Theory of Stochastic Processes
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DYNAMICS OF RANDOM CHAINS OF FINITE SIZE WITH AN INFINITE NUMBER OF ELEMENTS IN $\mathbb{R}^2$

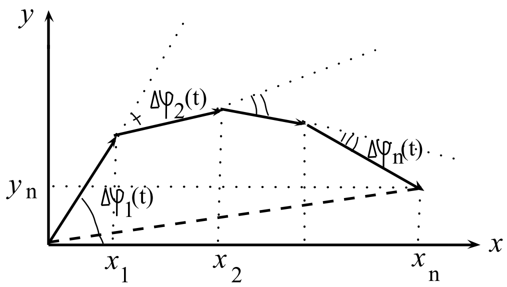
A finite chain with infinitely many units within the stochastic dynamical model in $\mathbb{R}^2$ is considered. The equation for the probability distribution density of chain lengths is constructed. This equation is a function of the parameter t which stands for the time. This research is a sequel to work [1].

1. Formulation of the problem

In Feller's book [2], the problem of the length of a random chain is considered. This chain is described in the following way: the number of elements is equal to n, the length of all its elements is equal to one, the angle of one component with respect to the previous is always the same up to a sign (the probability of each angle is equal to 1/2), the distance between the end points of the chain (length of the chain) is defined by means of the average square length

$$\mathbf{M}[L_n^2] = n\,\frac{1+\cos\alpha}{1-\cos\alpha} - 2\cos\alpha\,\frac{1-\cos^n\alpha}{(1-\cos\alpha)^2}.$$

We will consider the following chain: the length of the chain is finite, the number of components is infinite, the length of each component is a random variable, the angle of each component with respect to the previous one is also random.



The physical model can be a rope in a medium of Brownian particles. We can understand the modulus of the vector joining the starting point and the end point of the chain as the chain length.

Let $l \in [0, L]$ be a parameter, L a constant, $l_1, l_2, \dots$ the values of the parameter, $l_1 < l_2 < \dots \le L$, $\Delta = L/N$, $l_j = j \cdot \Delta$. We will consider the model of the chain described by the following system of equations:

(1) $$x_N(t) = \textstyle\sum_{s=1}^N a(l_s)\Delta \cdot \cos\varphi_s(t), \qquad y_N(t) = \sum_{s=1}^N a(l_s)\Delta \cdot \sin\varphi_s(t),$$

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where $a(l_s), \varphi_s(t)$ are, in general, random processes, $a(l) > 0$, $\varphi_s(t)$ is the angle between the s-th component with respect to the previous one, $\varphi_1(t)$ is the angle of the first component of the chain with respect to the positive direction of the x-axis.

If we denote the length of the chain consisting of n elements by $|x_n(t)|^2 + |y_n(t)|^2$, then the length $\triangle(l)$ of the real component of the chain is expressed by the variable

$$\triangle(l) = a(l)\Delta, \quad a(l) > 0, \quad \int_o^L a(l)dl = \mathcal{L}.$$

Models of type (1) describe the distribution of the length $\mathcal{L}(t)$ of the chain in the case where the following inequality is satisfied:

$$\mathcal{L}^2(t) = |x_N(t)|^2 + |y_N(t)|^2 \leq const.$$

From the point of view of the representation of the phenomenon of turbulent diffusion, model (1) can be useful for some generalizations of the passive displacement under the action of vortices of different sizes [3].

Let $n < N$ (i.e., we consider not the whole chain but its part), $N \to \infty$. Since the coordinates of the initial point and the end point of each component depend on time t and on the parameter l, we introduce some changes in model (1):

$$x_n(l;t) = \textstyle\sum_{s=1}^n a(l_s)\Delta \cdot \cos\varphi_s(t), \qquad y_n(l;t) = \sum_{s=1}^n a(l_s)\Delta \cdot \sin\varphi_s(t). \tag{2}$$

In this way, the random field $\{x_n(l;t); y_n(l;t)\}$ is a dynamical stochastic process. We will study its limit behavior as $n \to \infty$.

2. Assumptions on the model

In order to obtain the coefficient of the limit equation in analytical form, we restrict ourselves to the model satisfying the following assumptions:

$$\begin{gathered} a(l) > 0, \quad l \in [0, L], \\ \varphi_s(t) = \sum_{k=1}^{s} \eta(l_k;t)\Delta(w(l_k)), \quad t \in [0,T], \\ \eta(l_k;t) = \textstyle\int_0^t \sigma(l_k;\tau)\, dw_k(\tau), \end{gathered} \tag{3}$$

where $\triangle(w(l_k))$, $\triangle(w_k(\tau))$ are independent among themselves and for different s, and τ are anticipating increments of the corresponding Wiener processes defined on the product of independent probability spaces

$$\{\Omega_1,\, \Im_l,\, P_1\} \,\times\, \{\Omega_2,\, \Im_t(n),\, P_2\}.$$

Here, $\Im_l$, and $\Im_t(n)$ are the corresponding flows of sigma algebras generated by the processes $w(l)$ and $w(t) \in \mathbb{R}^n$; the functions $a(l) \in \mathbb{C}^1_{[0,L]}$ and $\sigma(l;t) \in \mathbb{C}^2_{[0,L]\times[0,T]}$ are deterministic functions depending on l and t, and $\eta(l_s;t)$ is the intensity of the angle.

Therefore, we have

$$\begin{aligned} x_n(l;t) &= \textstyle\sum_{s=1}^n a(l_s)\Delta \cdot \cos\left[\sum_{k=1}^{s} \left(\int_0^t \sigma(l_k;\tau)\, dw_k(\tau)\right) \triangle(w(l_k))\right] \\ y_n(l;t) &= \textstyle\sum_{s=1}^n a(l_s)\Delta \cdot \sin\left[\sum_{k=1}^{s} \left(\int_0^t \sigma(l_k;\tau)\, dw_k(\tau)\right) \triangle(w(l_k))\right]. \end{aligned} \tag{4}$$

Under the condition of a bounded length of the chain, we can define the limit for the random function $\varphi_s(t)$ as $n \to \infty$. In this context, the variable l appears as a parameter.

3. Transition to auxiliary processes

Let us transform (2) by means of the Euler representation:

$$\begin{aligned} x_n(l;t) &= \sum_{s=1}^n a(l_s)\Delta\cdot\cos\varphi_s(t) = \sum_{s=1}^n a(l_s)\Delta\cdot\tfrac{\exp\{i\varphi_s(t)\}+\exp\{-i\varphi_s(t)\}}{2} = \\ &= \tfrac12\sum_{s=1}^n a(l_s)\Delta\cdot\exp\{i\varphi_s(t)\} + \tfrac12\sum_{s=1}^n a(l_s)\Delta\cdot\exp\{-i\varphi_s(t)\}, \\ y_n(l;t) &= \sum_{s=1}^n a(l_s)\Delta\cdot\sin\varphi_s(t) = \sum_{s=1}^n a(l_s)\Delta\cdot\tfrac{\exp\{i\varphi_s(t)\}-\exp\{-i\varphi_s(t)\}}{2i} = \\ &= \tfrac{1}{2i}\sum_{s=1}^n a(l_s)\Delta\cdot\exp\{i\varphi_s(t)\} - \tfrac{1}{2i}\sum_{s=1}^n a(l_s)\Delta\cdot\exp\{-i\varphi_s(t)\}. \end{aligned}$$

By introducing the auxiliary process

$$z_1(s;t) = \exp\{-i\sum_{j=1}^s \triangle w(l_j)\int_0^t \sigma(l_j;\tau)\,dw_j(\tau)\},$$
$$z_{n,1}(l,t) = \sum_{s=1}^n a(l_s)\Delta\cdot\exp\{-i\varphi_s(t)\} = \sum_{s=1}^n a(l_s)\Delta\cdot z_1(s;t), \quad \Delta = O(n^{-1})$$

and using the Euler representation, we rewrite the process $\{x_n(l;t);\, y_n(l;t)\}$ in the form

$$x_n(l;t) = \frac12(z_{n,1}(l,t) + z^*_{n,1}(l,t)), \quad y_n(l;t) = \frac{i}{2}(z_{n,1}(l,t) - z^*_{n,1}(l,t)).$$

For the construction of the characteristic function of the random field $\{x_n(l;t);\, y_n(l;t)\}$, we define the form of the function $\exp\{i(\alpha x_n(l;t) + \beta y_n(l;t))\}$:

$$\begin{aligned} &\exp\{i(\alpha x_n(l;t) + \beta y_n(l;t))\} = \\ &= \exp\left\{i\alpha\tfrac{z_{n,1}(l;t)+z^*_{n,1}(l;t)}{2} - \beta\tfrac{z_{n,1}(l;t)-z^*_{n,1}(l;t)}{2}\right\} = \\ &= \sum_{m,r=1}^{\infty}\tfrac{(i\alpha-\beta)^m(i\alpha+\beta)^r}{2^{m+r}m!r!} z^m_{n,1}(l;t) z^{*^r}_{n,1}(l;t). \end{aligned}$$

Moreover, the analysis of the process $\{x_n(l;t); y_n(l;t)\}$ requires to study the process $z^m_{n,1}(l;t)z^{*^r}_{n,1}(l;t)$.

Since the summation and the integration operations have similar properties, we replace (in a symbolic form for $\Delta \to 0$, which corresponds to $n \to \infty$) the process

$$z_{n,1}(l,t) = \sum_{s=1}^n a(l_s)\Delta\cdot\exp\left\{-i\sum_{j=1}^s\left(\int_0^t \sigma(l_j;\tau)\,dw_j(\tau)\right)\triangle w(l_j)\right\}$$

by the process

$$z_{,1} = \sum_{s=1}^n a(l_s)\Delta\cdot\exp\left\{-\sum_{j=1}^s \eta(l_s,t)\triangle w(l_s)\right\} = ⨍_0^l a(u)\exp\left\{-i⨍_0^u \eta(\theta,t)dw(\theta)\right\}du,$$

where $\eta(u,t) = \int_0^t \sigma(u,\tau)d\tau$. We do not lose any generality in the analysis with this assumption, and, in what follows, we use the symbol $⨍$ instead of $\sum$.[1]

4. Degree transformation

By considering the continuity of the process $z_{n,1}(l;t)$ and, consequently, of the process $z_{,1}(l;t)$ with respect to both variables l and t, we define the degree transformation

$$\begin{aligned} z^m_{,1}(l;t) &= \left[⨍_0^l a(u)\exp\left\{-i⨍_0^u \eta(\theta;t)\,dw(\theta)\right\}\,du\right]^m = \\ &= m!\,⨍_0^l a(u_1)\,du_1\,\exp\left\{-i\,m⨍_0^{u_1}\eta(\theta_1;t)\,dw(\theta_1)\right\}\times \\ &\times ⨍_{u_1}^l a(u_2)\,du_1\,\exp\left\{-i(m-1)⨍_{u_1}^{u_2}\eta(\theta_2;t)\,dw(\theta_2)\right\}\times\ldots\times \\ &\times ⨍_{u_{m-1}}^l a(u_m)\,du_m\,\exp\left\{-i⨍_{u_{m-1}}^{u_m}\eta(\theta_m;t)\,dw(\theta_m)\right\}, \end{aligned} \tag{5}$$

where $0 < u_1 < \ldots < u_m < l$. In different intervals, we have different $dw(\theta)$ for every time instant t.

[1]This symbol does not concern known designations. It is a label only.

5. Determination of moments

Since the process $z^m_{,1}(l;t)z^{*r}_{,1}(l;t)$ depends on two variables, for the calculation of the mean $\mathbf{M}\,[z^m_{,1}(l;t)z^{*r}_{,1}(l;t)]$, it is necessary to carry out the averaging for t constant (on the space Ω_1) and then Ω_l.

5.1. **Averaging with respect to** l**.** On the disjoint intervals $[u_i, u_{i+1})$ $\ forall\ \ i$, for all i, the processes

$$f(u_i, u_{i+1}) = \int_{u_i}^{u_{i+1}} \eta(\theta_{i+1};t)\,dw(\theta_{i+1})$$

are independent by construction (since $dw(\theta)$ are Wiener processes, $\eta(\theta;t)$ for fixed t is a non-random function depending on θ). Because of this, since $a(l)$ is also a non-random function, the mathematical mean of each factor is defined in the following way:

$$\begin{aligned}\mathbf{M}_t\,[z^m_{,1}(l;t)] = m! \int_0^l a(u_1)\,du_1\,\mathbf{M}_t\left[\exp\left\{-i\,m \int_0^{u_1}\eta(\theta_1;t)\,dw(\theta_1)\right\}\right] \times \\ \times \int_{u_1}^l a(u_2)\,du_2\,\mathbf{M}_t\left[\exp\left\{-i(m-1)\int_{u_1}^{u_2}\eta(\theta_2;t)\,dw(\theta_2)\right\}\right] \times \ \ldots \times \\ \times \int_{u_{m-1}}^l a(u_m)\,du_m\,\mathbf{M}_t\left[\exp\left\{-i\int_{u_{m-1}}^{u_m}\eta(\theta_m;t)\,dw(\theta_m)\right\}\right].\end{aligned} \tag{6}$$

In this way, it is necessary to find the mean of an expression of the following type:

$$\exp\left\{-i(m-j)\int_{u_j}^{u_{j+1}} \eta(\theta;t)\,dw(\theta)\right\}.$$

Lemma 5.1. *The following equality holds:*

$$\mathbf{M}_t\,\left[\exp\left\{\alpha\int_a^b \eta(u;t)\,dw(u)\right\}\right] = \exp\left\{\frac{1}{2}\alpha^2\int_a^b \eta^2(u;t)\,du\right\}. \tag{7}$$

Proof. We denote

$$q(a,b;t) = \int_a^b \eta(u;t)\,dw(u) \tag{8}$$

and differentiate $q(a,b;t)$ with respect to the upper limit b. As a result, we obtain

$$d_b\,q(a,b;t) = \eta(b;t)\,dw(b).$$

Therefore, by the Itô formula, the stochastic differential of the expression

$$\exp\left\{\alpha\int_a^b \eta(u;t)\,dw(u)\right\} = \exp\left\{\alpha q(a,b;t)\right\}$$

with respect to b is equal to

$$d_b\,\exp\left\{\alpha q(a,b;t)\right\} = \exp\left\{\alpha q(b;t)\right\}\alpha\eta(b;t)\,dw(b) + \frac{1}{2}\alpha^2\eta^2(b;t)\exp\left\{\alpha q(a,b;t)\right\}db.$$

We compute the average with respect to l of the obtained expression:

$$d_b\,\mathbf{M}_t\,\left[\exp\left\{\alpha q(a,b;t)\right\}\right] = \frac{1}{2}\alpha^2\eta^2(b;t)\mathbf{M}_t\,\left[\exp\left\{\alpha q(a,b;t)\right\}\right]db. \tag{9}$$

We denote

$$I_1(a,b;t) = \mathbf{M}_t\,\left[\exp\left\{\alpha q(a,b;t)\right\}.\right] \tag{10}$$

Let $\eta(b;t)$ be independent of the stochastic process $w(u)$. In view of (9), we obtain the differential equation

$$\frac{dI_1(a,b;t)}{db} = \frac{1}{2}\alpha^2\eta^2(b;t)I_1(a,b;t).$$

Its solution

$$I_1(a,b;t) = \exp\left\{\frac{1}{2}\alpha^2 \fint_a^b \eta^2(u;t)\,du\right\}$$

satisfies the initial condition $I_1(a,a;t) = 1$. In view of (8), (10), the statement of the lemma is proved. □

As a consequence of Lemma 5.1, the mathematical mean (6) takes the form (for constant t)

$$\begin{aligned}\mathbf{M}_t\left[z^m_{,1}(l;t)\right] = m! \fint_0^l a(u_1)\,du_1 \exp\left\{-\tfrac{m^2}{2}\fint_0^{u_1}\eta^2(\theta;t)\,d\theta\right\}\times\\ \times \fint_{u_1}^l a(u_2)\,du_2 \exp\left\{-\tfrac{(m-1)^2}{2}\fint_{u_1}^{u_2}\eta^2(\theta;t)\,d\theta\right\}\times\ldots\times\\ \times\fint_{u_{m-1}}^l a(u_m)\,du_m \exp\left\{-\tfrac{1}{2}\fint_{u_{m-1}}^{u_m}\eta^2(\theta;t)\,d\theta\right\}.\end{aligned}$$

5.2. **Averaging with respect to t.** Now we average the process $z^m_{,1}(l;t)$ on the space Ω_2. Since $w_s(t)$ for all s are independent Wiener processes, $\eta^2(l_s;t)$ are independent, and the average of the product is therefore equal to the product of the means. As a result, we obtain

$$\begin{aligned}\mathbf{M}\left[\mathbf{M}_t\left[z^m_{,1}(l;t)\right]\right] = m! \fint_0^l a(u_1)\,du_1\, \mathbf{M}\left[\exp\left\{-\tfrac{m^2}{2}\fint_0^{u_1}\eta^2(\theta;t)\,d\theta\right\}\right]\times\\ \times\ldots\fint_{u_{m-1}}^l a(u_m)\,du_m\,\mathbf{M}\left[\exp\left\{-\tfrac{1}{2}\fint_{u_{m-1}}^{u_m}\eta^2(\theta;t)\,d\theta\right\}\right].\end{aligned}$$

Lemma 5.2. *The following relation holds:*

$$\mathbf{M}\left[\exp\left\{-\frac{\alpha^2}{2}\fint_a^b \eta^2(u;t)\,du\right\}\right] = \exp\left\{-\frac{\alpha^2}{4}\fint_a^b\left(\int_0^t \sigma^2(u;\tau)\,d\tau\right)du\right\}.$$

Here, $\eta(u;t) = \int_0^t \sigma(u;\tau)\,dw(\tau)$, *and* $\sigma(u;t)$ *is a non-random function.*

Proof. We use the representation

$$\fint_a^b \eta^2(u;t)\,du = \frac{b-a}{N}\sum_{k=1}^N \eta^2(u_k;t)$$

which is valid due to the model assumptions. By definition, the processes $\eta^2(u_k;t)$ ($k=\overline{1,N}$) are independent for different values of k. We introduce the notation

$$P_k(t) = \eta(u_k;t) = \int_0^t \sigma(u_k;\tau)\,dw_k(\tau),$$

where $\sigma(u_k;t)$ is a non-random function depending on u_k and t. We consider now two cases.

A. Let $\sigma(u_k;t)$ be constant. For the sake of simplicity, we assume that $\sigma(u_k;t) = 1$ and study the problem for the processes

$$\eta(u_k;t) = \int_0^t dw_k(\tau) = \tilde{P}_k(t). \tag{11}$$

By considering the representation of the integral in the form of sums, we carry out the transformation

$$\exp\left\{-\frac{\alpha^2(b-a)}{2N}\eta^2(u_k,t)\right\} = \exp\left\{-\frac{\alpha^2(b-a)}{2N}\left[\int_0^t dw_k(\tau)\right]^2\right\} = \\ = \exp\left\{-\frac{\alpha^2(b-a)}{2N}\tilde{P}_k^2(t)\right\}.$$

Therefore,

$$\exp\left\{-\frac{\alpha^2(b-a)}{2N}\sum_{k=1}^{N}\tilde{P}_k^2(t)\right\} = \prod_{k=1}^{N}\exp\left\{-\frac{\alpha^2(b-a)}{2N}\tilde{P}_k^2(t).\right\}$$

We denote

$$\exp\left\{-\frac{\alpha^2(b-a)}{2N}\sum_{k=1}^{N}\tilde{P}_k^2(t)\right\} = I_N(k,\alpha^2).$$

Since

$$\mathbf{M}\left[\exp\left\{\alpha q(a,b;t)\right\}\right] = \mathbf{M}\left[\exp\left\{-\alpha q(a,b;t)\right\}\right],$$

the following relations hold:

$$\mathbf{M}\left[\prod_{k=1}^{N}\exp\left\{-\tfrac{\alpha^2(b-a)}{2N}\tilde{P}_k^2(t)\right\}\right] =$$
$$= \prod_{k=1}^{N}\mathbf{M}\left[\exp\left\{-\tfrac{\alpha^2(b-a)}{2N}\tilde{P}_k^2(t)\right\}\right] \quad \to^{qm} \quad \mathbf{M}\left[\exp\left\{\alpha q(a,b;t)\right\}\right].$$

Then we carry out the Itô differentiation:

$$d\tilde{P}_k(t) = dw_k(t),$$

$$d\tilde{P}_k^2(t) = dt + 2w_k(t)\,dw_k(t) \tag{12}$$

and in view of (11) we have that:

$$d_t\exp\left\{-\tfrac{\alpha^2(b-a)}{2N}\,\tilde{P}_k^2(t)\right\} = -\exp\left\{-\tfrac{\alpha^2(b-a)}{2N}\,\tilde{P}_k^2(t)\right\}\tfrac{\alpha^2(b-a)}{2N}\,d\tilde{P}_k^2(t) +$$
$$+\tfrac{\alpha^4(b-a)^2}{2N^2}\,\tilde{P}_k^2(t)\exp\left\{-\tfrac{\alpha^2(b-a)}{2N}\,\tilde{P}_k^2(t)\right\}dt.$$

In the last differential, we introduce expression (12):

$$d_t\exp\left\{-\tfrac{\alpha^2(b-a)}{2N}\,\tilde{P}_k^2(t)\right\} = -\exp\left\{-\tfrac{\alpha^2(b-a)}{2N}\,\tilde{P}_k^2(t)\right\}\times$$
$$\times\left[\tfrac{\alpha^2(b-a)}{2N}\,2\tilde{P}_k(t)\,dw_k(t) - \tfrac{\alpha(b-a)}{2N}\,\left(-1+\tfrac{\alpha^2(b-a)}{N}\tilde{P}_k^2(t)\right)\,dt\right].$$

We calculate the mean for the last expression by denoting

$$I_2(t;\alpha^2) = \mathbf{M}\left[\exp\left\{-\frac{\alpha^2(b-a)}{2N}\,\tilde{P}_k^2(t)\right\}\right].$$

We obtain the equation

$$dI_2(t;\alpha^2) = I_2(t;\alpha^2)\,\frac{\alpha^2(b-a)}{2N}\,dt + \frac{\alpha^4(b-a)^2}{2N^2}\,\mathbf{M}\left[\tilde{P}_k^2(t)\exp\left\{-\frac{\alpha^2(b-a)}{2N}\,\tilde{P}_k^2(t)\right\}\right]\,dt.$$

By considering the differentiation of the expression

$$\exp\left\{-\frac{\alpha^2(b-a)}{2N}\,\tilde{P}_k^2(t)\right\}$$

with respect to α^2, the last equation can be represented as a partial differential equation with constant coefficients:

$$\frac{dI_2(t;\alpha^2)}{dt} = -\frac{\alpha^2(b-a)}{2N}\,I_2(t;\alpha^2) - \frac{\alpha^4(b-a)}{N}\,\frac{\partial}{\partial\alpha^2}\,I_2(t;\alpha^2). \tag{13}$$

The solution of this equation will be obtained by exploiting the properties of the stochastic processes. With this purpose, we evaluate the mean of the function

$$\exp\left\{-\frac{\alpha^2(b-a)}{2N}\,\tilde{P}_k^2(t)\right\}.$$

By considering that the process is a Wiener process (that is, a Gaussian process) we have

$$\mathbf{M}\left[\exp\left\{-\frac{\alpha^2(b-a)}{2N}\,\tilde{P}_k^2(t)\right\}\right] = \int_{-\infty}^{\infty}\exp\left\{-\frac{\alpha^2(b-a)}{2N}\,x^2\right\}\times\frac{1}{\sqrt{2\pi\,t}}\exp\left\{-\frac{x^2}{2t}\right\}dx =$$

$$= \frac{1}{\sqrt{2\pi\,t}}\int_{-\infty}^{\infty}\exp\left\{-\left(\frac{\alpha^2(b-a)}{2N}\,x^2+\frac{x^2}{2t}\right)\right\}dx.$$

Furthermore,

$$\mathbf{M}\left[\exp\left\{-\frac{\alpha^2(b-a)}{2N}\,\tilde{P}_k^2(t)\right\}\right] = \frac{1}{\sqrt{2\pi\,t}}\cdot\sqrt{2\pi}\cdot\sqrt{t\Big/\left(\frac{\alpha^2(b-a)}{2N}t+1\right)} =$$

$$= \left(\frac{\alpha^2(b-a)}{2N}t+1\right)^{-1/2}.$$

In this way, the obtained expression $I_2(t;\alpha^2) = \left(\frac{\alpha^2(b-a)}{2N}t+1\right)^{-1/2}$ is a solution of the differential equation (13). In addition, in view of Lemma 5.1, we have

$$\mathbf{M}_t\ [\exp\{-\alpha q(a,b;t)\}] = \exp\left\{-\frac{\alpha^2}{2}\int_a^b\eta^2(u,t)\right\} = \exp\left\{-\frac{\alpha^2(b-a)}{2N}\sum_{k=1}^{N}\eta^2(u_k,t)\right\}$$

$$= \prod_{k=1}^{N}\exp\left\{-\frac{\alpha^2(b-a)}{2N}\eta^2(u_k,t)\right\}.$$

Therefore,

$$\mathbf{M}\left[\mathbf{M}_t\ [\exp\{-\alpha q(a,b;t)\}]\right] \quad \to^{qm} \quad \prod_{k=1}^{N}\mathbf{M}\left[\exp\left\{-\tfrac{\alpha^2(b-a)}{2N}\tilde{P}_k^2(t)\right\}\right] =$$

$$= \prod_{k=1}^{N}\left(\tfrac{\alpha^2(b-a)}{2N}\,t+1\right)^{-1/2} = \left(\tfrac{\alpha^2(b-a)}{2N}\,t+1\right)^{-N/2}.$$

Passing to the limit, we obtain the complete averaging with respect to both components:

$$\mathbf{M}\ [I_1(a,b;t)] = \lim_{N\to\infty}\left(\frac{\alpha^2(b-a)}{2N}\,t+1\right)^{-N/2} = \exp\left\{-\frac{\alpha^2(b-a)t}{4}\right\}.$$

B. We now consider the case $\sigma(u_k;t)\neq 1$. For $P_k(t)$, we obtain the expression

$$d_t\,P_k(t) = \sigma(u_k;t)\,dw_k(t).$$

In this way, we have

$$d_tP_k^2(t) = \frac{\sigma^2(u_k;t)}{2}2dt + 2P_k(t)\,\sigma(u_k;t)\,dw_k(t)$$

and

$$d_t\exp\left\{-\tfrac{\alpha^2(b-a)}{2N}\,P_k^2(t)\right\} = -\exp\left\{-\tfrac{\alpha^2(b-a)}{2N}\,P_k^2(t)\right\}\times$$
$$\times\tfrac{\alpha^2(b-a)}{2N}\,\left[\sigma^2(u_k;t)\,dt + 2P_k(t)\,\sigma(u_k;t)\,dw_k(t)\right] +$$
$$+\tfrac{\alpha^4(b-a)^2}{2N^2}\,P_k^2(t)\,\sigma^2(u_k;t)\cdot\exp\left\{-\tfrac{\alpha^2(b-a)}{2N}\,P_k^2(t)\right\}dt.$$

Therefore,

$$d_tI_t(k;\alpha^2) = \partial_t\,\mathbf{M}\left[\exp\left\{-\tfrac{\alpha^2(b-a)}{2N}\,P_k^2(t)\right\}\right] =$$
$$= -\mathbf{M}\left[\exp\left\{-\tfrac{\alpha^2(b-a)}{2N}\,P_k^2(t)\right\}\tfrac{\alpha^2(b-a)}{2N}\,\sigma^2(u_k;t)\right]dt.$$

By exploiting the possibility of the differentiation with respect to the parameter α^2, we arrive at the equation

$$\text{(14)}\qquad \frac{\partial_tI_t(k;\alpha^2)}{\partial t} = -\frac{\alpha^2(b-a)}{2N}\,\sigma^2(u_k;t)\,I_t(k;\alpha^2) - \frac{\alpha^4(b-a)}{N}\,\sigma^2(u_k;t)\,\frac{\partial I_t(k;\alpha^2)}{\partial\alpha^2}.$$

We divide both terms by $\sigma^2(u_k;t)$ and denote

$$\theta(t) = \int_0^t \sigma^2(u_k;\tau)\,d\tau.$$

So, we pass to the auxiliary equation

$$\frac{\partial I_\theta(k;\alpha^2)}{\partial\theta} = -\frac{\alpha^2(b-a)}{2N}\,I_\theta(k;\alpha^2) - \frac{\alpha^4(b-a)}{N}\,\frac{\partial I_\theta(k;\alpha^2)}{\partial\alpha^2}. \tag{15}$$

Equation (15) is a differential equation with constant coefficients which is stochastically equivalent to Eq. (13) . Therefore, its solution has the form

$$I_\theta(k;\alpha^2) = \left(\frac{\alpha^2(b-a)}{2N}\,\theta + 1\right)^{-1/2}.$$

Hence, the solution of Eq. (14) reads

$$I_t(k;\alpha^2) = \left(\frac{\alpha^2(b-a)}{2N}\int_0^t \sigma^2(u_k;\tau)\,d\tau + 1\right)^{-1/2}.$$

As a consequence, we have

$$\mathbf{M}\left[I_N(k;\alpha^2)\right] = \prod_{k=1}^N I_t(k;\alpha^2) = \prod_{k=1}^N \left(\frac{\alpha^2(b-a)}{2N}\int_0^t \sigma^2(u_k;\tau)\,d\tau + 1\right)^{-1/2}. \tag{16}$$

In order to evaluate $I_1(a,b;t)$, we take the logarithm of (16):

$$\ln\mathbf{M}\left[I_N(k;\alpha^2)\right] = \ln\prod_{k=1}^N \left(\tfrac{\alpha^2(b-a)}{2N}\int_0^t \sigma^2(u_k;\tau)\,d\tau + 1\right)^{-1/2} =$$
$$= -\tfrac{1}{2}\sum_{k=1}^N \ln\left(\tfrac{\alpha^2(b-a)}{2N}\int_0^t \sigma^2(u_k;\tau)\,d\tau + 1\right).$$

By using the expansion of $\ln(x+1)$ in a series, we obtain

$$\ln\mathbf{M}\left[\tilde{I}_1(b;t)\right] = -\tfrac{1}{2}\sum_{k=1}^N \left[\tfrac{\alpha^2(b-a)}{2N}\int_0^t \sigma^2(u_k;\tau)\,d\tau - \right.$$
$$\left. -\tfrac{1}{2}\tfrac{\alpha^4(b-a)^2}{4N^2}\left(\int_0^t \sigma^2(u_k;\tau)\,d\tau\right)^2 + O(N^{-3})\right].$$

We calculate the limit as $N\to\infty$:

$$\lim_{N\to\infty}\ln\mathbf{M}\left[I_N(k,\alpha^2)\right] = -\tfrac{1}{2}\lim_{N\to\infty}\sum_{k=1}^N \left[\tfrac{\alpha^2(b-a)}{2N}\int_0^t \sigma^2(u_k;\tau)\,d\tau - \right.$$
$$\left. -\tfrac{1}{2}\cdot\tfrac{\alpha^4(b-a)^2}{4N^2}\left(\int_0^t \sigma^2(u_k;\tau)\,d\tau\right)^2 + O(N^{-3})\right] =$$
$$= -\tfrac{1}{2}\lim_{N\to\infty}\sum_{k=1}^N \tfrac{\alpha^2(b-a)}{2N}\int_0^t \sigma^2(u_k;\tau)\,d\tau = -\tfrac{\alpha^2}{4}\int_a^b\left(\int_0^t \sigma^2(u;\tau)\,d\tau\right)du.$$

By using the limit and by passing to the antilogarithm, we prove that Lemma 5.2 holds, and

$$\mathbf{M}\left[\exp\left\{-\frac{\alpha^2}{2}\int_a^b \eta^2(u;t)\,du\right\}\right] = \exp\left\{-\frac{\alpha^2}{4}\int_a^b\left(\int_0^t \sigma^2(u;\tau)\,d\tau\right)du\right\}. \quad \square$$

5.3. **Passage to the limiting process.** The field $\{x_n(l;t);\ y_n(l;t)\}$ is defined by the model assumption (3):

$$\eta(l_s;t) = \int_0^t \sigma(l_s;\tau)\,dw(\tau).$$

We change the model assumption

$$\tilde{\eta}(l_s;t) = \left(\frac{1}{2}\int_0^t \sigma^2(l_s;\tau)\,d\tau\right)^{1/2}$$

and consider the field $\{\hat{x}_n(l;t);\ \hat{y}_n(l;t)\}$ of the form

$$\begin{aligned} \hat{x}_n(l;t) &= \sum_{s=1}^{n} a(l_s)\cos\left[\sum_{j=1}^{s} \triangle(w(l_j))\ \left(\tfrac{1}{2}\textstyle\int_0^t \sigma^2(l_j;\tau)\,d\tau\right)^{1/2}\right]\ \Delta, \\ \hat{y}_n(l;t) &= \sum_{s=1}^{n} a(l_s)\sin\left[\sum_{j=1}^{s} \triangle(w(l_j))\ \left(\tfrac{1}{2}\textstyle\int_0^t \sigma^2(l_j;\tau)\,d\tau\right)^{1/2}\right]\ \Delta, \end{aligned} \tag{17}$$

where $\triangle(w(l_j))$ is an increment of the Wiener process on the interval $[l_j;l_{j+1}]$. This means that the variable t is not a random variable. From the analysis of the process on the flow of the σ-algebras $\Im_t(n)\oplus\Im_l$, it is possible to pass to the process defined on the flow of the σ-algebras $\Im(l)$, $\forall t = const$. Averaging with respect to t has already been carried out. We observe that the fields $\{x_n(l;t);\ y_n(l;t)\}$ and $\{\hat{x}_n(l;t);\ \hat{y}_n(l;t)\}$ are defined on different spaces. We consider the processes

$$z_2(k;t) = \exp\left\{-i\sum_{j=1}^{k}\triangle w(l_j)\ \left(\frac{1}{2}\int_0^t \sigma^2(l_j;\tau)\,d\tau\right)^{1/2}\right\},$$

$$z_{n,2}(l;t) = \sum_{k=1}^{n} z_2(k;t)\,a(l_k)\cdot\Delta, \quad \Delta = O(n^{-1}).$$

In view of the Euler representation, the components of fields (17) take the form

$$\begin{aligned} \hat{x}_n(l;t) &= \tfrac{1}{2}\left(z_{n,2}(l;t) + z^*_{n,2}(l;t)\right), \\ \hat{y}_n(l;t) &= \tfrac{i}{2}\left(z_{n,2}(l;t) - z^*_{n,2}(l;t)\right). \end{aligned}$$

We now construct the characteristic functions: $g_n(\alpha;\beta;t)$ for the field $\{x_n(l;t);y_n(l;t)\}$ and $\hat{g}_n(\alpha;\beta;t)$ for the field $\{\hat{x}_n(l;t);\hat{y}_n(l;t)\}$:

$$g_n(\alpha;\beta;t) = \mathbf{M}\left[\exp\left\{\frac{i}{2}(\alpha+i\,\beta)\,z_{n,1}(l;t) + \frac{i}{2}(\alpha-i\,\beta)\,z^*_{n,1}(l;t)\right\}\right],$$

$$\hat{g}_n(\alpha;\beta;t) = \mathbf{M}\left[\exp\left\{\frac{i}{2}(\alpha+i\,\beta)\,z_{n,2}(l;t) + \frac{i}{2}(\alpha-i\,\beta)\,z^*_{n,2}(l;t)\right\}\right].$$

For the continuation of the research, the next lemma is necessary.

Lemma 5.3. *Under the model assumptions for the random fields $\{x_n(l;t);y_n(l;t)\}$ and $\{\hat{x}_n(l;t);\hat{y}_n(l;t)\}$ and for a fixed integer m, there exists a number n' such that, for all $n>n'$, the following relations hold:*

$$\mathbf{M}[z^m_{,2}(l;t)] = \mathbf{M}[z^{*\,m}_{,2}(l;t)] = \mathbf{M}[z^m_{,1}(l;t)] = \mathbf{M}[z^{*\,m}_{,1}(l;t)]. \tag{18}$$

Proof. We calculate all increments in the series of equalities (18) by having in mind the lemmas. We introduce the notation $\tilde{\eta}(\theta,t)=\left(\frac{1}{2}\int_0^t\sigma^2(\theta,\tau)d\tau\right)^{1/2}$. We have

$$\begin{aligned}\mathbf{M}[z^m_{,2}(l;t)] = m!\, ⨍_0^l\, a(u_1)\,du_1\,\mathbf{M}\left[\exp\left\{-i\,m\, ⨍_0^{u_1}\tilde{\eta}(\theta_1,t)\,dw(\theta_1)\right\}\right]\times\\ \times\ \ldots\ \times ⨍_{u_m}^l\, a(u_m)\,du_m)\,\mathbf{M}\left[\exp\left\{-i\, ⨍_{u_{m-1}}^{u_m}\tilde{\eta}(\theta_m,t)\,dw(\theta_m)\right\}\right]=\\ = m!\left(⨍_0^l\, a(u_1)\,du_1\,\exp\left\{-\frac{m^2}{2}\, ⨍_0^{u_1}\tilde{\eta}^2(\theta_1,t)\,d\theta_1\right\}\right)\times\\ \times\ \ldots\ \times\left(⨍_{u_m}^l\, a(u_m)\,du_m)\,\exp\left\{-\frac{1}{2}\, ⨍_{u_{m-1}}^{u_m}\tilde{\eta}^2(\theta_m,t)\,d\theta_m\right\}\right)=\\ = m!\left(⨍_0^l\, a(u_1)\,du_1\,\exp\left\{-\frac{m^2}{2}\, ⨍_0^{u_1}\left(\frac{1}{2}\int_0^t\sigma^2(\theta_1,\tau)d\tau\right)\,d\theta_1\right\}\right)\times\ \ldots\ \times\\ \times\left(⨍_{u_m}^l\, a(u_m)\,du_m)\,\exp\left\{-\frac{1}{2}\, ⨍_{u_{m-1}}^{u_m}\left(\frac{1}{2}\int_0^t\sigma^2(\theta_m,\tau)d\tau\right)\,d\theta_m\right\}\right)=\mathbf{M}[z^m_{,1}(l;t)].\end{aligned}$$

Then, by virtue of (5) and (7), we have

$$\mathbf{M}[z^m_{,1}(l;t)]=\mathbf{M}[z^{*\,m}_{,1}(l;t)].$$

In this way, we complete the proof of the lemma. □

Lemma 5.4. *The characteristic functions of the fields* $\{x_n(l;t);\,y_n(l;t)\}$ *and* $\{\hat{x}_n(l;t);\,\hat{y}_n(l;t)\}$ *for* $n\to\infty$ *coincide for all* l *and* t.

Proof. The proof is based on the coincidence of the representations for the characteristic functions $g_n(\alpha,\beta,t)$ and $\hat{g}_n(\alpha,\beta,t)$ by means of the Maclaurin expansion (inside the mean) with respect to $z_{,1}(l;t)$ and $z^*_{,1}(l;t)$, and $z_{,2}(l;t)$, $z^*_{,2}(l;t)$, respectively, and on the conclusions of Lemma 5.3. □

Lemma 5.4 allows us to pass to the study of the limit behavior of the field $\{\hat{x}_n(l;t);\,\hat{y}_n(l;t)\}$ for $n\to\infty$ exclusively.

Theorem 5.1. *Let us assume that, for the field* $\{x_n(l;t);y_n(l;t)\}$, *the model assumptions* (17) *are satisfied:*

$$\begin{aligned}\hat{x}_n(l;t)&=\sum_{s=1}^n a(l_s)\cos\left[\sum_{j=1}^s\triangle(w(l_j))\,\left(\frac{1}{2}\int_0^t\sigma^2(l_j;\tau)\,d\tau\right)^{1/2}\right]\Delta,\\ \hat{y}_n(l;t)&=\sum_{s=1}^n a(l_s)\sin\left[\sum_{j=1}^s\triangle(w(l_j))\,\left(\frac{1}{2}\int_0^t\sigma^2(l_j;\tau)\,d\tau\right)^{1/2}\right]\Delta.\end{aligned}$$

We also assume that the field $\{x(l;t);y(l;t)\}$ *is defined in the following way:*

$$\begin{aligned}x(l;t)&=\int_0^l a(u)\cos\left[\int_0^u\frac{1}{2}\left(\int_0^t\sigma^2(\theta,\tau)d\tau\right)dw(\theta)\right]du,\\ y(l;t)&=\int_0^l a(u)\sin\left[\int_0^u\frac{1}{2}\left(\int_0^t\sigma^2(\theta,\tau)d\tau\right)dw(\theta)\right]du.\end{aligned}\tag{19}$$

Under these conditions, the characteristic functions of the processes $\{x(l;t);y(l;t)\}$ *and* $\{x_n(l;t);y_n(l;t)\}$ *coincide.*

Proof. The comparison of the characteristic functions for $\{\hat{x}_n(l;t),\hat{y}_n(l;t)\}$ and $\{x(l;t),\,y(l;t)\}$ for all values of $t\in[0,T]$, as $n\to\infty$ leads to the proof of the theorem. □

Theorem 5.2. *The stochastic process* $\{x(l;t);y(l;t)\}$ *is a solution to the Cauchy problem for the Itô stochastic differential equations*

(20)

$$\begin{aligned}d_lp(l;t)&=\left[p(l;t)\frac{\partial}{\partial l}\ln a(l)-\frac{p(l;t)}{4}\int\limits_0^t\sigma^2(l;\tau)d\tau\right]dl-\left(\frac{1}{2}\int\limits_0^t\sigma^2(l;\tau)d\tau\right)^{0,5}q(l;t)dw(l),\\ d_lq(l;t)&=\left[q(l;t)\frac{\partial}{\partial l}\ln a(l)-\frac{q(l;t)}{4}\int\limits_0^t\sigma^2(l;\tau)d\tau\right]dl+\left(\frac{1}{2}\int\limits_0^t\sigma^2(l;\tau)d\tau\right)^{0,5}p(l;t)dw(l),\\ &d_lx(l;t)=q(l;t)\,dl,\qquad d_ly(l;t)=p(l;t)\,dl\end{aligned}$$

satisfying the boundary conditions

$$x(0;t) = 0, \quad y(0;t) = 0, \quad p(0;t) = a(0), \quad q(0;t) = 0.$$

Proof. We differentiate $x(l;t)$ and $y(l;t)$ in (19) with respect to l:

$$\frac{\partial x(l;t)}{\partial l} = a(l) \sin \left[\int_0^l \frac{1}{2} \left(\int_0^t \sigma^2(\theta, \tau) d\tau \right) dw(\theta) \right] = q(l;t), \tag{21}$$

$$\frac{\partial y(l;t)}{\partial l} = -a(l) \cos \left[\int_0^l \frac{1}{2} \left(\int_0^t \sigma^2(\theta, \tau) d\tau \right) dw(\theta) \right] = p(l;t). \tag{22}$$

The obtained expressions are now differentiated by the Itô formula with respect to the variable l:

$$d_l \left(\frac{\partial x(l;t)}{\partial l} \right) = \frac{1}{a(l)} \frac{\partial a(l)}{\partial l}\, a(l) \cos \left[\int_0^l \tfrac{1}{2} \left(\int_0^t \sigma^2(\theta, \tau) d\tau \right) dw(\theta) \right] dl -$$
$$- a(l) \sin \left[\int_0^l \tfrac{1}{2} \left(\int_0^t \sigma^2(\theta, \tau) d\tau \right) dw(\theta) \right] \cdot \tfrac{1}{2} \left(\int_0^t \sigma^2(\theta, \tau) d\tau \right) dw(l) -$$
$$- \tfrac{1}{2} \cos \left[\int_0^l \tfrac{1}{2} \left(\int_0^t \sigma^2(\theta, \tau) d\tau \right) dw(\theta) \right] \cdot \left(\tfrac{1}{2} \int_0^t \sigma^2(\theta, \tau) d\tau \right)^2 dl.$$

Taking (21) and (22) into account, we obtain the last equation of system (20) . In a similar way, we get the second expression of the system. The functions $x(l;t)$, $y(l;t)$, $p(l;t)$, $q(l;t)$ defined by (19), (21), and (22) satisfy the given initial conditions. □

Within the framework of the given formulation (L is constant), we have found $F_t(x;y;L)$ for different values of t.

Theorem 5.3. *The distribution function of the process* $\{x(l;t); y(l;t)\}$ *can be obtained by integrating the Kolmogorov equation of system* (20) *with respect to the variables* p *and* q.

Proof. After the enlargement of the space obtained by introducing the new variables p and q, the compound process $\{x(l;t); y(l;t); p(l;t); q(l;t)\}$ becomes a Markov process. This means that it is possible to obtain a Kolmogorov equation for the density function $\rho(x, y, p, q, l, t)$ and then, by integrating with respect to p and q, infer the density function of the distribution $\rho(x, y, l, t)$ for all l and t. □

Theorem 5.4. *The distribution function of the original process* $\{x_n(l;t); y_n(l;t)\}$ *under the model conditions* (4) *coincides with the distribution function of the Markov process* $\{x(l;t); y(l;t)\}$ (19).

Proof. The proof is based on the conclusions of Theorem 5.1 and Theorem 5.2. □

Remark 5.1. The character of the analysis doesn't substantially changes when, for example, $a = a(l,t)$ (vibrating chain), $\sigma(l;t)$ is a non-anticipating measurable random function with respect to independent flows of σ-algebras governed by independent Wiener processes $w(l)$ and $w(t)$.

In this way, we arrive at a coherent representation of distribution: the parameter t defines also the structure of the chain.

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