$

Here,

$$G_g(u; x) = \int_{\mathbb{R}^d} g(v)G_{u,x}(dv), \quad g \in C_3(\mathbb{R}^d). \tag{5}$$

The above negligible terms $\theta_a^\varepsilon, \theta_c^\varepsilon$, and θ_g^ε satisfy the condition

$$\sup_{x\in E} |\theta_\cdot^\varepsilon(u; x)| \to 0, \quad \varepsilon \to 0.$$

L4: *Balance condition*:

$$\int_E \rho(dx)a_1(u; x) = 0.$$

In addition, the following conditions are used:

C3: *Uniform square integrability*:

$$\lim_{c\to\infty} \sup_{x\in E} \int_{|v|>c} vv^* G_{u,x}(dv) = 0.$$

C4: *Linear growth*: there exists a positive constant L such that

$$|a(u; x)| \leq L(1 + |u|), \quad \text{and} \quad |c(u; x)| \leq L(1 + |u|^2),$$

and, for any real-valued non-negative function $f(v), v \in \mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d\setminus\{0\}} (1 + f(v))|v|^2 dv < \infty,$$

we have

$$|G_{u,x}(v)| \leq Lf(v)(1 + |u|).$$

The main result of our work is the following.

Theorem 2.1. *Under conditions* **C1** $-$ **C4**, *the weak convergence*

$$\xi^\varepsilon(t) \Rightarrow \xi^0(t), \quad \varepsilon \to 0$$

takes place.

The limit process $\xi^0(t), t \geq 0$ *is a Lévy process defined by a generator* $\mathbf{L}$ *of the form*

$$\mathbf{L}\varphi(u) = (\widehat{a}(u) - \widehat{a}_0(u))\varphi'(u) + \frac{1}{2}\sigma^2(u)\varphi''(u) + \lambda(u)\int_{\mathbb{R}^d} [\varphi(u + v) - \varphi(u)]G_u^0(dv), \tag{6}$$

where

$$\widehat{a}(u) = q\int_E \rho(dx)a(u;x),\ \ \widehat{a}_0(u) = \int_E vG_u(dv),$$

$$G_u(dv) = q\int_E \rho(dx)G_{u,x}(dv),\ \ \widehat{a_1^2}(u) = q\int_E \rho(dx)a_1^2(u;x),$$

$$\widetilde{a}_1(u;x) := q(x)\int_E P(x, dy)a_1(u;x),\ \ c_0(u;x) = \int_E vv^* G_{u,x}(dv),$$

$$\sigma^2(u) = 2\int_E \pi(dx)\{\widetilde{a}_1(u;x)\widetilde{R}_0\widetilde{a}_1^*(u;x) + \frac{1}{2}[c(u;x) - c_0(u;x)]\} - \widehat{a_1^2}(u), \sigma^2(u) \geq 0$$

$$\lambda(u) = G_u(\mathbb{R}^d),\ \ G_u^0(dv) = G_u(dv)/\lambda(u),$$

and $\widetilde{R}_0$ *is the potential operator of the embedded Markov chain.*

Remark 2.1. The limit Lévy process consists of three parts: deterministic drift, diffusion part, and Poisson part.

There are some possible cases:

1: If $\widehat{a}(u) - \widehat{a}_0(u) = 0$, then the limit process has no deterministic drift.

2: If $\sigma^2(u) = 0$, then the limit process has no diffusion part. As a variant of this case, we note that if $c(u;x) = c_0(u;x)$, then also $a_1(u;x) = 0$, and we obtain the conditions of the Poisson approximation after the renormalization $\varepsilon^2 = \widetilde{\varepsilon}$ (see, e.g., Chapter 7 in [5]).

Remark 2.2. In work [5] (Theorem 9.3), an analogous result was obtained for an impulsive process with Markov switching. If we study an ordinary impulsive process without switching, we should obtain $\sigma^2 = \mathbf{E}(\alpha_k^\varepsilon)^2 - (\mathbf{E}(\alpha_k^\varepsilon))^2 = (c - c_0) - a_1^2$. This result correlates with the similar results from [4, Chapter IX]. In case of our Theorem, this may be easily shown, but it is not obvious in [5] (Theorem 9.3).

The difference is that we used $\widetilde{R}_0$ − the potential operator of an embedded Markov chain instead of R_0 − the potential operator of a Markov process. Due to this circumstance, our result obviously correlates with other well-known results.

Remark 2.3. The asymptotics of the second moment in condition **L1** contains the second modified characteristic $c(u;x)$ (see Corollary 4.2 at p.555 in [4]). This characteristic in limit contains both the second moment of the Poisson part and the dispersion of the diffusion part, namely $c = c_0 + \sigma^2$.

3. Proof of Theorem 2.1

The proof of Theorem 2.1 is based on the semimartingale representation of the impulsive process (4).

We split up the proof of Theorem 2.1 into following two steps.

Step 1. At this step, we establish the relative compactness of the family of processes $\xi^\varepsilon(t), t \geq 0, \varepsilon > 0$ by using the approach developed in [7]. We recall that the space of all probability measures defined on the standard space $(E, \mathcal{E})$ is also a Polish space; so, the relative compactness and the tightness are equivalent.

First, we need the following lemma.

Lemma 3.1. *Under assumption* **C4**, *there exists a constant $k > 0$, independent of ε and dependent on T, such that*

$$\mathbf{E} \sup_{t \leq T} |\xi^\varepsilon(t)|^2 \leq k_T, \forall T > 0.$$

Corollary 3.1. *Under assumption* **C4**, *the following compact containment condition (CCC) holds:*

$$\lim_{c \to \infty} \sup_{\varepsilon \leq \varepsilon_0} \mathbf{P} \left\{ \sup_{t \leq T} |\xi^\varepsilon(t)| > c \right\} = 0, \forall \varepsilon_0 > 0, T > 0.$$

Proof: The proof of this corollary follows from Kolmogorov's inequality.

Proof of Lemma 3.1: (following [7]). The impulsive process (4) has the semimartingale representation

$$\xi^\varepsilon(t) = u + B_t^\varepsilon + M_t^\varepsilon, \tag{7}$$

where $u = \xi_0^\varepsilon$; B_t^ε is the predictable drift,

$$B_t^\varepsilon = \sum_{k=1}^{\nu(t/\varepsilon^2)} a^\varepsilon(\xi_{k-1}^\varepsilon, x_{k-1}^\varepsilon) = A_1^\varepsilon(t) + A^\varepsilon(t) + \theta_a^\varepsilon(t),$$

here

$$A_1^\varepsilon(t) := \varepsilon \sum_{k=1}^{\nu(t/\varepsilon^2)} a_1(\xi_{k-1}^\varepsilon, x_{k-1}^\varepsilon), A^\varepsilon(t) := \varepsilon^2 \sum_{k=1}^{\nu(t/\varepsilon^2)} a(\xi_{k-1}^\varepsilon, x_{k-1}^\varepsilon),$$

and M_t^ε is the locally square integrable martingale, where

$$\langle M^\varepsilon \rangle_t = \varepsilon^2 \sum_{k=1}^{\nu(t/\varepsilon^2)} \int_{\mathbb{R}^d \setminus \{0\}} vv^* G(\xi_{k-1}^\varepsilon, dv; x_{k-1}^\varepsilon) + \theta_c^\varepsilon(t) = \varepsilon^2 \sum_{k=1}^{\nu(t/\varepsilon^2)} c(\xi_{k-1}^\varepsilon; x_{k-1}^\varepsilon) + \theta_c^\varepsilon(t), \tag{8}$$

and for every finite $T > 0$

$$\sup_{0 \leq t \leq T} |\theta_\cdot^\varepsilon(t)| \to 0, \varepsilon \to 0.$$

To verify the compactness of the process $\xi^\varepsilon(t)$, we split up it into two parts.

The first part of the order of ε,

$$A_1^\varepsilon(t) = \varepsilon \sum_{k=1}^{\nu(t/\varepsilon^2)} a_1(\xi_{k-1}^\varepsilon; x_{k-1}^\varepsilon),$$

can be characterized by the compensating operator

$$\mathbf{L}^\varepsilon \varphi(u; x) = \varepsilon^{-2} q(x) [\mathbf{A}_1^\varepsilon(x) P - I] \varphi(u; x),$$

where

$$\mathbf{A}_1^\varepsilon(x)\varphi(u) = \varphi(u + \varepsilon a_1(u;x)) = \varepsilon a_1(u;x)\varphi'(u) + \varepsilon\theta^\varepsilon\varphi(u).$$

After simple calculations, we can rewrite the operator:

$$\mathbf{L}^\varepsilon = \varepsilon^{-2}\mathbf{Q} + \varepsilon^{-1}\mathbf{A}_1(x)P + \theta^\varepsilon,$$

where $\mathbf{A}_1(x)\varphi(u) = \varepsilon a_1(u;x)\varphi'(u)$.

The corresponding martingale characterization is as follows:

$$\mu_{n+1}^\varepsilon = \varphi(A_{1,n+1}^\varepsilon, x_{n+1}^\varepsilon) - \varphi(A_{1,0}^\varepsilon, x_0^\varepsilon) - \sum_{m=0}^{\nu_n} \theta_{m+1}^\varepsilon \mathbf{L}^\varepsilon \varphi(A_{1,m}^\varepsilon, x_m^\varepsilon).$$

Using the results from [5], Chapter 1, we obtain the last martingale in the form

$$\widetilde{\mu}_t^\varepsilon = \varphi^\varepsilon(A_1^\varepsilon(t), x_t^\varepsilon) + \varphi^\varepsilon(A_1^\varepsilon(0), x_0^\varepsilon) - \int_0^t \mathbf{L}^\varepsilon\varphi^\varepsilon(A_1^\varepsilon(s), x_s^\varepsilon)ds,$$

where $x_t^\varepsilon := x(t/\varepsilon^2)$.

Thus (see, e.g., Theorem 1.2 in [5]), it has quadratic characteristic

$$< \widetilde{\mu}^\varepsilon >_t = \int_0^t \left[\mathbf{L}^\varepsilon(\varphi^\varepsilon(A_1^\varepsilon(s), x_t^\varepsilon))^2 - 2\varphi^\varepsilon(A_1^\varepsilon(s), x_s^\varepsilon)\mathbf{L}^\varepsilon\varphi^\varepsilon(A_1^\varepsilon(s), x_s^\varepsilon)\right] ds.$$

Applying the operator $\mathbf{L}^\varepsilon = \varepsilon^{-2}\mathbf{Q}+\varepsilon^{-1}\mathbf{A}_1(x)P+\theta^\varepsilon$ to the test-function $\varphi^\varepsilon = \varphi+\varepsilon\varphi_1$, we obtain the integrand of the form

$$Q\varphi_1^2 - 2\varphi_1 Q\varphi_1 + \theta^\varepsilon\varphi^\varepsilon.$$

Thus, the integrand is bounded. The boundedness of the quadratic characteristic provides the compactness of $\widetilde{\mu}_t^\varepsilon$. Thus, $\varphi(A_1^\varepsilon(t))$ is compact too and bounded uniformly by ε. By the results from [2, Chapter 3], we obtain the compactness of $A_1^\varepsilon(t)$, because the test-function $\varphi(u)$ belongs to the measure-determining class.

Now we should study the second part of the order of ε^2.

For the process $y(t)$, $t \geq 0$, let us define the process

$$y^\dagger(t) = \sup_{s\leq t} |y(s)|.$$

Then, from (7), we have

$$((\xi^\varepsilon(t))^\dagger)^2 \leq 4[u^2 + ((A^\varepsilon(t))^\dagger)^2 + ((M_t^\varepsilon)^\dagger)^2]. \tag{9}$$

Now we can apply the result of Section 2.3 [5], namely

$$\sum_{k=1}^{\nu(t)} a(\xi_{k-1}^\varepsilon, x_{k-1}^\varepsilon) = \int_0^t a(\xi^\varepsilon(s), x^\varepsilon(s))d\nu(s).$$

Condition **C4** implies that, for sufficiently large ε,

$$(A^\varepsilon(t))^\dagger = \varepsilon^2\int_0^{t/\varepsilon^2} a(\xi^\varepsilon(s), x^\varepsilon(s))d\nu(s) \leq L\varepsilon^2\int_0^{t/\varepsilon^2} (1 + (\xi^\varepsilon(s))^\dagger)d\nu(s). \tag{10}$$

Now by (8), condition **C4**, and Doob's inequality (see, e.g., [8, Theorem 1.9.2]),

$$\mathbf{E}((M_t^\varepsilon)^\dagger)^2 \leq 4|\mathbf{E}\langle M^\varepsilon\rangle_t|,$$

we obtain

$$|\langle M^\varepsilon\rangle_t| = \left|\varepsilon^2\int_0^{t/\varepsilon^2}\int_{\mathbb{R}^d\setminus\{0\}} vv^* G(\xi^\varepsilon(s), dv; x_s^\varepsilon)d\nu(s)\right| = \left|\varepsilon^2\int_0^{t/\varepsilon^2} c(\xi^\varepsilon(s); x^\varepsilon(s))d\nu(s)\right| \leq L\varepsilon^2\int_0^{t/\varepsilon^2}[1 + ((\xi^\varepsilon(s))^\dagger)^2]d\nu(s). \tag{11}$$

Inequalities (9)-(11), condition (3), and Cauchy–Buniakowski–Schwarz inequality, ($[\int_0^t \varphi(s)ds]^2 \le t\int_0^t \varphi^2(s)ds$), imply

$$\mathbf{E}((\xi^\varepsilon(t))^\dagger)^2 \le k_1 + k_2\varepsilon^2 \int_0^{t/\varepsilon^2} \mathbf{E}[((\xi^\varepsilon(s))^\dagger)^2 d\nu(s)] \le k_1 + k_2 l_1 \varepsilon^2 \int_0^{t/\varepsilon^2} \mathbf{E}((\xi^\varepsilon(s))^\dagger)^2 ds =$$

$$k_1 + k_2 l_1 \int_0^t \mathbf{E}((\xi^\varepsilon(s))^\dagger)^2 ds,$$

where k_1, k_2, and l_1 are positive constants independent of ε.

By Gronwall's inequality (see, e.g., [2, p. 498]), we obtain

$$\mathbf{E}((\xi^\varepsilon(t))^\dagger)^2 \le k_1 \exp(k_2 l_1 t).$$

Thus, both parts of $\xi^\varepsilon(t)$ are compact and bounded, so

$$\mathbf{E}\sup_{t\le T} |\xi^\varepsilon(t)|^2 \le k_T.$$

Hence, the lemma is proved.

Lemma 3.2. *Under assumption* **C4**, *there exists a constant $k > 0$ independent of ε and such that*

$$\mathbf{E}|\xi^\varepsilon(t) - \xi^\varepsilon(s)|^2 \le k|t-s|.$$

Proof: In the same manner with (9), we can write

$$|\xi^\varepsilon(t) - \xi^\varepsilon(s)|^2 \le 2|B_t^\varepsilon - B_s^\varepsilon|^2 + 2|M_t^\varepsilon - M_s^\varepsilon|^2.$$

By using Doob's inequality, we obtain

$$\mathbf{E}|\xi^\varepsilon(t) - \xi^\varepsilon(s)|^2 \le 2\mathbf{E}\{|B_t^\varepsilon - B_s^\varepsilon|^2 + 8|\langle M^\varepsilon\rangle_t - \langle M^\varepsilon\rangle_s|\}.$$

Now (11), condition (3), and assumption **C4** yield

$$|B_t^\varepsilon - B_s^\varepsilon|^2 + 8|\langle M^\varepsilon\rangle_t - \langle M^\varepsilon\rangle_s| \le k_3[1 + ((\xi^\varepsilon(T))^\dagger)^2]|t-s|,$$

where k_3 is a positive constant independent of ε.

From the last inequality and Lemma 3.1, the desired conclusion emerges.

The conditions proved in Corollary 3.1 and Lemma 3.2 are necessary and sufficient for the compactness of the family of processes $\xi^\varepsilon(t), t \ge 0, \varepsilon > 0$.

Step 2. At the next step of the proof, we apply the problem of singular perturbation to the generator of the process $\xi^\varepsilon(t)$. To do this, we recall the following theorem. Let $C_0^2(\mathbb{R}^d \times E)$ be the space of real-valued functions twice continuously differentiable with respect to the first argument, defined on $\mathbb{R}^d \times E$, and vanishing at infinity. Let $C(\mathbb{R}^d \times E)$ be the space of real-valued continuous bounded functions defined on $\mathbb{R}^d \times E$.

Theorem 3.1. *(*[5, Theorem 6.3]*) Let the following conditions hold for a family of Markov processes $\xi^\varepsilon(t), t \ge 0, \varepsilon > 0$:*

- **CD1:** *There exists a family of test functions $\varphi^\varepsilon(u,x)$ in $C_0^2(\mathbb{R}^d \times E)$, such that*

  $$\lim_{\varepsilon\to 0} \varphi^\varepsilon(u,x) = \varphi(u),$$

  uniformly on u, x.
- **CD2:** *The following convergence holds:*

  $$\lim_{\varepsilon\to 0} \mathbf{L}^\varepsilon \varphi^\varepsilon(u,x) = \mathbf{L}\varphi(u),$$

  uniformly on u, x. The family of functions $\mathbf{L}^\varepsilon\varphi^\varepsilon, \varepsilon > 0$, is uniformly bounded, $\mathbf{L}\varphi(u)$ and $\mathbf{L}^\varepsilon\varphi^\varepsilon$ belong to $C(\mathbb{R}^d \times E)$.
- **CD3:** *The quadratic characteristic of the martingale that characterizes a coupled Markov process $\xi^\varepsilon(t), x^\varepsilon(t), t \ge 0, \varepsilon > 0$ has the representation $\langle\mu^\varepsilon\rangle_t = \int_0^t \zeta^\varepsilon(s)ds$,*

where the random functions $\zeta^\varepsilon, \varepsilon > 0$, *satisfy the condition*

$$\sup_{0\le s\le T} \mathbf{E}|\zeta^\varepsilon(s)| \le c < +\infty.$$

CD4: *The convergence of the initial values holds, and*

$$\sup_{\varepsilon>0} \mathbf{E}|\zeta^\varepsilon(0)| \le C < +\infty.$$

Then the weak convergence

$$\xi^\varepsilon(t) \Rightarrow \xi(t), \quad \varepsilon \to 0,$$

takes place.

We consider the extended Markov renewal process

$$\xi_n^\varepsilon, x_n^\varepsilon, \tau_n^\varepsilon, n \ge 0, \tag{12}$$

where $x_n^\varepsilon = x^\varepsilon(\tau_n^\varepsilon), x^\varepsilon(t) := x(t/\varepsilon^2), \xi_n^\varepsilon = \xi^\varepsilon(\tau_n^\varepsilon)$ and $\tau_{n+1}^\varepsilon = \tau_n^\varepsilon + \varepsilon^2\theta_n^\varepsilon, n \ge 0$, and

$$\mathbf{P}(\theta_{n+1}^\varepsilon \le t | x_n^\varepsilon = x) = F_x(t) = \mathbf{P}(\theta_x \le t).$$

Definition 3.1. [10] The *compensating operator* $\mathbf{L}^\varepsilon$ of the Markov renewal process (12) is defined by the relation

$$\mathbf{L}^\varepsilon\varphi(\xi_0^\varepsilon, x_0, \tau_0) = q(x_0)\mathbf{E}[\varphi(\xi_1^\varepsilon, x_1, \tau_1) - \varphi(\xi_0^\varepsilon, x_0, \tau_0)|\mathcal{F}_0],$$

where

$$\mathcal{F}_t := \sigma(\xi^\varepsilon(s), x^\varepsilon(s), \tau^\varepsilon(s); 0 \le s \le t).$$

Using Lemma 9.1 from [5], we obtain that the compensating operator of the extended Markov renewal process from Definition 3.1 can be defined by the relation (see also Section 2.8 in [5])

$$\mathbf{L}^\varepsilon\varphi(u, v; x) = \varepsilon^{-2}q(x)\left[\int_E P(x, dy)\int_{\mathbb{R}^d} G_{u,x}^\varepsilon(dz)\varphi(u+z, v; y) - \varphi(u, v; x)\right]. \tag{13}$$

By analogy with [5, Lemma 9.2], we can prove the following result:

Lemma 3.3. *The main part in the asymptotic representation of the compensating operator (13) is*

$$\mathbf{L}^\varepsilon\varphi(u, v, x) = \varepsilon^{-2}\mathbf{Q}\varphi(\cdot, \cdot, x) + \varepsilon^{-1}a_1(u; x)\mathbf{Q}_0\varphi'_u(u, \cdot, \cdot) + [a(u; x) - a_0(u; x)]\mathbf{Q}_0\varphi'_u(u, \cdot, \cdot) +$$
$$\frac{1}{2}[c(u; x) - c_0(u; x)]\mathbf{Q}_0\varphi''_{uu}(u, \cdot, \cdot) + \mathbf{G}_{u,x}\mathbf{Q}_0\varphi(u, \cdot, \cdot),$$

where

$$\mathbf{Q}_0\varphi(x) := q(x)\int_E P(x, dy)\varphi(y), \mathbf{G}_{u,x}\varphi(u) := \int_{\mathbb{R}^d}[\varphi(u+z) - \varphi(u)]G_{u,x}(dz),$$
$$a_0(u; x) = \int_E vG_{u,x}(dv), c_0(u; x) = \int_E vv^*G_{u,x}(dv).$$

Proof of this Lemma is analogous to the proof of [5, Lemma 9.2].

The solution of the singular perturbation problem at the test functions $\varphi^\varepsilon(u, x) = \varphi(u) + \varepsilon\varphi_1(u, x) + \varepsilon^2\varphi_2(u, x)$ in the form

$$\mathbf{L}^\varepsilon\varphi^\varepsilon = \mathbf{L}\varphi + \theta^\varepsilon\varphi \tag{14}$$

can be found in the same manner with Lemma 9.3 in [5].

To simplify the formula, we refer to the embedded Markov chain. The corresponding generator $\widetilde{\mathbf{Q}} := P - I$, and the potential operator satisfies the relation $\widetilde{R}_0(P-I) = \widetilde{\Pi} - I$.

From (14), we obtain

$$\widetilde{\mathbf{Q}}\varphi = 0,$$
$$\widetilde{\mathbf{Q}}\varphi_1 + \mathbf{A}_1(x)P\varphi = 0,$$

$$\widetilde{\mathbf{Q}}\varphi_2 + \mathbf{A}_1(x)P\varphi_1 + (\mathbf{A}(x) + \mathbf{C}(x) + \mathbf{G}_{u,x})P\varphi = m(x)\mathbf{L}\varphi,$$

where

$$\mathbf{A}(x)\varphi(u) := [a(u;x) - a_0(u;x)]\varphi'(u), \mathbf{A}_1(x)\varphi(u) := a_1(u;x)\varphi'(u),$$
$$\mathbf{C}(x) := \frac{1}{2}[c(u;x) - c_0(u;x)]\varphi''_{uu}(u).$$

The second equation yields $\varphi_1 = \widetilde{R}_0\mathbf{A}_1(x)\varphi$. Substituting it into the last equation, we have

$$\widetilde{\mathbf{Q}}\varphi_2 + \mathbf{A}_1(x)P\widetilde{R}_0\mathbf{A}_1(x)\varphi + (\mathbf{A}(x) + \mathbf{C}(x) + \mathbf{G}_{u,x})\varphi = m(x)\mathbf{L}\varphi.$$

Since $P\widetilde{R}_0 = \widetilde{R}_0 + \widetilde{\Pi} - I$, we finally obtain

$$q^{-1}\mathbf{L} = \widetilde{\Pi}[(\mathbf{A}(x) + \mathbf{C}(x) + \mathbf{G}_{u,x}) + \mathbf{A}_1(x)\widetilde{R}_0\mathbf{A}_1(x) - \mathbf{A}_1^2(x)]\widetilde{\Pi}. \tag{15}$$

Simple calculations give us (6) from (15).

Now Theorem 3.1 can be applied.

We see from (13) and (15) that the solution of the singular perturbation problem for $\mathbf{L}^\varepsilon\varphi^\varepsilon(u,v;x)$ satisfies conditions **CD1, CD2**. Condition **CD3** of this theorem implies that the quadratic characteristic of the martingale corresponding to a coupled Markov process is relatively compact. The same result follows from the CCC (see Corollary 3.1 and Lemma 3.2) by [4, Chapter 9]. Thus, condition **CD3** follows from Corollary 3.1 and Lemma 3.2. Due to **L1**, condition **CD4** is also satisfied. Thus, all the conditions of Theorem 3.1 are satisfied, so the weak convergence $\xi^\varepsilon(t) \Rightarrow \xi^0(t)$ takes place.

Theorem 2.1 is proved.

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Institute of Mathematics, Ukrainian National Academy of Science, Kiev, Ukraine
E-mail address: korol@imath.kiev.ua

Laboratoire de Mathématiques Appliquées, Université de Technologie de Compiègne, France
E-mail address: nikolaos.limnios@utc.fr

Institute of Mathematics, Ukrainian National Academy of Science, Kiev, Ukraine
E-mail address: isamoil@imath.kiev.ua