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ON A DIFFUSION PROCESS ON A HALF-LINE WITH FELLER–WENTZEL BOUNDARY CONDITION THAT CORRESPONDS TO REFLECTION AND JUMPS

An operator semigroup that describes a diffusion process on a half-line such that its behavior on a boundary is defined by the Feller–Wentzel boundary condition with the integral term is constructed using classical potential theory.

1. Introduction and statement of the problem

Consider a domain $D = \mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ on a line $\mathbb{R}$. By $\partial D = \{0\}$ and $\overline{D} = D \cup \partial D$, we denote the boundary and the closure of the domain D, respectively. Let continuous bounded functions $a(x)$ and $b(x)$ be given in $\overline{D}$, and let $b(x) \geq 0$. Let L be a corresponding differential operator that acts on a set $C^2(\overline{D})$ of all twice continuously differentiable functions bounded along with their derivatives,

$$L\varphi(x) = \frac{1}{2} b(x) \frac{d^2\varphi(x)}{dx^2} + a(x) \frac{d\varphi(x)}{dx}. \tag{1}$$

If the functions $b(x)$ and $a(x)$ can be interpreted as the diffusion coefficient and the drift coefficient of some homogeneous diffusion process given on D, respectively, then L is called a generating differential operator of this process [1], [2]. We assume also that a Feller–Wentzel operator of the following kind is given:

$$L_0\varphi(0) = q\frac{d\varphi(0)}{dx} - \int_D (\varphi(0) - \varphi(y))\mu(dy). \tag{2}$$

Here, q is a nonnegative constant, and $\mu(\cdot)$ is a nonnegative measure on D such that

$$\int_{D \setminus D_\delta} y\mu(dy) + \mu(D_\delta) < \infty, \tag{3}$$

where $D_\delta = \{x \in D : x > \delta > 0\}$. The numbers q and $m = \mu(D)$ are not equal to zero simultaneously. We recall that all continuations of the diffusion process after it reaches the boundary are described with the Feller–Wentzel boundary operator (see [1] – [5]). In our case, such continuations can be only reflections and jumps into the domain.

We will define a problem to find an operator semigroup $(T_t)_{t>0}$ that generates the Feller process on $\overline{D}$ such that it coincides at points of D with the diffusion process generated by the operator L, and its behavior in $\{0\}$ is determined by the boundary condition

$$L_0\varphi(0) = 0. \tag{4}$$

The construction of the required semigroup is provided with the use of analytical methods, by solving the corresponding initial-boundary problem for a second-order linear

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parabolic equation. This problem is to find a function $u(t, x)$ $(t > 0,\ x \in D)$ that satisfies the conditions

$$\frac{\partial u}{\partial t} = Lu, \quad t > 0,\ x \in D, \tag{5}$$

$$u(0, x) = \varphi(x), \quad x \in D, \tag{6}$$

$$L_0 u(t, 0) = 0, \quad t > 0. \tag{7}$$

The fundamental solution of the parabolic equation and the generated heat potentials are used to solve problem (5) – (7) (see [2], [3]). By using the potentials, problem (5) – (7) turns into a Volterra second-type integral equation, whose solution is obtained by the successive approximation method.

Note that a similar problem was considered in [7] in case where the parameter $q = 0$ in (2), and the measure μ is finite. The multidimensional analog of this case was considered in [8] and was studied with the use of function analysis methods.

Further, we use the following notations and definitions: D_t^r and D_x^p are the symbols of a partial derivative with respect to t of order r and partial derivative with respect to x of order p, respectively, where r and p are integer nonnegative numbers; $\mathcal{B}(\mathbb{R})$ is a Banach space of all real-valued bounded measurable functions on $\mathbb{R}$ with norm $\|\varphi\| = \sup_{x\in\mathbb{R}} |\varphi(x)|$; $C^l(D)$ $(C^l(\overline{D}))$, $l = 0, 1, 2$, $(C^0(D) = C(D))$ is a set of all functions continuous in D (in $\overline{D}$) with continuous derivatives D_x^p, $p \le l$ in D (in $\overline{D}$), where D is a subset in $\mathbb{R}$; T is some fixed positive number; $\mathbb{R}_T^2 = (0, T] \times \mathbb{R}$, $\mathbb{R}_T^\infty = (0, \infty) \times \mathbb{R}$; (t, x) is a point in $\overline{\mathbb{R}_\infty^2}$; $C^{m,l}(\Omega)$, $m = 0, 1$, $l = 0, 1, 2$, is a set of all functions continuous in Ω with continuous derivatives D_t^r and D_x^p, $r \le m$, $p \le l$, in Ω, where Ω is a subset of $\mathbb{R}_\infty^2$; and $H^\alpha(\mathbb{R})$, $\alpha \in (0, 1)$, is a Hölder space with norm $\|\varphi\| = \sup_{x\in\mathbb{R}} |\varphi(x)| + \sup_{x,y\in\mathbb{R}} \frac{|\varphi(x)-\varphi(y)|}{|x-y|^\alpha}$, as in [6, p. 16]. Everywhere else, by C and c, we denote some constants that do not depend on (t, x), in exact values of which we are not interested. Other definitions and notations will be described, when they appear for the first time.

We assume that the following assumptions about the coefficients of the operator L from (1) and the parameter q and the measure μ from (2) hold:

a) functions $b(x)$, $a(x)$ are defined on $\mathbb{R}$ and $a, b \in H^\alpha(\mathbb{R})$;
b) there exist such constants b_0, b_1 that $0 < b_0 \le b_1$ and $b_0 \le b(x) \le b_1$ for all $x \in \mathbb{R}^1$;
c) $q > 0$

Notice that relation (3) yields the inequality

$$\int_{D\setminus D_\delta} y\mu(dy) \le \lambda(\delta), \tag{8}$$

where $\lambda(\delta) \to 0$, as $\delta \to 0$.

It is known (see [1], [2], and [6]) that, under assumptions a) and b), there exists a fundamental solution for Eq. (5), a function $g(t, x, y)$, $(t > 0,\ x, y \in \mathbb{R}^1)$. For $x \ne y$, it satisfies the equation and is of form

$$g(t, x, y) = g_0(t, x, y) + g_1(t, x, y), \tag{9}$$

where

$$g_0(t, x, y) = (2\pi b(y) t)^{-\frac{1}{2}} \exp\left(-\frac{(x-y)^2}{2b(y)t}\right), \tag{10}$$

and $g_1(t, x, y)$ is in the form of integral operator with the kernel g_0 and the density Φ_0 that is determined from some Volterra second-order integral equation.

Notice the following estimations (see [6]):

$$|D_t^r D_x^p g(t,x,y)| \le Ct^{-\frac{1+2r+p}{2}} \exp\left(-c\frac{|x-y|^2}{t}\right), \quad 2r+p \le 2,\ t \in (0,T], \tag{11}$$

$$|D_t^r D_x^p g_1(t,x,y)| \le Ct^{-\frac{1+2r+p-\alpha}{2}} \exp\left(-c\frac{|x-y|^2}{t}\right), \quad 2r+p \le 2,\ t \in (0,T], \tag{12}$$

We will also need some properties of potentials generated by the fundamental solution g. Consider the integrals

$$u_0(t,x) = \int_{\mathbb{R}^1} g(t,x,y)\varphi(y)dy, \tag{13}$$

$$u_1(t,x) = \int_0^t g(t-\tau,x,0)V(\tau)d\tau. \tag{14}$$

Here, $\varphi(x)$, $x \in \mathbb{R}$ and $V(t)$, $t > 0$, are the given functions. In the theory of parabolic equations, the function u_0 is called a Poisson potential, and the function u_1 is a simple layer potential ([6]).

From the properties of the fundamental solution g, we obtain that when the function $\varphi(x)$ in (13) is bounded and continuous on $\mathbb{R}$, then $u_0(t,x)$ satisfies Eq. (5) and the initial condition $u_0(0,x) = \varphi(x)$, $x \in \mathbb{R}$. In addition, in $\mathbb{R}_T^2$, the following inequality holds:

$$|D_t^r D_x^p u_0(t,x)| \le C\|\varphi\| t^{-\frac{2r+p}{2}}, \quad 2r+p \le 2. \tag{15}$$

Assume that the function $V(t)$ from (14) is bounded and continuous on $[0,\infty)$. Then the function $u_1(x)$ is continuous on $\overline{\mathbb{R}}_\infty^2$ and satisfies Eq. (5) in domains $(t,x) \in (0,\infty)\times D$ and $(t,x) \in (0,\infty) \times (\mathbb{R} \setminus \overline{D})$ and the initial condition $u(0,x) = 0$, $x \in \mathbb{R}$.

As for a behavior of the derivative of the simple layer potential with respect to x in the neighborhood of a point, ∂D, the so-called formula of potential jump holds (see [6, sec. IV, §15]. In our case, it has the form

$$\frac{\partial u_1(t,0\pm)}{\partial x} = \int_0^t \frac{\partial g_1(t-\tau,0,0)}{\partial x} V(\tau)d\tau \mp \frac{V(t)}{b(0)}, \quad t > 0. \tag{16}$$

Notice that the existence of the integral in (16) follows from the inequality

$$\left|\frac{\partial g_1(t-\tau,0,0)}{\partial x}\right| \le C(t-\tau)^{-1+\alpha/2}, \quad 0 < \tau < t \le T, \tag{17}$$

that is a corollary of estimation (12), where we put $r = 0$, $p = 1$, and $x = y = 0$, and t is changed into $t - \tau$.

2. Solving the initial-boundary problem for a parabolic equation with nonlocal boundary conditions

Consider the initial-boundary problem (5) – (7) under assumptions that the coefficients of the operator L from (1) and the parameter q and the measure μ from (2) satisfy conditions a) – c). As far as the initial function φ from (6) is concerned, we assume that it is defined on $\mathbb{R}$ and satisfies the condition

$$\varphi \in C(\mathbb{R}) \cup \mathcal{B}(\mathbb{R}). \tag{18}$$

We will bring problem (5)-(7) into the solution of some singular Volterra equation. For this, we will find a solution $u(t,x)$ of problem (5)-(7) in a view of a sum of two potentials, $u_0(t,x)$ from (13) and $u_1(t,x)$ from (14), with the unknown density $V(t)$:

$$u(t,x) = u_0(t,x) + u_1(t,x), \quad (t,x) \in (0,\infty) \times D. \tag{19}$$

Using the boundary condition (7) and relation (17), we obtain the integral equation for $V(t)$,

$$V(t) = \int_0^t K(t-\tau)V(\tau)d\tau + \Phi(t), \quad t>0, \tag{20}$$

where

$$K(t-\tau) = b(0)\left(\frac{\partial g_1(t-\tau,0,0)}{\partial x} + \int_D (g(t-\tau,y,0) - g(t-\tau,0,0))\mu_0(dy)\right),$$

$$\Phi(t) = b(0)\left(\frac{\partial u_0(t,0)}{\partial x} + \int_D (u_0(t,y) - u_0(t,0))\mu_0(dy)\right), \quad \mu_0(dy) = \frac{1}{q}\mu(dy).$$

Equation (20) is a Volterra integral equation of the second kind. Consider firstly its right part, i.e., the function $\Phi(t)$. For this, we use the relation

$$\begin{aligned}\Phi(t) &= b(0)\left(\frac{\partial u_0(t,0)}{\partial x} + \int_{D\setminus D_\delta} \frac{\partial u_0(t,x)}{\partial x}\Big|_{x=\Theta y} y\mu_0(dy) + \right.\\ &+ \left.\int_{D_\delta} (u_0(t,y) - u_0(t,0))\mu_0(dy)\right).\end{aligned}$$

Then relations (15), (3), and (8) yield

$$|\Phi(t)| \le C\|\varphi\|\left(t^{-\frac{1}{2}} + t^{-\frac{1}{2}}\frac{1}{q}\lambda(\delta) + \mu_0(D_\delta)\right), \quad t\in(0,T]. \tag{21}$$

Inequality (8) guarantees the existence of some positive number $\delta = \delta_0$ such that, for a constant $d_0 = \frac{1}{q}\lambda(\delta_0)$, the inequality $0 < C_0 < 1$ holds. Setting $\delta = \delta_0$ in (21), we obtain

$$|\Phi(t)| \le C_T\|\varphi\|t^{-\frac{1}{2}}, \quad t\in(0,T], \tag{22}$$

Now we estimate the singularity in Eq. (20). Thereto, we write $K(t-\tau)$ as

$$K(t-\tau) = K_1(t-\tau) + K_2(t-\tau), \tag{23}$$

where

$$\begin{aligned}K_1(t-\tau) &= b(0)\left(\frac{\partial g_1(t-\tau,0,0)}{\partial x} + \int_{D_{\delta_0}} (g(t-\tau,y,0) - g(t-\tau,0,0))\mu_0(y) + \right.\\ &+ \left.\int_{D\setminus D_{\delta_0}} (g_1(t-\tau,y,0) - g_1(t-\tau,0,0))\mu_0(dy)\right),\\ K_2(t-\tau) &= b(0)\int_{D\setminus D_{\delta_0}} (g_0(t-\tau,y,0) - g_0(t-\tau,0,0))\mu_0(dy).\end{aligned}$$

When estimating $K_1(t-\tau)$, we use inequalities (11), (12), and (3). In addition, we will use the finite variation formula for $g_1(t-\tau,y,0) - g_1(t-\tau,0,0)$ in the second integral. For $0\le\tau<t\le T$, we obtain

$$|K_1(t-\tau)| \le N_T(t-\tau)^{-1+\frac{\alpha}{2}}, \tag{24}$$

where N_T is some constant.

Similarly, we can estimate $K_2(t-\tau)$. As a corollary, we will obtain an estimation that implies that this function has a nonintegrable singularity. Despite this fact, we will prove that the method of successive approximations can be applied to Eq. (20).

So we will try to find a solution of the integral equation (20) as a series

$$V(t) = \sum_{k=0}^{\infty} V^{(k)}(t), \quad t>0, \tag{25}$$

where

$$V^{(0)}(t) = \Phi(t),$$
$$V^{(k)}(t) = \int_0^t K(t-\tau)V^{(k-1)}(\tau)d\tau, \quad k = 1, 2, \dots$$

We will prove convergence of series (25). To estimate $V^{(1)}(t)$, we will use the formula

$$V^{(1)}(t) = \int_0^t K(t-\tau)V^{(0)}(\tau)d\tau = \int_0^t K_1(t-\tau)\Phi(\tau)d\tau +$$
$$+ \int_0^t K_2(t-\tau)\Phi(\tau)d\tau = V^{(11)}(t) + V^{(12)}(t). \tag{26}$$

Using (24) and (22), we obtain

$$\left|V^{(11)}(t)\right| \leq C_T\|\varphi\|K_T\int_0^t (t-\tau)^{-1+\frac{\alpha}{2}}\tau^{-\frac{1}{2}}d\tau =$$
$$= C_T\|\varphi\|\frac{K_T\Gamma(\frac{\alpha}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{1+\alpha}{2})}t^{-\frac{1}{2}+\frac{\alpha}{2}}, \tag{27}$$

where $t \in (0, T)$, and $\Gamma(t)$ is the gamma function.

Before estimating $V^{(12)}(t)$, we will put it in the form

$$\begin{aligned}
V^{(12)} &= \frac{b(0)}{\sqrt{2\pi b(0)}}\int_{D\setminus D_{\delta_0}}\mu_0(dy)\int_0^t (t-\tau)^{-\frac{1}{2}}\left(e^{-\frac{y^2}{2b(0)(t-\tau)}} - 1\right)\Phi(\tau)d\tau = \\
&= \frac{b(0)}{\sqrt{2\pi b(0)}}\int_{D\setminus D_{\delta_0}}\mu_0(dy)\int_0^t (t-\tau)^{-\frac{1}{2}}\Phi(\tau)\left(\int_0^1 \frac{\partial}{\partial\Theta}e^{-\frac{y^2\Theta}{2b(0)(t-\tau)}}d\Theta\right)d\tau = \\
&= -\frac{1}{2\sqrt{2\pi b(0)}}\int_0^1 d\Theta\int_{D\setminus D_{\delta_0}} y^2\mu_0(dy)\int_0^t (t-\tau)^{-\frac{3}{2}}e^{-\frac{y^2\Theta}{2b(0)(t-\tau)}}\Phi(\tau)d\tau = \\
&= -\frac{1}{2\sqrt{2\pi b(0)}}\int_0^1 d\Theta\int_{D\setminus D_{\delta_0}} y^2 e^{-\frac{y^2\Theta}{2b(0)t}}\mu_0(dy)\int_0^t (t-\tau)^{-\frac{3}{2}}e^{-\frac{y^2\Theta}{2b(0)t}\frac{\tau}{t-\tau}}\Phi(\tau)d\tau.
\end{aligned}$$

Then (21) and (8) yield

$$\begin{aligned}
&\left|V^{(12)}\right| \\
&\leq C_T\|\varphi\|\frac{1}{2\sqrt{2\pi b(0)}}\int_0^1 d\Theta\int_{D\setminus D_{\delta_0}} y^2 e^{-\frac{y^2\Theta}{2b(0)t}}\mu_0(dy)\int_0^t (t-\tau)^{-\frac{3}{2}}\tau^{-\frac{1}{2}}e^{-\frac{y^2\Theta}{2b(0)t}\frac{\tau}{t-\tau}}d\tau = \\
&= C_T\|\varphi\|\frac{1}{2\sqrt{2\pi b(0)}}\int_0^1 d\Theta\int_{D\setminus D_{\delta_0}} y^2 e^{-\frac{y^2\Theta}{2b(0)t}}\frac{\sqrt{2\pi b(0)}}{\sqrt{ty^2\Theta}}\mu_0(dy) = \\
&= C_T\|\varphi\|t^{-\frac{1}{2}}\frac{1}{2}\int_0^1 \Theta^{-\frac{1}{2}}d\Theta\int_{D\setminus D_{\delta_0}} y e^{-\frac{y^2\Theta}{2b(0)t}}\mu_0(dy) \leq \\
&\leq C_T\|\varphi\|d_0 t^{-\frac{1}{2}}, \quad t \in (0, T],
\end{aligned} \tag{28}$$

where $0 < C_0 < 1$.

Gathering inequalities (27) and (28), we obtain

$$\left|V^{(1)}\right| \leq C_T\|\varphi\|t^{-\frac{1}{2}}\left(\frac{K_T\Gamma(\frac{\alpha}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{1+\alpha}{2})}t^{\frac{\alpha}{2}} + d_0\right), \quad t \in (0, T]. \tag{29}$$

Further, using the induction, we establish

$$\left|V^{(k)}\right| \leq C_T\|\varphi\|t^{-\frac{1}{2}}\sum_{m=0}^{k}\binom{m}{k}a_t^{(k-m)}(d_0)^m, \quad k = 0, 1, 2, \dots,$$

where

$$\binom{m}{k} = \frac{k!}{n!(k-n)!}, \quad a_t^{(m)} = \frac{\left(N_T\Gamma(\frac{\alpha}{2})\right)^m \Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+m\frac{\alpha}{2})} t^{m\frac{\alpha}{2}}, \quad m = 0,1,2,\dots$$

The last estimate yields

$$\begin{aligned}
\sum_{k=0}^{\infty} \left|V^{(k)}(t)\right| &\leq C_T\|\varphi\|t^{-\frac{1}{2}} \sum_{k=0}^{\infty}\sum_{m=0}^{k} \binom{m}{k} a_t^{(k-m)}(d_0)^m = \\
&= C_T\|\varphi\|t^{-\frac{1}{2}} \sum_{k=0}^{\infty} a_t^{(k)} \sum_{m=0}^{\infty} \binom{k+m}{m}(d_0)^m = \\
&= C_T\|\varphi\|t^{-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{a_t^{(k)}}{(1-d_0)^{k+1}} = \\
&= C_T\|\varphi\|t^{-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{\left(N_T\Gamma(\frac{\alpha}{2})\right)^k \Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+k\frac{\alpha}{2})(1-d_0)^{k+1}} t^{k\frac{\alpha}{2}} \qquad (30)
\end{aligned}$$

Inequality (30) guarantees the convergence of series (25) and gives the estimate for $V(t)$,

$$|V(t)| \leq C\|\varphi\|t^{-\frac{1}{2}}, \quad t \in (0,T], \qquad (31)$$

where C is some constant.

So we have constructed the solution of the integral equation (20) that is obviously continuous for all $t > 0$, and, in every domain $t \in (0,T]$, estimation (31) is allowed.

We will also note another property of solution (20) that we will use in Section 3. If the function $\varphi(x)$ satisfies condition (18), and $\varphi_n(x)$ is such a sequence of bounded continuous real-valued functions on $\mathbb{R}$ that $\varphi_n(x) \to \varphi(x)$ for every $x \in \mathbb{R}$ when $n \to \infty$ and $\sup_n \|\varphi_n\| < \infty$, then $\lim_{n\to\infty} V(t,\varphi_n) = V(t,\varphi)$, $t > 0$. Here, by $V(t,\varphi_n)$ and $V(t,\varphi)$, we denote a solution of Eq. (20) corresponding to the functions φ_n and φ, respectively. To prove this, it is enough to notice that we can pass to the termwise limit in series (25).

The obtained estimate (31) together with (11) for $r = p = 0$ guarantee the existence of integral (14) and the validity of the following inequality for the function $u_1(t,x)$:

$$|u_1(t,x)| \leq C\|\varphi\|, \quad (t,x) \in [0,T] \times \overline{D}. \qquad (32)$$

The additional analysis of series (25) and integral (14) implies that the function $u_1(t,x)$ is continuous at $t = 0$ and $u_1(0,x) = 0$, $x \in \overline{D}$. It is obvious that the same inequality as (32) also holds for the function $u_0(t,x)$ from (13) and $u(0,x) = \varphi(x)$, $x \in \overline{D}$. This means that the following theorem of existence for problem (5)-(7) was proved

Theorem 2.1. *Under assumptions (3), a) – c), and (18), there exists a classical solution of problem (5) – (7) that is continuous in the domain* $\overline{\Omega} = [0,\infty) \times \overline{D}$ *and satisfies the estimation*

$$|u(t,x)| \leq C\|\varphi\|, \quad (t,x) \in [0,T] \times \overline{D}. \qquad (33)$$

Now we will prove the uniqueness of the solution.

Theorem 2.2. *Under conditions of Theorem 2.1, there exists at most one solution of problem (5) – (7) in the class* $C(\overline{\Omega})$.

Proof. Assume that $u_1(t,x)$ and $u_2(t,x)$ are two different solutions of problem (5) – (7) which are continuous on $\overline{\Omega}$ from the class $C(\overline{\Omega})$, for which estimation (33) holds. Then the function $u(t,x) = u_1(t,x) - u_2(t,x)$ is a solution of the second parabolic boundary-value problem

$$\frac{\partial u(t,x)}{\partial t} - Lu(t,x) = 0, \quad (t,x) \in (0,\infty) \times D, \tag{34}$$

$$u(0,x) = 0, \quad x \in D, \tag{35}$$

$$\frac{\partial u(t,0)}{\partial x} = \Psi(t), \quad t > 0, \tag{36}$$

where

$$\Psi(t) = \int_D ((u_1(t,0) - u_2(t,0)) - (u_1(t,y) - u_2(t,y)))\,\mu(dy).$$

But, under our assumptions, the solution of the second boundary-value problem can be given as

$$u(t,x) = \int_0^t g(t-\tau, x, 0) V(\tau) d\tau, \tag{37}$$

where the density V is uniquely defined from the boundary condition (36). As $\Phi(x)$ is continuous for $t > 0$, and it satisfies inequality (31), the second boundary-value problem (34) – (36) has the unique solution continuous on $\overline{\Omega}$ (see, e.g., [9][sec. V]), so representation (37) is unique. Notice that, after the use of operations like those in the proof of Theorem 2.1, the function $V(t)$ is a solution of the homogeneous equation (20) with $\Phi(t) \equiv 0$. Solving this equation by the method of successive approximations, we obtain $V \equiv 0$. Hence, $u \equiv 0$ and $u_1 \equiv u_2$. □

3. Construction of the process desired

It follows from Theorem 2.1 that, using a solution of problem (5) – (7,) we can define a family of linear operators $(T_t)_{t \geq 0}$ that act in the space $\mathcal{B}(\mathbb{R}) \cap C(\mathbb{R})$. For $t > 0$, $x \in \overline{D}$, and $\varphi \in \mathcal{B}(\mathbb{R})$, we set

$$T_t\varphi(x) = \int_{\mathbb{R}} g(t,x,y)\varphi(y)dy + \int_0^t g(t-\tau, x, o) V(\tau, \phi) d\tau, \tag{38}$$

where $V(t,\phi) \equiv V(t)$ is a solution of the Volterra second-type integral equation (20) determined by formula (25). In addition, for $T_t\varphi(x)$ in a domain $(t,x) \in [0,T] \times \overline{D}$, the following estimate holds:

$$|T_t\varphi(x)| \leq C\|\varphi\|.$$

The existence of the integral representation (38) allows us to quite easily validate the following conditions:

(1) if $\varphi_n \in \mathcal{B}(\mathbb{R}) \cap C(\mathbb{R})$, $n = 1, 2, \ldots$, $\sup_n \|\varphi_n\| < \infty$ and, for all $x \in \mathbb{R}$, $\lim_{n\to\infty} \varphi_n(x) = \varphi(x)$, then, for all $t \geq 0$, $x \in \overline{D}$, the relation $\lim_{n\to\infty} \varphi_n(x) = T_t\varphi(x)$ holds;
(2) for all $t_1 \geq 0$, $t_2 \geq 0$, the relation $T_{t_1+t_2} = T_{t_1}T_{t_2}$ holds;
(3) $T_t\varphi(x) \geq 0$ for all $t \geq 0$, $x \in \overline{D}$, whenever the function $\varphi \in \mathcal{B}(\mathbb{R}) \cap C(\mathbb{R})$ is such that $\varphi(x) \geq 0$;
(4) $\|T_t\| \leq 1$ for all $t \geq 0$.

We will mention only the main points in the proofs of these properties. In particular, property 1) is a corollary from properties of the solution of Eq. (20) and the Lebesgue theorem on the passage to a limit under the integral sign. We recall that property 2) is called a semigroup property of the operators T_t. It can be implied from Theorem 2.2. Property 3) can be verified similarly to [2, p. 82], by using the maximum principle for parabolic equations. Finally, property 4) means that, for every $t \geq 0$, the operator T_t

is a contractive operator. To prove property 4), it is enough to notice with regard for property 3) that $T_t\varphi_0(x) \equiv 1$ for all $t \geq 0$, $x \in \overline{D}$, when $\varphi_0(x) \equiv 1$.

As a corollary from properties 1) – 4), we have that the constructed operator semigroup T_t, $t \geq 0$ defines some homogeneous Feller process on $\overline{D}$. We denote its transition function by $P(t, x, dy)$, i.e.,

$$T_t\varphi(x) = \int_{\overline{D}} P(t, x, dy)\varphi(y).$$

So we have proved the following statement:

Theorem 3.1. *For coefficients of the operator L from (1) and for the measure $\mu(\cdot)$ from (2), let assumptions (3), a) – c), hold. Then the solution of the parabolic boundary-value problem (5) – (7) uniquely defines the operator family T_t, $t \geq 0$, that describes a homogeneous Feller process on $\overline{D}$ such that it coincides in D with the diffusion process controlled by the operator L, and its behavior on the domain boundary is determined by the boundary condition (4).*

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