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# STOCHASTIC DYNAMICS VIA EQUATIONS AND INCLUSIONS IN TERMS OF MEAN DERIVATIVES AND INFINITESIMAL GENERATORS

This is a survey of recent results on stochastic differential equations and inclusions given in terms of mean derivatives and infinitesimal generators of stochastic processes. We pay the main attention to equations and inclusions on manifolds.

The notion of mean derivatives was introduced by Edward Nelson in [20, 21, 22] for the needs of stochastic mechanics (a version of quantum mechanics). The equation of motion in this theory (called the Newton–Nelson equation) was the first example of equations in mean derivatives. Later on, it turned out that the equations in mean derivatives arose also in the description of motion of a viscous incompressible fluid, mechanical systems with random perturbations, Navier–Stokes vortices, *etc.* (see, e.g., [13, 14, 15] and [18]).

In all above-mentioned cases, the solutions of the equations were supposed to be Itô diffusion type processes (or even Markov diffusion processes), whose diffusion summand was given *a priori* since the classical Nelson's mean derivatives yield, roughly speaking, only the drift term (forward, backward, *etc.*) of a stochastic process. In [1], as a slight modification of some Nelson's idea, the mean derivative of a new sort, called a quadratic one, was introduced so that it became possible, in principle, to recover a process from its mean derivatives.

This paper contains an introduction into this theory and a survey of results obtained on first-order equations and inclusions starting from paper [1], with focus on systems on manifolds (see [1, 2, 3, 16, 17]). In particular, we discuss the relations between mean derivatives and infinitesimal generators.

For results related to the second-order case, we refer the reader to [13, 14, 15, 5, 6], where, in particular, the list of mean derivatives for processes from a broad class can be found, as well as explicit examples of second-order equations and inclusions with mean derivatives arising in mathematical physics.

Preliminaries from the geometry of manifolds can be found in [15], from the general theory of stochastic differential equations on manifolds in [10] and from set-valued analysis in [8].

Some remarks on notations. The space of $n \times n$ matrices is denoted by $L(\mathbb{R}^n, \mathbb{R}^n)$. By $\mathrm{S}(n)$, we denote the linear space of symmetric $n\times n$ matrices, i.e., a subspace in $L(\mathbb{R}^n, \mathbb{R}^n)$. The symbol $\mathrm{S}_+(n)$ denotes the set of positive definite symmetric $n \times n$ matrices that is a convex open set in $\mathrm{S}(n)$. Its closure, i.e., the set of positive semidefinite symmetric $n \times n$ matrices, is denoted by $\bar{\mathrm{S}}_+(n)$.

Everywhere below for a set $B$ in $\mathbb{R}^n$ or in $L(\mathbb{R}^n, \mathbb{R}^n)$, we use the norm introduced by the usual formula $\|B\| = \sup_{y\in B} \|y\|$.

In spite of the fact that we do not use the concept of strong solution, we say by analogy with the theory of stochastic differential equations that an equation (inclusion)

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has a weak solution, if there exists a probability space and a process given on that space, for which the equation (inclusion) is satisfied.

## 1. Mean Derivatives

Consider a stochastic process $\xi(t)$ in $\mathbb{R}^n$, $t \in [0, T]$ given on a certain probability space $(\Omega, \mathcal{F}, \mathsf{P})$ and such that $\xi(t)$ is an $L_1$ random element for all $t$. It is known that such a process determines 3 families of $\sigma$-subalgebras of the $\sigma$-algebra $\mathcal{F}$:

(i) "the past" $\mathcal{P}_t^\xi$ generated by preimages of Borel sets from $\mathbb{R}^n$ under all mappings $\xi(s) : \Omega \to \mathbb{R}^n$ for $0 \le s \le t$;

(ii) "the future" $\mathcal{F}_t^\xi$ generated by preimages of Borel sets from $\mathbb{R}^n$ under all mappings $\xi(s) : \Omega \to \mathbb{R}^n$ for $t \le s \le T$;

(iii) "the present" ("now") $\mathcal{N}_t^\xi$ generated by preimages of Borel sets from $\mathbb{R}^n$ under the mapping $\xi(t) : \Omega \to \mathbb{R}^n$.

All the above families are supposed to be complete, i.e., containing all sets of probability zero. Analogous notions are used for processes on manifolds as well.

For the sake of convenience, we denote, by $E_t^\xi$, the conditional expectation $E(\cdot|\mathcal{N}_t^\xi)$ with respect to the "present" $\mathcal{N}_t^\xi$ for $\xi(t)$.

According to Nelson's ideas, we introduce the following notions of forward and backward mean derivatives:

**Definition 1.1.** (i) The forward mean derivative $D\xi(t)$ of $\xi(t)$ at the time instant $t$ is an $L_1$ random element of the form

$$D\xi(t) = \lim_{\triangle t \to +0} E_t^\xi\Big(\frac{\xi(t + \triangle t) - \xi(t)}{\triangle t}\Big), \tag{1}$$

where the limit is supposed to exist in $L_1(\Omega, \mathcal{F}, \mathsf{P})$, and $\triangle t \to +0$ means that $\triangle t$ tends to 0 and $\triangle t > 0$.

(ii) The backward mean derivative $D_*\xi(t)$ of $\xi(t)$ at $t$ is the $L_1$-random element

$$D_*\xi(t) = \lim_{\Delta t \to +0} E_t^\xi\Big(\frac{\xi(t) - \xi(t - \Delta t)}{\Delta t}\Big), \tag{2}$$

where (as well as in (i)) the limit is assumed to exist in $L^1(\Omega, \mathcal{F}, \mathsf{P})$, and $\Delta t \to +0$ means that $\Delta t \to 0$ and $\Delta t > 0$.

*Remark* 1.1. If $\xi(t)$ is a Markov process, then $E_t^\xi$ can be evidently replaced by $E(\cdot|\mathcal{P}_t^\xi)$ in (1) and by $E(\cdot|\mathcal{F}_t^\xi)$ in (2). In initial Nelson's works, there were two versions of the definition of mean derivatives: as in our Definition 1.1 and with conditional expectations with respect to "past" and "future" that coincide for Markov processes. Here, we deal with mean derivatives with respect to "present" taking the physical principle of locality into account: the derivative should be determined by the present state of the system, not by its past or future. An alternative theory of equations and inclusions with mean derivatives relative to the "past" filtration (called $\mathcal{P}$-*mean derivatives*) is discussed in [3].

Rather often, the following generalizations of the notions of forward and backward mean derivatives arise. The forward mean derivative $D^\xi\eta(t)$ and the backward derivative of $D_*^\xi\eta(t)$ of $\eta(t)$ with respect to $\xi(t)$ at the time instant $t$ are $L_1$ random elements of the form

$$D^\xi\eta(t) = \lim_{\triangle t \to +0} E_t^\xi\big(\tfrac{\eta(t+\triangle t)-\eta(t)}{\triangle t}\big) \quad \text{and} \quad D_*^\xi\eta(t) = \lim_{\Delta t \to +0} E_t^\xi\big(\tfrac{\eta(t)-\eta(t-\Delta t)}{\Delta t}\big),$$

where the limits are supposed to exist in $L_1(\Omega, \mathcal{F}, \mathsf{P})$, and $\triangle t \to +0$ means that $\triangle t$ tends to 0 and $\triangle t > 0$.

Introduce a differential operator $D_2$ that differentiates an $L_1$ random process $\xi(t)$, $t \in [0,T]$ according to the rule

$$D_2\xi(t) = \lim_{\triangle t \to +0} E_t^{\xi}\Big(\frac{(\xi(t+\triangle t) - \xi(t)) \otimes (\xi(t+\triangle t) - \xi(t))}{\triangle t}\Big).$$

It can be also described as

$$D_2\xi(t) = \lim_{\triangle t \to +0} E_t^{\xi}\Big(\frac{(\xi(t+\triangle t) - \xi(t))(\xi(t+\triangle t) - \xi(t))^*}{\triangle t}\Big),$$

where $(\xi(t+\triangle t)-\xi(t))$ is considered as a column vector (vector in $\mathbb{R}^n$), $(\xi(t+\triangle t)-\xi(t))^*$ is a row vector (transposed or conjugate vector), and the limit is supposed to exists in $L_1(\Omega, \mathcal{F}, \mathsf{P})$. We emphasize that the matrix product of a column on the left and a row on the right is a matrix so that $D_2\xi(t)$ is a symmetric semipositive definite matrix function on $[0,T] \times \mathbb{R}^n$.

**Definition 1.2.** $D_2$ is called quadratic mean derivative.

*Remark* 1.2. From the properties of a conditional expectation, it follows that there exist Borel mappings $a(t,x)$, $a_*(t,x)$ and $\alpha(t,x)$ from $R \times \mathbb{R}^n$ to $\mathbb{R}^n$ and to $\bar{\mathrm{S}}_+$, respectively, such that $D\xi(t) = a(t,\xi(t))$, $D_*\xi(t) = a_*(t,\xi(t))$ and $D_2\xi(t) = \alpha(t,\xi(t))$. We call $a(t,x)$, $a_*(t,x)$ and $\alpha(t,x)$ the regressions.

Recall that the Itô process is a process of the form $\xi(t) = \xi_0 + \int\limits_0^t a(s)ds + \int\limits_0^t A(s)dw(s)$.

**Definition 1.3.** An Itô process $\xi(t)$ is called a process of the diffusion type if $a(t)$ and $A(t)$ are not anticipating with respect to $\mathcal{P}_t^{\xi}$, and the Wiener process $w(t)$ is adapted to $\mathcal{P}_t^{\xi}$. If $a(t) = a(t,\xi(t))$ and $A(t) = A(t,\xi(t))$, where $a(t,x)$ and $A(t,x)$ are Borel measurable mappings from $[0,T] \times \mathbb{R}^n$ to $\mathbb{R}^n$ and to $L(\mathbb{R}^n, \mathbb{R}^n)$, respectively, the Itô process is called a diffusion process.

In view of the properties of conditional expectation and the fact that $\mathcal{N}_t^{\xi}$ is a $\sigma$-subalgebra in $\mathcal{P}_t^{\xi}$, it is clear that, for any martingale $\eta(t)$ with respect to $\mathcal{P}_t^{\xi}$, the equality $D^{\xi}\eta(t) = 0$ holds. Since, for a diffusion-type process, the integral $\int_0^t A(s)dw(s)$ is a martingale with respect to $\mathcal{P}_t^{\xi}$, the following statement takes place:

**Theorem 1.1.** *For an Itô diffusion type process $\xi(t)$, the mean derivative $D\xi(t)$ exists and equals $E_t^{\xi}(a(t))$. In particular, if $\xi(t)$ is a diffusion process, $D\xi(t) = a(t,\xi(t))$.*

**Theorem 1.2.** *Let $\xi(t)$ be a diffusion type process. Then $D_2\xi(t) = E_t^{\xi}[\alpha(t)]$, where $\alpha(t) = A(t)A^*(t)$ is the diffusion coefficient. In particular, if $\xi(t)$ is a diffusion process, $D_2\xi(t) = \alpha(t,\xi(t))$, where $\alpha(t,x) = A(t,x)A^*(t,x)$ is the diffusion coefficient.*

**Theorem 1.3.** $D_2\xi(t) = 0$ *if and only if* $A = 0$, *and so* $\xi(t)$ *is a deterministic process.*

**Definition 1.4.** The derivative $D_S = \frac{1}{2}(D+D_*)$ is called the symmetric mean derivative. The derivative $D_A = \frac{1}{2}(D - D_*)$ is called the antisymmetric mean derivative.

Consider the vectors $v^{\xi}(t,x) = \frac{1}{2}(a(t,x) + a_*(t,x))$ and $u^{\xi}(t,x) = \frac{1}{2}(a(t,x) - a_*(t,x))$.

**Definition 1.5.** The quantity $v^{\xi}(t) = v^{\xi}(t,\xi(t)) = D_S\xi(t)$ is called the current velocity of the process $\xi(t)$; and $u^{\xi}(t) = u^{\xi}(t,\xi(t)) = D_A\xi(t)$ is called the osmotic velocity of the process $\xi(t)$.

The physical meaning of $v^{\xi}$ and $u^{\xi}$ is as follows. Let $\xi(t)$ describe the motion of a physical process, say the motion of a particle (we are sure that all physical motions are random with a very small dispersion, so that it usually looks natural to omit the randomness from consideration). Then the current velocity $v^\xi$ is what we usually consider as the ordinary physical velocity while the osmotic velocity $u^\xi$ shows how fast the particle "diffuses" into the enveloping continuum, i.e., how fast the "randomness" is changing. This interpretation has the following mathematical motivation discovered by Nelson.

Consider an autonomous smooth field of non-degenerate linear operators $A(x) : \mathbb{R}^n \to \mathbb{R}^n$, $x \in \mathbb{R}^n$. Suppose that $\xi(t)$ is a diffusion-type process, whose diffusion integrand is $A(\xi(t))$. Then its diffusion coefficient $A(x)A^*(x)$ is a smooth field of symmetric positive definite matrices $\alpha(x) = (\alpha^{ij}(x))$. Since all those matrices are non-degenerate, the field of inverse matrices $(\alpha_{ij})$ exists and is smooth. Moreover, at any $x$, the matrix $(\alpha_{ij})(x)$ is symmetric and positive definite. Thus, it defines a new Riemannian metric $\alpha(\cdot,\cdot) = \alpha_{ij}dx^i dx^j$ on $\mathbb{R}^n$. Consider the Riemannian volume form of this Riemannian metric $\Lambda_\alpha = \sqrt{\det(\alpha_{ij})}dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$.

Denote, by $\rho^\xi(t,x)$, the probability density of $\xi(t)$ with respect to the volume form $dt \wedge \Lambda_\alpha = \sqrt{\det(\alpha_{ij})}dt \wedge dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$ on $[0,T] \times \mathbb{R}^n$. In other words, for any continuous bounded function $f : [0,T] \times \mathbb{R}^n \to R$. the relation

$$\int_0^T E(f(t,\xi(t)))dt = \int_0^T (\int_\Omega f(t,\xi(t))d\mathsf{P})dt = \int_{[0,T]\times\mathbb{R}^n} f(t,x)\rho^\xi(t,x)dt \wedge \Lambda_\alpha$$

holds. Then

$$u^\xi(t,x) = \frac{1}{2} Grad \log \rho^\xi(t,x) = Grad \log \sqrt{\rho^\xi(t,x)},$$

where $Grad$ denotes the gradient with respect to the Riemannian metric $\alpha(\cdot,\cdot)$. For $v^\xi(t,x)$ and $\rho^\xi(t,x)$, the so-called equation of continuity

$$\frac{\partial \rho^\xi(t,x)}{\partial t} = -Div(v^\xi(t,x)\rho^\xi(t,x))$$

holds, where $Div$ denotes the divergence with respect to the Riemannian metric $\alpha(\cdot,\cdot)$.

Let $Y(t,m)$, $t \in [0,l]$, be a smooth time-dependent vector field on $\mathbb{R}^n$. We define the forward derivative $DY$ and the backward derivative $D_*Y$ of $Y$ along $\xi(t)$ as follows:

$$DY(t,\xi(t)) = \lim_{\Delta t \to +0} E_t^\xi \frac{Y(t+\Delta t, \xi(t+\Delta t)) - Y(t,\xi(t))}{\Delta t},$$

$$D_*Y(t,\xi(t)) = \lim_{\Delta t \to +0} E_t^\xi \frac{Y(t,\xi(t)) - Y(t-\Delta t, \xi(t-\Delta t))}{\Delta t}.$$

Suppose that the process $\xi$ has the diffusion coefficient $\frac{\sigma^2}{2}I$. Then we obtain that

$$DY = \left(\frac{\sigma^2}{2}\Delta + X \cdot \nabla + \frac{\partial}{\partial t}\right)Y \qquad \text{and} \qquad D_*Y = \left(-\frac{\sigma^2}{2}\Delta + X_* \cdot \nabla + \frac{\partial}{\partial t}\right)Y,$$

where $\Delta$ is the Laplace operator, $\nabla = \left(\partial/\partial q^1, \ldots, \partial/\partial q^n\right)$, and the dot denotes the inner product in $\mathbb{R}^n$.

## 2. Mean Derivatives on Manifolds

Let $M$ be a finite-dimensional smooth manifold. Specify a certain connection on $M$. Consider an $M$-valued stochastic process $\xi(t)$. Let $m$ be a point of the manifold $M$. Consider the normal chart $U_m$ at this point $m$ with respect to the above connection. For any $m'$ from this chart, we can compute the regression

$$Y^0(t,m')_{|U_m} = \lim_{\triangle t \to 0} E((\frac{\xi(t+\triangle t) - \xi(t)}{\triangle t})_{U_m} | \xi(t) = m').$$

Construct a vector field $Y^0(t,\cdot)$ such that, at every point $m \in M$, it is equal to

$$Y^0(t,m)|_{U_m}$$

computed in the normal chart $U_m$. Thus, we have that $Y^0$ is a measurable section of the tangent bundle $TM$.

**Definition 2.1.** $D\xi(t) = Y^0(t, \xi(t))$ is called the mean forward derivative of a process $\xi(t)$ on $M$ at the time instant $t$.

The mean backward derivative is defined analogously, and it depends on the choice of connection as well.

Introduce also another derivative for the stochastic process $\xi(t)$ on $M$. Take any chart $U$ and, in it, consider the $L_1$ random variable determined by the rule

$$D_2\xi(t) = \lim_{\triangle t \to +0} E^\xi_t\left(\frac{(\xi(t+\triangle t) - \xi(t)) \otimes (\xi(t+\triangle t) - \xi(t))}{\triangle t}\right). \tag{3}$$

**Definition 2.2.** $D_2\xi(t)$ is called the quadratic mean derivative of the process $\xi(t)$ on $M$ at the time instant $t$.

Notice that the forward and backward derivatives depend on the choice of the connection while the quadratic mean derivative does not. It takes values in $(2,0)$-tensors.

Let $A(t,m)$ be a field of linear maps from a certain Euclidean space $\mathbb{R}^k$ to tangent spaces $T_mM$. We recall that the Itô equation on a manifold is a section of the fiber bundle that we call the Itô bundle. Over any chart $\mathrm{U}_\alpha$ in a manifold $M$, it has the form of a direct product $\mathrm{U}_\alpha \times (\mathbb{R}^n \times L(\mathbb{R}^k, \mathbb{R}^n))$. Under a transition to another chart $\mathrm{U}_\beta$ with a change of the coordinates $\varphi_{\beta\alpha}$, the point $(m^\alpha, (a^\alpha, A^\alpha))$ is transformed by the rule

$$(m^\alpha, (a^\alpha, A^\alpha)) \mapsto (\varphi_{\beta\alpha} m^\alpha, (\varphi'_{\beta\alpha} a^\alpha + \frac{1}{2} tr\, \varphi''_{\beta\alpha}(A^\alpha, A^\alpha), \varphi'_{\beta\alpha} A^\alpha)).$$

We denote Itô equations as couples $(\hat{a}, A)$. For such a couple, the stochastic differential equation defined in charts by the relation

$$d\xi(t) = \hat{a}(t, \xi(t))dt + A(t, \xi(t))dw(t), \tag{4}$$

where $w(t)$ is a Wiener process in $\mathbb{R}^k$, is well-posed on the entire $M$.

The couple $(a, A)$, where $a$ is a vector field on $M$, and $A$ is as above, is called the Itô vector field.

**Definition 2.3.** An Itô equation $(\hat{a}, A)$ and the Itô vector field $(a, A)$ such that, in any chart $\hat{a}(t,m) = a(t,m) - \frac{1}{2} tr \Gamma_m(A(t,m), A(t,m))$, where $\Gamma(\cdot,\cdot)$ is a local coefficient of the connection $\mathcal{H}$, are said to be canonically corresponding to each other with respect to the connection $\mathcal{H}$.

There are the descriptions of solutions of (4) by means of Itô vector fields, known as Itô equations in the Belopol'skaya–Daletskii form and in the Baxendale form (see, e.g., [7]). The former is given globally on a manifold (exponential map of the connection is involved into construction) and the latter is given in charts and plays the role of a local description of the former. For the sake of simplicity, we use the description in charts: a solution of (4) in a chart satisfies the equation

$$d\xi(t) = a(t, \xi(t))dt - \frac{1}{2} tr \Gamma_{\xi(t)}(A(t, \xi(t), \xi(t))dt + A(t, \xi(t))dw(t),$$

where $(a, A)$ canonically corresponds to $(\hat{a}, A)$ with respect to the connection $\mathcal{H}$.

**Theorem 2.1.** *Specify any connection $\mathcal{H}$. Let $(a, A)$ be the Itô vector field canonically corresponding to the Itô equation $(\hat{a}, A)$ with respect to $\mathcal{H}$, and the forward mean derivative $D$ is defined also with respect to normal charts of $\mathcal{H}$. Then, for a solution $\xi(t)$ of (4), the equality $D\xi(t) = a(t, \xi(t))$ holds.*

Thus, changing a connection, we change both the canonically corresponding Itô vector field and the forward mean derivative, but anyhow $D\xi(t) = a(t, \xi(t))$. On the one hand, this means that the Itô equation (4) has invariant sense, rather than its presentation in the Belopol'skaya–Daletskii or Baxendale form. On the other hand, we can try to choose "the best" connection and use it both for the canonical correspondence and the mean derivative.

**Theorem 2.2.** *For a solution $\xi(t)$ of (4), the relation $D_2\xi = AA^*$ holds.*

As above, if $A$ is smooth, autonomous, and non-degenerate, then $AA^*$ generates a Riemannian metric on $M$. Its Levi-Civita connection looks most convenient for applications. In this section, we use this connection.

It follows from Nelsons's results that the current velocity of a process is a vector and can be determined without the use of connections.

The second-order tangent vector $\mathcal{A}$ is a second-order differential operator of the form $\mathcal{A} = b^i\frac{\partial}{\partial q^i} + \beta^{ij}\frac{\partial^2}{\partial q^j \partial q^j}$, where the matrix $(\beta^{ij})$ is symmetric since $\frac{\partial^2 f}{\partial q^j \partial q^j} = \frac{\partial^2 f}{\partial q^j \partial q^i}$ for a smooth real-valued $f$. The space of second-order tangent vectors at $m \in M$ is called the second-order tangent space and denoted by $\tau_m M$. The second-order tangent bundle is denoted by $\tau M$. Thus, $\frac{\partial}{\partial x^i}$ and $\frac{\partial^2}{\partial x^i \partial x^j}$, $i, j = 1, 2, \dots, n$ form a basis in $\tau_m M$. A detailed description of the theory of second-order tangent vectors and differential forms on manifolds is contained in [11, 19, 23]

At every $m \in M$, there is a canonical isomorphism between the space $T_m M \odot T_m M$ (where $\odot$ denotes the symmetric tensor product) and the quotient space $\tau_m M/T_m M$, and so between $TM \odot TM$ and $\tau M/TM$. Taking the above factorization into account, we construct the morphism $\mathcal{Q} : \tau M \to TM \odot TM$, i.e., the field of linear projectors $\mathcal{Q}_m : \tau_m M \to T_m M \odot T_m M$ such that

$$\mathcal{Q}B(t, m) = \mathcal{Q}(b^i \frac{\partial}{\partial q^i} + \beta^{ij} \frac{\partial^2}{\partial q^i \partial q^j}) = \beta^{ij} \frac{\partial}{\partial q^i} \otimes \frac{\partial}{\partial q^j}. \tag{5}$$

Every connection $\mathcal{H}$ on $M$ determines a linear operator from $\tau_m M$ to $T_m M$ at any point $m \in M$ as follows:

$$\mathcal{H}(b^k \frac{\partial}{\partial q^k} + \beta^{ij} \frac{\partial^2}{\partial q^j \partial q^j}) = (b^k + \Gamma^k_{ij}\beta^{ij})\frac{\partial}{\partial q^k}, \tag{6}$$

where $\Gamma^k_{ij}$ are the Christoffel symbols of connection $\mathcal{H}$ (see [11, 19, 23]). Thus, the connections and only they are smooth cross-sections of $Hom(\tau M, TM)$, the bundle of fiber-wise linear operators from $\tau M$ to $TM$.

Let $(\hat{a}, A)$ be an Itô equation. The example of a second-order tangent vector is its infinitesimal generator $L = \hat{a}^k\frac{\partial}{\partial q^k} + \frac{1}{2}\alpha^{ij}\frac{\partial^2}{\partial q^i \partial q^j}$, where $(\alpha^{ij})$ is the matrix of $AA^*$. We emphasize that both $(\hat{a}, A)$ and its generator are invariant objects that are not related to connections.

**Theorem 2.3** (see [2, 17])**.** *Let $\xi(t)$ be a solution of $(\hat{a}, A)$. Then (i) $D_2\xi(t) = 2\mathcal{Q}L(\xi(t))$, (ii) $D\xi(t) = \mathcal{H}L(\xi(t))$. (iii) If $L_*$ is the backward generator, then $D_*\xi(t) = \mathcal{H}L_*(\xi(t))$.*

## 3. Differential Equations with Mean Derivatives

Everywhere below for the sake of simplicity, we consider equations, their solutions, *etc.*, on a finite time interval, $t \in [0, T]$.

Let Borel measurable maps $a(t, x)$ and $\alpha(t, x)$ from $[0, T] \times \mathbb{R}^n$ to $\mathbb{R}^n$ and to $\bar{S}_+(n)$, respectively, be given.

Consider the system

$$\begin{cases} D\xi(t) = a(t, \xi(t)), \\ D_2\xi(t) = \alpha(t, \xi(t)), \end{cases} \tag{7}$$

We call it the first-order differential equation with forward mean derivatives.

With regard for Remark 1.2, Theorem 1.1, and Theorem 1.2, we mainly look for weak solutions of (7) in the class of diffusion-type processes, since the first equation of (7) determines a drift, and, by the second one, we can try to determine the diffusion term of an Itô diffusion-type equation. To do this, we use the following two technical statements:

**Lemma 3.1.** *Let $\alpha(t,x)$ be a jointly continuous (measurable, smooth) mapping from $[0,T]\times\mathbb{R}^n$ to $\mathrm{S}_+(n)$. Then there exists a jointly continuous (measurable, smooth, respectively) mapping $A(t,x)$ from $[0,T]\times\mathbb{R}^n$ to $L(\mathbb{R}^n,\mathbb{R}^n)$ such that, for all $t\in R$, $x\in\mathbb{R}^n$, the equality $A(t,x)A^*(t,x)=\alpha(t,x)$ holds.*

To prove Lemma 3.1, we use the so-called Gauss decomposition of symmetric non-degenerate matrices (see details in [1]). If $\alpha(t,x)$ is degenerate, the corresponding $A$ exists as well. But it is not unique, and there is a problem to prove that it is continuous, smooth, measurable, *etc.* In this case, applying results of [12], it is possible to obtain a continuous $A(t,x)$ by another construction:

**Lemma 3.2.** *If $\alpha(t,x)$ is a $C^2$-smooth map from $[0,T]\times\mathbb{R}^n$ to $\bar{S}_+(n)$, there exists a jointly continuous map $A(t,x)$ from $[0,T]\times\mathbb{R}^n$ to $L(\mathbb{R}^n,\mathbb{R}^n)$ such that, for all $t\in R$, $x\in\mathbb{R}^n$, the equality $A(t,x)A^*(t,x)=\alpha(t,x)$ holds.*

**Theorem 3.1.** *Let $\alpha(t,x)$ in (7) be jointly continuous, positive definite (i.e., for all $t\in[0,T]$, $x\in\mathbb{R}^n$, it belongs to $S_+(n)$) and satisfy the estimate*

$$\|tr\,\alpha(t,x)\| < K(1+\|x\|)^2 \tag{8}$$

*for a certain $K>0$. Let $a(t,x)$ be Borel measurable and satisfy the estimate*

$$\|a(t,x)\| < K(1+\|x\|) \tag{9}$$

*for a certain $K>0$. Then, for any initial condition $\xi(0)=x_0\in\mathbb{R}^n$, Eq. (7) has a weak solution well-posed on the entire interval $[0,T]$.*

**Theorem 3.2.** *Let $\alpha(t,x)$ be $C^2$-smooth semipositive definite (i.e., for all $t\in[0,T]$, $x\in\mathbb{R}^n$, it belongs to $\bar{S}_+(n)$) and satisfy (8). Let $a(t,x)$ be continuous and satisfy (9). Then, for any initial condition $\xi(0)=x_0\in\mathbb{R}^n$, Eq. (7) has a weak solution that is well-posed on the entire interval $[0,T]$.*

As is mentioned in Section 1, the meaning of current velocities is analogous to that of the ordinary velocity for a non-random process. Thus, the case of equations and inclusions with current velocities is probably the most natural from the physical point of view.

The system

$$\begin{cases} D_S\xi(t) = v(t,\xi(t)) \\ D_2\xi(t) = \alpha(t,\xi(t)) \end{cases} \tag{10}$$

is called a first-order differential equation with current velocities.

**Theorem 3.3.** *Let $v:[0,T]\times\mathbb{R}^n\to\mathbb{R}^n$ be smooth, and let $\alpha:\mathbb{R}^n\to\mathrm{S}_+(n)$ be smooth and autonomous (so, it determines the Riemannian metric $\alpha(\cdot,\cdot)$ on $\mathbb{R}^n$ introduced in Section 1). Let also they satisfy the estimates*

$$\|v(t,x)\| < K(1+\|x\|),$$
$$tr\ \alpha(x) < K(1+\|x\|^2)$$

*for some $K>0$. Let $\xi_0$ be a random element with values in $\mathbb{R}^n$, whose probability density $\rho_0$ with respect to the volume form $\Lambda_\alpha$ of $\alpha(\cdot,\cdot)$ on $\mathbb{R}^n$ (see Section 1) is smooth and nowhere equal to zero. Then, for the initial condition $\xi(0)=\xi_0$, Eq. (10) has a weak solution that is well-posed on the entire interval, $t\in[0,T]$.*

**Lemma 3.3.** *Let $\alpha(x)$, $\rho(t,x)$ and $\Lambda_\alpha$ be the same as in Theorem 3.3. Let also the vector field $v$ from Theorem 3.3 be autonomous. Then the flow $\hat{g}_t$ of the vector field $(1, v(x))$ on $[0,T]\times\mathbb{R}^n$ preserves the volume form $\rho(t,x)dt\wedge\Lambda_\alpha$ (i.e., $\hat{g}_t^*(\rho(t,x)dt\wedge\Lambda_\alpha) = \rho_0(x)dt\wedge\Lambda_\alpha$, where $\hat{g}_t^*$ is the pull back) and so, for any measurable set $Q \subset \mathbb{R}^n$ and for any $t \in [0,T]$,*

$$\int_Q \rho_0(x)\Lambda_\alpha = \int_{g_t(Q)} \rho(t,x)\Lambda_\alpha.$$

## 4. Differential Inclusions with Forward Mean Derivatives and Infinitesimal Generators

Stochastic differential inclusions naturally arise in many problems. Stochastic equations turn into inclusions by the same reasons as ordinary ones (see details, e.g., in [8]). For example, if the coefficients of an equation are (even) not measurable (say, the equation describes the motion in complicated stochastic media, or dry friction is present), there exists a method of passing from the equation to some inclusion. For ordinary equations, this approach was suggested by A. Filippov. In stochastic case, it was first considered by Conway [9] in 1971. Another well-known sort of inclusions describes the systems with feedback control. In this case, one considers the right-hand side of the equation for all values of a controlling parameter and so obtains the set-valued right-hand side.

Among stochastic differential inclusions, those with mean derivatives are ideologically the closest to ordinary differential inclusions, since they are formulated in the differential form, not in terms of stochastic integrals of measurable selectors as in the classical approach. Many explicit inclusions arisen in mathematical physics are second-order ones (see, e.g., [13, 14, 15, 5, 6]) that we do not consider here. But below, we describe some possible economic meaning of first-order inclusions with mean derivatives.

Let $\mathbf{a}(t,x)$ and $\boldsymbol{\alpha}(t,x)$ be set-valued mappings from $[0,T]\times\mathbb{R}^n$ to $\mathbb{R}^n$ and to $\bar{\mathrm{S}}_+(n)$, respectively. The system of the form

$$\begin{cases} D\xi(t) \in \mathbf{a}(t,\xi(t)), \\ D_2\xi(t) \in \boldsymbol{\alpha}(t,\xi(t)) \end{cases} \tag{11}$$

is called a first-order differential inclusion with forward mean derivatives.

Besides the physical meaning (motion in a complicated medium or under feedback control, see above), (11) may have economic interpretation if $\mathbf{a}(t,x)$ and $\boldsymbol{\alpha}(t,x)$ are obtained from estimates of the expected growth rate of profits and the covariation matrix, respectively, predicted by experts for the vector $x$ of assets of a portfolio at a time instant $t$. Then the solution of (11) is a portfolio that satisfies the experts' predictions. An explicit inclusion of this sort with economic meaning is presented in [4].

**Theorem 4.1.** *Suppose that $\mathbf{a}(t,x)$ is a uniformly bounded, Borel measurable set-valued mapping from $[0,T]\times\mathbb{R}^n$ to $\mathbb{R}^n$ with closed values. Let $\boldsymbol{\alpha}(t,x)$ be a uniformly bounded, Borel measurable set-valued mapping from $[0,T]\times\mathbb{R}^n$ to $\mathrm{S}_+(n)$ with closed values, and let there exist $\varepsilon_0 > 0$ such that, for all $t,x$, the $\varepsilon_0$-neighborhood of $\boldsymbol{\alpha}(t,x)$ in $\mathrm{S}(n)$ does not intersect the set $\mathrm{S}_0(n)$ of symmetric degenerate $n\times n$ matrices.*

*Then, for any initial condition $\xi(0) = \xi_0 \in \mathbb{R}^n$, inclusion (11) has a weak solution that is well-posed on the entire interval, $t \in [0,T]$.*

Theorem 4.1 is proved by the application of Lemma 3.1 and Krylov's theorem on the existence of weak solutions for stochastic differential equations with measurable coefficients.

**Theorem 4.2.** *Let $\mathbf{a}(t,x)$ be an upper semicontinuous set-valued mapping with closed convex values from $[0,T]\times\mathbb{R}^n$ to $\mathbb{R}^n$, and let it satisfy the estimate*

$$\|\mathbf{a}(t,x)\|^2 < K(1+\|x\|^2)$$

*for some $K > 0$.*

*Let $\boldsymbol{\alpha}(t,x)$ be an upper semicontinuous set-valued mapping with closed convex values from $[0,T]\times\mathbb{R}^n$ to $\bar{\mathrm{S}}_+(n)$ such that, for each $\alpha(t,x)\in\boldsymbol{\alpha}(t,x)$, the estimate*

$$\|tr\,\alpha(t,x)\| < K(1+\|x\|^2)$$

*takes place for some $K > 0$.*

*Then, for any initial condition $\xi(0)=\xi_0\in\mathbb{R}^n$, inclusion (11) has a weak solution that is well-posed on the entire interval, $t\in[0,T]$.*

Here, we consider the systems of continuous $\varepsilon_i$-approximations ($\varepsilon_i\to+0$) for $\boldsymbol{\alpha}(t,x)$ and, for $\mathbf{a}(t,x)$ that piecewise converge to Borel selections, show that the measures corresponding to the solutions of equations with those selections as coefficients, are weakly compact, take a limit measure, and show that this measure corresponds to a solution to the equation with limit selections as coefficients.

**Theorem 4.3.** *Suppose that $\boldsymbol{\alpha}(t,x)$ takes values in the space $\bar{\mathrm{S}}_+(n)$ of positive semidefinite symmetric matrices, has closed convex images, and is lower semicontinuous. We also assume that, for each $\alpha\in\boldsymbol{\alpha}(t,x)$, the following estimate*

$$\|tr\,\alpha(t,x)\| < K(1+\|x\|)^2$$

*holds for some $K > 0$. Let also $\mathbf{a}(t,x)$ be Borel measurable set-valued mapping and satisfy the estimate*

$$\|\mathbf{a}(t,x)\| < K(1+\|x\|)$$

*for some $K > 0$. Then, for any initial condition $\xi(0)=\xi_0$, there exists a weak solution of (11) that is well-posed on the entire interval, $t\in[0,T]$.*

Analogous statements take place for the inclusions with $\mathcal{P}$-mean derivatives [3].

Now consider inclusions in mean derivatives on manifolds. Let $\mathbf{a}(t,m)$ be a set-valued vector field on a manifold $M$, i.e., for every point $m\in M$, a certain set $\mathbf{a}(t,m)\subset T_mM$ is specified. Let also $\boldsymbol{\alpha}(t,m)$ be a set-valued symmetric positive semidefinite $(2,0)$-tensor field on $M$ (this means that, for all $t$, $m$, any tensor from the set $\boldsymbol{\alpha}(t,m)$ is symmetric and positive semidefinite). Consider the problem

$$\left\{\begin{array}{l} D\xi(t)\in\mathbf{a}(t,\xi(t)),\\ D_2\xi(t)\in\boldsymbol{\alpha}(t,\xi(t)).\end{array}\right. \tag{12}$$

We recall that a function $M\to\mathbb{R}$ is called proper if the preimage of every compact set in $\mathbb{R}$ is compact in $M$.

By $\mathcal{A}_{\mathbf{a},\boldsymbol{\alpha}}(t,m)$, we denote the set-valued second-order vector field with images

$$\begin{aligned}\mathcal{A}_{\mathbf{a},\boldsymbol{\alpha}}(t,m)=\{\mathcal{A}_{a,\alpha}(t,m)\mid \mathcal{H}\mathcal{A}_{a,\alpha}(t,m)=a(t,m),\\ \mathcal{Q}\mathcal{A}_{a,\alpha}(t,m)=\alpha(t,m)\text{ for } a\in\mathbf{a}(t,m),\alpha\in\boldsymbol{\alpha}(t,m)\},\end{aligned}$$

where the mappings $\mathcal{Q}$ and $\mathcal{H}$ are introduced by formulae (5) and (6), respectively.

**Theorem 4.4.** *Let $\boldsymbol{\alpha}(t,m)$ and $\mathbf{a}(t,m)$ be upper semicontinuous set-valued symmetric positive semidefinite $(2,0)$-tensor field and vector field on $M$, respectively, with closed convex images. In addition, let, for every compact set $\mathbf{K}\subset M$, the sets $\mathbf{a}([0,T],\mathbf{K})$ and $\boldsymbol{\alpha}([0,T],\mathbf{K})$ are compact, and let there exist a proper function $\varphi : M\to\mathbb{R}$ such that, at every $(t,m)$, the generator $\mathcal{A}_{a,\alpha}$ from a certain neighborhood $\mathcal{V}$ of the the graph of $\mathcal{A}_{\mathbf{a},\boldsymbol{\alpha}}(t,m)$, satisfies the condition $|\mathcal{A}\varphi| < C$ for some constant $C > 0$. Then, for any initial condition $\xi(0)=m_0$, there exists a weak solution of (12) well-posed on the entire interval $[0,T]$.*

For a process $\xi(t)$ with values in a manifold $M$ (in particular, in $\mathbb{R}^n$), we introduce the generator as a field of second-order semielliptic differential operators acting on the smooth enough function $f$ according to the rule

$$(13) \qquad \mathcal{A}(t,m)f = \lim_{\Delta t \to +0} E\big(\frac{f(\xi((t+\Delta t)) \wedge \theta_m) - f(\xi(t \wedge \theta_m))}{\Delta t} \mid \xi(t) = m\big),$$

where $\theta_m$ is the Markov time of the first hit of $\xi$ to the boundary of a certain chart containing $m$. The difference between (13) and the ordinary definition of a generator is that here we use the conditional expectation instead of the unconditional one. Note that if $\xi(t)$ is Markovian, both (13) and the ordinary definition introduce the same object. Obviously, the generator defined by (13) is a second-order tangent vector.

Let a field of set-valued second-order tangent vectors $\mathsf{A}(t,m)$ be given, i.e., in every second-order tangent space $\tau_m M$ to the manifold $M$, there is a certain set $\mathsf{A}(t,m)$ depending on $t \in [0,\infty)$. We want to find a stochastic process $\xi(\cdot)$ such that, for every $t$, its generator $L(t,m)$ introduced in (13), a.s. satisfies the inclusion

$$(14) \qquad L(t,\xi(t)) \in \mathsf{A}(t,\xi(t)).$$

Problems of this sort naturally arise if the process is described in terms of its generator.

**Theorem 4.5.** *Let $\mathsf{A}(t,m)$, $t \in [0,T]$, be an upper semicontinuous set-valued second-order vector field on a manifold $M$ with closed convex values such that: (i) for every $t \in [0,T]$, $m \in M$ for each $\mathcal{A} \in \mathsf{A}(t,m)$, the $(2,0)$-tensor $\mathcal{Q}_m\mathcal{A}$ is symmetric and positive semidefinite; (ii) for every compact set $\mathbf{K} \in M$, the set $\mathsf{A}([0,T],\mathbf{K})$ is compact in $\tau M$; (iii) there exist a proper function $\psi : M \to \mathbb{R}$, a constant $C > 0$, and a neighborhood $\mathcal{V}$ of the graph of $\mathsf{A}$ in $[0,T] \times \tau(M)$ such that, for every $(t,m,\mathcal{A}) \in \mathcal{V}$, the inequality $|\mathcal{A}\psi| < C$ holds.*

*Then, for every $m_0 \in M$, there exists a certain probability space and a stochastic process $\xi(t)$ with initial condition $\xi(0) = m_0$, well-posed for all $t \in [0,T]$, given on that probability space and taking values in $M$, such that, for its infinitesimal generator, inclusion (14) is a.s. satisfied.*

In [2], a certain existence theorem for (14) in $\mathbb{R}^n$ is obtained under some conditions of another sort.

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